On simple recognizing of bounded sets

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Abstract. We characterize those uniform spaces and commutative topological groups the bounded subsets of which can be recognized by using only one uniformly continuous pseudometric.

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The usual concept of boundedness of subsets of general metric space is not of great importance. It is neither an invariant of uniformly homeomorphic mappings, every metric space can be easily remetrized to become bounded without changing its uniform structure. Therefore other, more useful definitions of boundedness were introduced. We will work with the concept of boundedness (see Definition 1 below) studied in [1], [2] and by other authors (see [4] for references). This definition is meaningful in every uniform space, coincides with the usual boundedness in Euclidean spaces and in locally convex topological linear spaces, and every totally bounded set, hence every compact or finite set is bounded.

Let d be a pseudometric on a set X, $A \subset X$. If the function d is bounded on $A \times A$, we say that A is bounded for d. There are two important properties of the bounded subsets (see Theorems 1.12, 1.14 in [2]): If X is a uniform space, then $A \subset X$ is bounded in X iff (1) A is bounded for every pseudometric uniformly continuous on X, or iff (2) every uniformly continuous real-valued function $f : X \to \mathbb{R}$ is bounded on A (i.e. the set $f[A]$ is bounded in the usual sense).

Thus to recognize whether a subset is bounded it "suffices" to consider all uniformly continuous pseudometrics, real-valued functions respectively. The aim of this paper is to characterize those uniform spaces, commutative topological groups, topological linear spaces in which one pseudometric suffices for recognizing bounded sets. Let us agree on terminology.

For uniform spaces, the basic terminology from [5] will be used. Thus a uniformity $\mathcal{U}$ on a set X is a certain collection of entourages. If $V \in \mathcal{U}$, we put $V^1 := V$, $V^{n+1} := V \circ V^n$ for any $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. We say that a collection $\mathcal{C}$ of subsets of X is V-discrete provided $V[A] \cap B = \emptyset$ for any A, B in $\mathcal{C}$. A collection which is V-discrete for some V in $\mathcal{U}$ is called uniformly discrete. A uniformly discrete decomposition of X is called a uniform partition of X. If d is a pseudometric on X, $r > 0$, we put $B_d(r) := \{(x, y); d(x, y) < r\}$. The symbols d-diam A, d-dist(x, A) are certainly clear.

As usual, any commutative topological group G will be considered as a uniform space with the base of the uniformity consisting of all $U' := \{(x,y) \in G \times G; y - x \in U\}$ where U is a neighbourhood of the neutral element, which will be always denoted by o. Then $U'[A] = A + U := \{z + u; z \in A, u \in U\}$.

Let us recall some definitions.

Definition 1 ([2])**.** Let $(X,\mathcal{U})$ be a uniform space. A set $A \subset X$ is *bounded in* $(X,\mathcal{U})$ (shortly "bounded") if for each U in $\mathcal{U}$ there exist a finite set $K \subset X$ and $n \in \mathbb{N}$ such that $A \subset U^n[K]$. A set will be called *σ-bounded* if it is the union of countably many bounded sets.

A subset, the union of two bounded sets are bounded. A set bounded in a space need not be bounded in a subspace.

Definition 2 ([3])**.** Let $(X,\mathcal{U})$ be a uniform space. An entourage $U \in \mathcal{U}$ is *B-conserving* if for each bounded subset A of X the set $U[A]$ is bounded too. If there exists a B-conserving entourage we say that $(X,\mathcal{U})$ is *B-conservative*.

A uniform space $(X,\mathcal{U})$ is called uniformly locally bounded if there is U in $\mathcal{U}$ such that $U[x]$ is bounded for each $x \in X$. Clearly, every B-conservative uniform space is uniformly locally bounded. The converse is not true, in general (see Example in [3]). However, we have

Proposition 1. *Every locally bounded commutative topological group is B-conservative.*

Proof: Let U be a bounded neighbourhood of o in a group G. Let us show that U' is B-conserving. If $A \subset G$ is bounded, then $U'[A] = A + U$ is bounded by 2.11 in [2]. □

Before stating the basic definition let us present, without proof, the following simple proposition, which shows that it is no matter whether the boundedness is recognized by pseudometric or by functions. The trivial case of the void space, which is bounded, may be omitted.

