On variations of functions of one real variable

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Abstract. We discuss variations of functions that provide conceptually similar descriptive definitions of the Lebesgue and Denjoy-Perron integrals.

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The conceptual affinity between the Denjoy-Perron and Lebesgue integrals was established vis-à-vis their Riemannian definitions more than twenty years ago in the works of Henstock [6], Kurzweil [8], and McShane [10]. Yet, until recently, the descriptive definitions of these integrals have little in common. Modifying the variational measures of Thomson [15] and elaborating on a new result of Bongiorno, Di Piazza, and Skvortsov [2], we shall elucidate the similarities between the contemporary descriptive definitions of the Lebesgue integral, Denjoy-Perron integral, and $\mathcal{F}$-integral of [12, Chapter 11].

Our ambient space is the real line $\mathbf{R}$. The interior, diameter, and the Lebesgue measure of a set $E \subset \mathbf{R}$ are denoted by $\operatorname{int} E$, $d(E)$, and $|E|$, respectively. A set $E \subset \mathbf{R}$ with $|E| = 0$ is called *negligible*. The terms "almost everywhere" and "absolutely continuous" always refer to the Lebesgue measure in $\mathbf{R}$. For $x \in \mathbf{R}$ and $\varepsilon \geq 0$, we let $U(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$.

A *cell* is a compact nondegenerate subinterval of $\mathbf{R}$, and a *figure* is a finite (possibly empty) union of cells. We say figures A and B *overlap* if their interiors meet. With each nonempty figure A, we associate two numbers: the *perimeter* $\|A\|$ equal to twice the number of connected components of A, and the *regularity*

$$r(A) = \frac{|A|}{d(A)\|A\|}\,.$$

For completeness, we let $\|A\| = r(A) = 0$ whenever A is the empty figure. Note that a figure A is a cell whenever $r(A) \geq 1/4$, in which case $r(A) = 1/2$.

Unless specified otherwise, all functions we shall consider are real-valued. If F is a function defined on a cell A and B is a subfigure of A whose connected components are the cells $[a_1, b_1], \dots, [a_n, b_n]$, we let

$$F(B) = \sum_{i=1}^{n} \bigl[F(b_i) - F(a_i)\bigr]\,.$$

Clearly, $F(B \cup C) = F(B) + F(C)$ whenever B and C are nonoverlapping subfigures of A. Denoting by the same symbol both the function of points and the associated function of figures will lead to no confusion.

A nonnegative function δ on a set $E \subset \mathbf{R}$ is called a *gage* on E whenever its null set $N_\delta = \{x \in E : \delta(x) = 0\}$ is countable. A *partition* is a collection (possibly empty) $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ such that $A_1, \dots, A_p$ are nonoverlapping figures, and $x_i \in A_i$ for $i = 1, \dots, p$. Given $\varepsilon > 0$, $E \subset \mathbf{R}^m$, and a gage δ on E, we say that P is

1. *cellular* if each A_i is a cell;
2. *ε-regular* if $r(A_i) > \varepsilon$ for $i = 1, \dots, p$;
3. *in* E if $\bigcup_{i=1}^{p} A_i \subset E$;
4. *anchored in* E if $\{x_1, \dots, x_p\} \subset E$;
5. *δ-fine* if it is anchored in E and $d(A_i) < \delta(x_i)$ for $i = 1, \dots, p$.

Given a positive gage δ on A, a collection $Q = \{(B_1, y_1), \dots, (B_q, y_q)\}$ is called a δ-fine *McShane partition* in A if $B_1, \dots, B_q$ are nonoverlapping subcells of A, each y_i is a point in A, and $d(B_i \cup \{y_i\}) < \delta(y_i)$ for $i = 1, \dots, q$. If each y_i belongs to a set $E \subset A$, we say Q is anchored in E.

Proposition 1. *A function f on a cell A is Lebesgue integrable in A if and only if there is a function F on A satisfying the following condition: given $\varepsilon > 0$, we can find a positive gage δ on A so that*

$$\sum_{i=1}^{p} \big| f(x_i)|A_i| - F(A_i) \big| < \varepsilon$$

for each δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. The function F is the indefinite Lebesgue integral of f in A; in particular, F is continuous.