Proposition 2. *Let X be a set, $a \in X$, $A \subset X$. If d is a pseudometric on X, then A is bounded for d if and only if the function $g_d := x \mapsto d(a,x)$ is bounded on A. If $f : X \to \mathbb{R}$ is a function on X, then f is bounded on A if and only if A is bounded for the pseudometric $e_f := (x,y) \mapsto |f(x) - f(y)|$. Moreover, if X is a uniform space, then g_d, e_f are uniformly continuous provided d, f respectively are uniformly continuous.*

Definition 3. Let X be a uniform space. Let d be a uniformly continuous pseudometric on X with the following property: a set $A \subset X$ is bounded if and only if A is bounded for d. Then we will say that d is a *B-recognizing* pseudometric on X. We will say that X is *B-simple* provided there exists a B-recognizing pseudometric on X.

Clearly, $\mathbb{R}$, any normed linear space, any bounded uniform space are B-simple. Notice we can also work with B-recognizing functions, see Proposition 2.

Proposition 3. *Every B-simple uniform space is B-conservative and σ-bounded.*

Proof: Let d be a B-recognizing pseudometric on a uniform space X. We may suppose $X \neq \emptyset$, choose $a \in X$. Put, for $n \in \mathbb{N}$, $X_n := \{x \in X; d(a,x) \leq n\}$. Then X_n are bounded, $\bigcup\{X_n; n \in \mathbb{N}\} = X$, hence X is σ-bounded. Let us prove that $V := B_d(1)$ is a B-conserving entourage. If $A \subset X$ is bounded, then d-diam$\,A < \infty$ and clearly d-diam$\,V[A] \leq d$-diam$\,A + 2 < \infty$, hence $V[A]$ is bounded. □

Proposition 4. *If a uniform space $(X,\mathcal{U})$ is σ-bounded, then every uniform partition of X is countable.*

Proof: There are bounded sets X_n such that $X = \bigcup\{X_n; n \in \mathbb{N}\}$. Let $\mathcal{P}$ be an infinite uniform partition of X. Let $U \in \mathcal{U}$ be such that $\mathcal{P}$ is U-discrete. Then for each n there exist $m(n) \in \mathbb{N}$ and a finite set K_n such that $X_n \subset U^{m(n)}[K_n]$. If $x \in P \in \mathcal{P}$, $k \in \mathbb{N}$, then $U^k[x] \subset P$, hence each X_n meets only finitely many sets from $\mathcal{P}$, therefore $\mathcal{P}$ is countable. □

Remark 1. If a uniform space X is connected, more generally chained — see [2], then every uniform partition of X consists of one set X only, thus it is countable. Let us present a simple example of a connected non-σ-bounded space. Denote by $\mathbb{R}^{\mathbb{N}}$ the cartesian product of $\aleph_0$ copies of $\mathbb{R}$. Suppose $\mathbb{R}^{\mathbb{N}} = \bigcup\{X_n; n \in \mathbb{N}\}$ where X_n are bounded. Each j-th projection $\mathrm{pr}_j[X_n]$ of any X_n is bounded in $\mathbb{R}$. Choose, for each n, $a_n \in \mathbb{R} \setminus \mathrm{pr}_n[X_n]$. The sequence (a_n) belongs to $\mathbb{R}^{\mathbb{N}}$ but to no X_n. Thus the converse of Proposition 4 is far from being true.

Theorem 1. *Let $(X,\mathcal{U})$ be a uniform space. Then the following properties are equivalent:*

(a) *$(X,\mathcal{U})$ is B-simple,*
(b) *$(X,\mathcal{U})$ is B-conservative and σ-bounded,*
(c) *$(X,\mathcal{U})$ is B-conservative and every uniform partition of X is countable.*

Proof: (a) ⇒ (b) follows from Proposition 3, (b) ⇒ (c) follows from Proposition 4. To prove (c) ⇒ (b), suppose $U \in \mathcal{U}$ is a symmetric B-conserving entourage. Define an equivalence $\sim$ on X: $x \sim y$ means $y \in U^n[x]$ for some $n \in \mathbb{N}$. Let $\mathcal{D}$ be the decomposition of X defined by $\sim$. Clearly, $\mathcal{D}$ is a uniform partition, hence $\mathcal{D}$ is countable. If $D \in \mathcal{D}$, $c \in D$, then $D = \bigcup\{U^n[c]; n \in \mathbb{N}\}$, hence D is σ-bounded. Therefore $(X,\mathcal{U})$ is σ-bounded and (b) holds.

It remains to prove (b) ⇒ (a). Suppose that $X = \bigcup\{X_n; n \in \mathbb{N}\}$, where X_n are bounded, $X_1 \neq \emptyset$ and $U \in \mathcal{U}$ is a B-conserving entourage, d is a uniformly continuous pseudometric on X such that $B_d(1) \subset U$. Put $Y_0 := X_1$, $Y_n := U[Y_{n-1} \cup X_n]$ for each $n \in \mathbb{N}$. Put

$$f_n(x) := \min\{1, d\text{-dist}(x, Y_{n-1})\} \quad \text{for} \quad n \in \mathbb{N},$$

$$f(x) := \sum (f_n(x); n \in \mathbb{N}).$$

If $x \in Y_m$, then $f_n(x) = 0$ for all $n > m$, hence $f(x) \leq m$. Thus f is a real-valued function. If $x \in X \setminus Y_m$, then $f_n(x) = 1$ for all $n \leq m$ hence $f(x) \geq m$.

Let us prove that f is uniformly continuous. Let $x, y \in X$, $d(x, y) < 1$. We are going to show that $|f(x) - f(y)| \leq 2d(x, y)$. If $y \in Y_0 = X_1$, then $f(y) = 0$ and $x \in U[X] = Y_1$, thus $0 \leq f(x) = f_1(x) \leq d\text{-dist}(x, X_1) \leq d(x, y)$, hence $|f(x) - f(y)| \leq d(x, y)$. Let $x \in Y_m \setminus Y_{m-1}$, $y \in Y_n \setminus Y_{n-1}$. Then $d(x, y) < 1$ implies $|m - n| \leq 1$. If $n = m$, we have $|f(x) - f(y)| = |f_m(x) - f_m(y)| \leq d(x, y)$. If $n = m+1$, then $0 \leq f(y) - f(x) = f_{m+1}(y) + f_m(y) - f_m(x)$; further $f_m(y) = 1$, $d\text{-dist}(y, Y_m) \leq d(y, x) < 1$, hence $f_{m+1}(y) = d\text{-dist}(y, Y_m)$. If $f_m(x) = 1$, then $f(y) - f(x) = d\text{-dist}(y, Y_m) + 1 - 1 \leq d(x, y)$. If $f_m(x) = d\text{-dist}(x, Y_{m-1})$, then $f_{m+1}(y) \leq d(x, y)$, $1 \leq d\text{-dist}(y, Y_{m-1}) \leq d(y, x) + d\text{-dist}(x, Y_{m-1}) = d(x, y) + f_m(x)$, hence $f_{m+1}(y) + 1 - f_m(x) \leq 2d(y, x)$, i.e. $|f(x) - f(y)| \leq 2d(x, y)$.

Of course, if $A \subset X$ is bounded, then f is bounded on A. If $A \subset X$ is not bounded, then $A \setminus Y_n$ holds for no n. Thus, for arbitrary n, there is an a_n in $A \setminus Y_n$. But $f(a_n) \geq n$, therefore f is not bounded on A. The pseudometric e_f from Proposition 2 is just the required B-recognizing pseudometric. □

If a uniform space is chained (see 1.1 in [2]), specially connected, or separable, then every uniform partition of this space is countable. Thus we obtain

Corollary. *Let a uniform space be chained or separable. Then it is B-simple if and only if it is B-conservative.*

Remark 2. Let X be an uncountable set endowed with the uniformity generated by the metric $d(x, y) := 1$ whenever $x \neq y$. Then the bounded subsets of X are exactly the finite subsets, X is B-conservative and is not σ-bounded. Example in [3] exhibits a uniformly locally bounded space S which is not B-conservative but is σ-bounded. Hence the both conditions in Theorem 1 (b) are independent. Notice there are examples of countable spaces which are not uniformly locally bounded or which are uniformly locally bounded but are not B-conservative.