Proof: The continuity of F at $x \in A$ is easily established by choosing a sufficiently small positive gage δ on A and considering a δ-fine partition

$$\{(A \cap [x - \eta, x + \eta], x)\}$$

(see [12, Corollary 2.3.2] for details).

Suppose the condition of the proposition is satisfied, and select a δ-fine McShane partition $\{(B_1, y_1), \dots, (B_q, y_q)\}$ in A. Denote by $x_1, \dots, x_p$ the distinct points among $y_1, \dots, y_q$, and let $C_i = \bigcup \{B_j : y_j = x_i\}$. As F is continuous, there is a δ-fine cellular partition $\{(D_1, x_1), \dots, (D_p, x_p)\}$ in A such that

$$\sum_{i=1}^{p} \Big[\big| f(x_i) \big| \cdot |D_i| + \big| F(D_i) \big| \Big] < \varepsilon$$

and

$$\sum_{i,k=1}^{p} \Big[\big| f(x_i) \big| \cdot |C_i \cap D_k| + \big| F(C_i \cap D_k) \big| \Big] < \varepsilon \,.$$

If $A_i = D_i \cup \bigl(C_i - \bigcup_{k=1}^{p} D_k\bigr)$, then $\bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ is a δ-fine partition in A, and we have

$$\begin{aligned}\varepsilon &> \sum_{i=1}^{p}\bigl[f(x_i)|A_i| - F(A_i)\bigr] = \sum_{i=1}^{p}\bigl[f(x_i)|D_i| - F(D_i)\bigr] \\ &\quad + \sum_{i=1}^{p}\bigl[f(x_i)|C_i| - F(C_i)\bigr] - \sum_{i,k=1}^{p}\bigl[f(x_i)|C_i \cap D_k| - F(C_i \cap D_k)\bigr] \\ &> \sum_{j=1}^{q}\bigl[f(y_j)|B_j| - F(B_j)\bigr] - 2\varepsilon\,.\end{aligned}$$

From this inequality we deduce $\sum_{j=1}^{q}\bigl|f(y_j)|B_j| - F(B_j)\bigr| < 6\varepsilon$.

Conversely, suppose we can find a positive gage δ on A so that

$$\sum_{j=1}^{q}\bigl|f(y_j)|B_j| - F(B_j)\bigr| < \varepsilon$$

for each δ-fine McShane partition in A, and select a δ-fine partition $\bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ in A. If $A_{i,1}, \dots, A_{i,n_i}$ are the connected components of A_i, then

$$\bigl\{(A_{i,j}, x_i) : j = 1, \dots, n_i \text{ and } i = 1, \dots, p\bigr\}$$

is a δ-fine McShane partition in A, and we have

$$\sum_{i=1}^{p}\bigl|f(x_i)|A_i| - F(A_i)\bigr| \le \sum_{i=1}^{p}\sum_{j=1}^{n_i}\bigl|f(x_i)|A_{i,j}| - F(A_{i,j})\bigr| < \varepsilon\,.$$

Thus the condition of the theorem is equivalent to f being McShane integrable in A, and the proposition follows from [5, Theorem 10.9]. □

In Proposition 1, a positive gage is needed to assure the continuity of F. If F is assumed continuous and a positive gage is replaced by an arbitrary gage, the condition of Proposition 1 defines an integral that is closed with respect to the formation of improper integrals, and thus slightly more general than the Lebesgue integral.

Proposition 2. *A function f on a cell A is Denjoy-Perron integrable in A if and only if there is a continuous function F on A satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ on A so that*

$$\sum_{i=1}^{p}\bigl|f(x_i)|A_i| - F(A_i)\bigr| < \varepsilon$$

for each δ-fine cellular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. The function F is the indefinite Denjoy-Perron integral of f in A.