Theorem 2. *A* (*pseudo*)*metrizable uniform space is B-simple if and only if it is* (*pseudo*)*metrizable by a B-recognizing* (*pseudo*)*metric.*

Proof: Let (X, e) be (pseudo)metric space. Let d be a B-recognizing pseudometric on (X, e), d is uniformly continuous. Then $e + d$ is again uniformly continuous. Now, $e \leq e + d$ implies that $e + d$ generates the same uniformity as e and $d \leq e + d$ implies that $e + d$ is B-recognizing. □

Proposition 1 and Theorem 1 just imply

Theorem 3. *Let G be a commutative topological group. Then the following properties are equivalent:*

(a) *G is B-simple,*
(b) *G is locally bounded and σ-bounded,*
(c) *G is locally bounded and every uniform partition of G is countable.*

Corollary 1. *Let G be a chained or separable commutative topological group. Then G is B-simple if and only if it is locally bounded.*

Corollary 2. *Let E be a topological linear space. Then E is B-simple if and only if it is locally bounded.*

Corollary 3. *Let E be a separated locally convex topological linear space. Then E is B-simple if and only if E is normable.*

Proof: Locally bounded locally convex spaces are normable by an old result of Kolgomorov (see the beginning of the Section 3 in [2] for details and references). □

Remark 3. Local convexity is essential in Corollary 3. Example: Let P be the space of measurable functions on the interval $[0,1]$ metrized by

$$d(x,y) := \int_0^1 \frac{|x(t)-y(t)|}{1+|x(t)-y(t)|}\,dt.$$

Then P is bounded (see Example 1 in [4]) and any uncountable product P^M is bounded (hence B-simple) and non-metrizable.

Let us formulate the result for frequently used locally compact groups.

Proposition 3. *A locally compact commutative topological group G is B-simple if and only if it is σ-compact.*

Proof: The condition is sufficient by Theorem 3 (b). Let V be a compact symmetric neighbourhood of o in G. Put $V_1 := V$, $V_{n+1} := V + V_n$ for $n \in \mathbb{N}$. Then all V_n are compact (see e.g. 2.11 in [2]), hence $G_0 := \bigcup\{V_n; n \in \mathbb{N}\}$ is σ-compact. Further, G_0 is an open subgroup, G/G_0 is a uniform partition. If G is B-simple, this partition is countable by Theorem 3, hence G is σ-compact. □

When considering topological groups, one often desires to use invariant pseudometrics. Let us recall briefly some well-known facts (see e.g. [2, 2.1–2.4]). Let G be a commutative topological group. If d is an invariant (i.e. $d(x+z,y+z) = d(x,y)$ for any x,y,z) pseudometric on G, then the function $r := x \mapsto d(x,o)$ is (1) $\mathbb{R}$-valued, non-negative, $r(o) = 0$, (2) even: $r(-x) = r(x)$ and (3) subadditive: $r(x+y) \le r(x) + r(y)$, all for any x,y. We will call such a function r a pseudonorm on G. On the other hand, given a pseudonorm r on G, the function $d := (x,y) \mapsto r(x-y)$ is an invariant pseudometric on G. Clearly, if r is generated by d, then r generates the original d. The pseudometric d is uniformly continuous iff the corresponding pseudonorm r is uniformly continuous or iff r is continuous or iff r is continuous at o. For any neighbourhood U of o, there is a continuous pseudonorm r such that $\{x \in G; r(x) < 1\} \subset U$. Every (pseudo)metrizable group can be (pseudo)metrized by an invariant (pseudo)metric.

Now we are ready to prove the following important

Theorem 4. *A commutative topological group G is B-simple if and only if there exists an invariant B-recognizing pseudometric on G.*

Proof: Suppose G is B-simple. Using the above mentioned facts, we will construct a continuous B-recognizing pseudonorm on G. Use Theorem 3 and choose a symmetric bounded neighbourhood U of o in G. Put $U_1 := U$, $U_{n+1} := U_n + U$ for $n \in \mathbb{N}$. Then all U_n are bounded (see [2, 2.11]) and $G_0 := \bigcup\{U_n; n \in \mathbb{N}\}$ is the subgroup of G generated by U. Let q be a continuous pseudonorm on G such that $\{x; q(x) < 1\} \subset U$. First, we construct another pseudonorm on G_0. If $x \in G_0$ put $s(x) := \inf \sum_{i=1}^{\ell} q(x_i)$ where the infimum is taken over all finite sequences $x_1, \dots, x_\ell$ such that $x_i \in U$ for each i and $x_1 + \dots + x_\ell = x$. In view of subadditivity of q, we may consider only such sequences that $x_i + x_j \in U$ for no $i \neq j$. It is easy to prove that s is a pseudonorm on G_0. As $s(x) = q(x)$ for $x \in U$, s is continuous at o. Let us show that, if $x \in G_0$, $s(x) < k \in \mathbb{N}$, then $x \in U_{2k+1}$. In fact, there are $\ell \in \mathbb{N}$ and $x_1, \dots, x_\ell$ in U such that $x = x_1 + \dots + x_\ell$, $q(x_1) + \dots + q(x_\ell) < k$ and $x_i + x_j \in U$ for no $i \neq j$. Thus $q(x_i) < 1/2$ for one indice i at most, hence $(\ell - 1)/2 \le k$, $\ell \le 2k + 1$, $x \in U_{2k+1}$.

Now, we are going to extend the pseudonorm s onto the whole group G. First, G/G_0 is a U-discrete partition of G, by Theorem 3 it is countable. Choose a single element in each coset of the subgroup G_0 and arrange them into a countable (finite or infinite) sequence $a_1, a_2, \dots$. Let, for any $n \in \mathbb{N}$, G_n be the subgroup of G generated by G_0 and all a_i with $i < n$. We will define by induction a pseudonorm r_n on G_n for each n such that $r_1 = s$, $r_j(x) = r_n(x)$ whenever $j \le n$ and $x \in G_j$. Put $r_1 := s$ and suppose that r_n has been defined. If $G_{n+1} = G_n$ put $r_{n+1} := r_n$. Suppose $a_n \notin G_n$. Then every element y of G_{n+1} can be written in the form

$$y = x + pa_n \tag{i}$$

where $x \in G_n$, p is an integer. Two cases should be distinguished:

(1) $pa_n \notin G_n$ for any $p \neq 0$; then the expression (i) is unique and we denote by Z_n the set of all integers;

(2) there is a smallest integer $m_n > 1$ such that $m_n a_n \in G_n$; then the expression (i) is unique provided $p \in Z_n$ where Z_n denotes the set $\{0, 1, \dots, m_n - 1\}$. We put

in the case (1): $r_{n+1}(x + pa_n) := r_n(x) + |p| \cdot n$,

in the case (2): $r_{n+1}(x + pa_n) := r_n(x) + \max\{n, r_n(m_n a_n)\}$ for $p > 0$,
$r_{n+1}(x) := r_n(x)$.

We must prove that r_{n+1} is a pseudonorm on G_{n+1} which has all required properties. Let us verify only the subadditivity in the case (2). Let $x_1, x_2 \in G_n$, $p_1, p_2 \in Z_n$. If $p_1 > 0$, $p_2 > 0$, then

$$\begin{aligned} r_{n+1}(x_1 + x_2 + (p_1 + p_2)a_n) &\le r_n(x_1 + x_2) + r_n(m_n a_n) + \max\{n, r_n(m_n a_n)\} \le \\ &\le r_n(x_1) + r_n(x_2) + 2\max\{n, r_n(m_n a_n)\} = r_{n+1}(x_1 + p_1 a_n) + r_{n+1}(x_2 + p_2 a_n). \end{aligned}$$