Proof: In view of [5, Chapter 11], it suffices to show that if the condition of the proposition holds, it holds already for a positive gage δ_+. To this end, enumerate the null set N_δ of δ as $z_1, z_2, \ldots$, and find $\theta_n > 0$ so that

$$\bigl|f(z_n)\bigr| \cdot |C| + \bigl|F(C)\bigr| < 2^{-n}\varepsilon$$

for each cell $C \subset U(z_n, \theta_n)$ and $n = 1, 2, \ldots$. Now let

$$\delta_+(x) = \begin{cases} \theta_n & \text{if } x = z_n \text{ for an integer } n \geq 1, \\ \delta(x) & \text{if } x \in A - N_\delta. \end{cases}$$

Given a δ_+-fine cellular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$, observe that

$$\sum_{i=1}^{p} \bigl| f(x_i)|A_i| - F(A_i)\bigr| < \sum_{\delta(x_i)>0} \bigl| f(x_i)|A_i| - F(A_i)\bigr| + \sum_{n=1}^{\infty} 2^{-n}\varepsilon < 2\varepsilon,$$

which establishes the proposition. □

According to [5, Chapter 11], a gage in Proposition 2 can be replaced by a positive gage, in which case the continuity of F can be deduced as in Proposition 1. However, a slight modification of [12, Example 12.3.5] shows that Proposition 2 is false when cellular partitions, which are (1/4)-regular partitions, are replaced by α-regular partitions with $\alpha < 1/4$.

Propositions 1 and 2 lead to the definition of the $\mathcal{F}$-integral, which lies properly in between the Lebesgue and Denjoy-Perron integrals. It was introduced in [13] as a coordinate free multidimensional integral that integrates partial derivatives of differentiable functions (cf. [11]).

Definition 3. A function f on a cell A is called *$\mathcal{F}$-integrable* in A whenever there is a continuous function F on A satisfying the following condition: given $\varepsilon > 0$, we can find a gage δ on A so that

$$\sum_{i=1}^{p} \bigl| f(x_i)|A_i| - F(A_i)\bigr| < \varepsilon$$

for each δ-fine ε-regular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. The function F, uniquely determined by f, is called the *indefinite $\mathcal{F}$-integral* of f in A.

We note that the additivity properties of the $\mathcal{F}$-integral depend on the use of arbitrary, not necessarily positive, gages.

Remark 4. One may also consider the integrals defined by means of α-regular partitions, where $0 < \alpha < 1/4$ is a *fixed* number. Whether different α's produce different integrals is unclear, however, the work of Jarník and Kurzweil [9] suggests this may be the case. We do not study these integrals, since they may not be invariant with respect to diffeomorphisms (a diffeomorphic image of an α-regular figure need not be α-regular).

Let F be a function defined on a cell A, and let $E \subset A$ be an arbitrary set. Elaborating on the ideas of B.S. Thomson [15, Chapter 3], we define variations of F corresponding to the integrals discussed earlier.

Lebesgue variation:

$$V^L F(E) = \inf_{\delta} \sup_{P} \sum_{i=1}^{p} \left| F(A_i) \right|$$

where δ is a positive gage on E and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a δ-fine partition in A anchored in E.

Denjoy-Perron variation:

$$V^{DP} F(E) = \inf_{\delta} \sup_{P} \sum_{i=1}^{p} \left| F(A_i) \right|$$

where δ is a gage on E and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a δ-fine cellular partition in A anchored in E.

***$\mathcal{F}$*-variation:**

$$V^{\mathcal{F}} F(E) = \sup_{\alpha} \inf_{\delta} \sup_{P} \sum_{i=1}^{p} \left| F(A_i) \right|$$

where $\alpha > 0$, δ is a gage on E, and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a δ-fine α-regular partition in A anchored in E.

Arguments analogous to those of [15, Theorems 3.7 and 3.15] reveal that the extended real-valued functions $V^L F$, $V^{DP} F$, and $V^{\mathcal{F}} F$ are *Borel regular measures* in A (cf. [12, Lemma 3.3.14] and [3, Lemma 4.6]). We shall use this important fact in the proof of Proposition 6 below. The inequalities

$$V^{DP} F \leq V^{\mathcal{F}} F \leq V^L F \tag{1}$$

follow directly from the definitions.

Remark 5. Let F be a *continuous* function on a cell A. Employing ideas which proved Proposition 1, it is easy to show that in defining $V^L F(E)$ we can use δ-fine *McShane partitions*. Similarly, $V^{DP} F(E)$ can be defined by *positive gages* (cf. [2, Proposition 6] and the proof of Proposition 2).