If $p_1 > 0$, $p_2 = 0$, then

$$\begin{aligned} r_{n+1}(x_1 + x_2 + p_1 a_n) &= r_n(x_1 + x_2) + \max\{n, r_n(m_n a_n)\} \le \\ &\le r_n(x_1) + r_n(x_2) + \max\{n, r_n(m_n a_n)\} = r_{n+1}(x_1 + p_1 a_n) + r_{n+1}(x_2). \end{aligned}$$

Finally, if $x \in G$, choose n such that $x \in G_n$ and put $r(x) := r_n(x)$. The function $r : G \to \mathbb{R}$ is defined correctly and r is a pseudonorm. It is continuous at o as $r(x) = s(x)$ for $x \in G_0 \supset U$ and hence continuous on G.

Let $A \subset G$ and let r be bounded on A. Choose a positive c such that $r(x) < c$ for each x in A. The construction of r_n's just implies that $A \subset G_k$ for some $k \leq c$. Consider $G_0 \neq G_k$. Let $n_1 < n_2 < \cdots < n_h$ be the sequence of all natural numbers n less than k and such that $G_{n+1} \neq G_n$. Then $G_{n_{j+1}} = G_{n_j+1}$, $G_k = G_{n_h+1}$. From the expression (i), it follows by induction that every element z of G_k can be uniquely expressed in the form

$$z = x + \sum_{j=1}^{h} p_j a_{n_j}$$

where $x \in G_0$ and $p_j \in Z_{n_j}$ for $j = 1, \dots, h$. Then

$$r(z) = r(x + \sum_{j=1}^{h-1} p_j a_{n_j}) + r(p_h a_{n_h}) = \cdots = r(x) + \sum_{j=1}^{h} r(p_j a_{n_j}).$$

If $z \in A$, then $r(z) < c$ and $r(p_j a_{n_j}) < c$ for each j. The definition of r_{n_j+1} just implies that for each j there is only a finite number of possibilities for the numbers p_j. Suppose that $A \cap (a_n + G_0) \neq \emptyset$ for infinitely many indices n. Then there must exist $n' \neq n''$, $z' \in A \cap (a_{n'} + G_0)$, $z'' \in A \cap (a_{n''} + G_0)$ such that

$$z' = x' + \sum_{j=1}^{h} p_j a_{n_j}, \quad z'' = x'' + \sum_{j=1}^{h} p_j a_{n_j},$$

where $x', x'' \in G_0$ (with the same integers p_j). But then $z' - z'' = x' - x'' \in G_0$, hence z', z'' belong to the same coset. This contradiction proves that there exists a finite $K \subset \mathbb{N}$ such that $A = A \cap G_k = \bigcup\{A \cap (a_n + G_0); n \in K\}$. This is also true in the case $G_0 = G_k$, of course. Let us show that each set in the union is bounded. If $n \in K$, $x \in A \cap (a_n + G_0)$, then $x - a_n \in G_0$ and $s(x - a_n) = r(x - a_n) < c + r(a_n)$. Choose $k' \in \mathbb{N}$ such that $c + r(a_n) < k'$ for each $n \in K$. Then, as proved above, $x - a_n \in U_{2k'+1}$, $U_{2k'+1}$ is bounded, $a_n + U_{2k'+1}$ are also bounded. Clearly, $A \subset \bigcup\{a_n + U_{2k'+1}; n \in K\}$, hence A is bounded. □

Theorem 5. *A (pseudo)metrizable commutative topological group is B-simple if and only if it can be (pseudo)metrized by an invariant B-recognizing (pseudo)metric.*

Proof is quite analogous to the proof of Theorem 2. Use invariant pseudometrics, or pseudonorms, and replace Theorem 1 by Theorem 4.

Open questions: (1) One may consider, instead of one B-recognizing pseudometric, a B-recognizing system of pseudometrics. There are non-B-simple spaces with a countable B-recognizing system. Is there a characterization of such spaces?

(2) Is it reasonable to consider B-recognizing by pseudometrics (or functions) which are not uniformly continuous?

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