If F is a function on a cell A, we denote by $VF(B)$ the *usual variation* of F over a figure $B \subset A$ [5, Chapter 4].

Proposition 6. *If F is a continuous function in a cell A, then*

$$V^{DP}F(B) = V^{\mathcal{F}}F(B) = VF(B) \tag{2}$$

for each figure $B \subset A$, and $V^L F(A) = VF(A)$. Moreover, $V^{DP}F = V^{\mathcal{F}}F$ whenever $V^{\mathcal{F}}F$ is σ-finite, and $V^{\mathcal{F}}F = V^L F$ whenever $V^L F$ is σ-finite.

Proof: Equality (2), which is an easy consequence of generalized Cousin's lemma [7, Lemma 6], was established in [1, Proposition 4.8].

If $V^{\mathcal{F}}F$ is σ-finite, then $V^{DP}F$ and $V^{\mathcal{F}}F$ vanish on all but countably many singletons. Thus it is not difficult to deduce from (2) that $V^{DP}F(U) = V^{\mathcal{F}}F(U)$ for each relatively open set $U \subset A$ (see [12, Lemma 3.4.4] for details). As $V^{DP}F$ and $V^{\mathcal{F}}F$ are σ-finite Borel regular measures in A, they coincide.

Let B be a subfigure of A, and let $\operatorname{int}_A B$ be the relative interior of B in A. Choose a positive gage δ on $\operatorname{int}_A B$ so that $A \cap U\big(x, \delta(x)\big) \subset B$ for each $x \in \operatorname{int}_A B$, and let $\big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ be a δ-fine partition in A anchored in $\operatorname{int}_A B$. By the choice of δ, each A_i is contained in B, and so if $A_{i,1}, \dots, A_{i,k_i}$ are the connected components of A_i, then

$$\sum_{i=1}^{p} \big|F(A_i)\big| \leq \sum_{i=1}^{p} \sum_{j=1}^{k_i} \big|F(A_{i,j})\big| \leq VF(B)\,.$$

From this and (1), we obtain

$$V^{\mathcal{F}}F(\operatorname{int}_A B) \leq V^L F(\operatorname{int}_A B) \leq VF(B)\,; \tag{3}$$

in particular, $V^L F(A) = VF(A)$ by (2). Using (3), the proof is completed by the argument employed in the previous paragraph. □

Lemma 7. *Let F be a function on a cell A. If $VF^L\big(\{x\}\big) = 0$ for each $x \in A$, then $VF^L(A) < +\infty$.*

Proof: Observe first F is continuous at $x \in A$ whenever $V^L F\big(\{x\}\big) = 0$. According to Remark 5, for each $y \in A$, there is an $\eta_y > 0$ such that $\sum_{j=1}^{q} \big|F(B_j)\big| < 1$ for every η_y-fine McShane partition $\big\{(B_1, y_1), \dots, (B_q, y_q)\big\}$ in A anchored in $\{y\}$, i.e., with $y_1 = \cdots = y_q = y$. Since A is compact, we can find points $z_1, \dots, z_n$ in A so that A is covered by $U(z_1, \eta_{z_1}), \dots, U(z_n, \eta_{z_n})$. Define a positive gage δ on A as follows: given $x \in A$, select a $\delta(x) > 0$ so that $U\big(x, \delta(x)\big)$ is contained in some $U(z_k, \eta_{z_k})$. Now each δ-fine McShane partition $\big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ in A is the disjoint union of families $P_1, \dots, P_n$ such that $A_i \subset U(z_k, \eta_k)$ whenever $(A_i, x_i) \in P_k$. It follows that $\big\{(A_i, z_k) : (A_i, x_i) \in P_k\big\}$ is an η_{z_k}-fine McShane partition in A anchored in $\{z_k\}$, and so

$$\sum_{i=1}^{p} \big|F(A_i)\big| = \sum_{k=1}^{n} \sum_{(A_i, x_i) \in P_k} \big|F(A_i)\big| < n\,.$$

In view of this and Remark 5, we have $VF^L(A) \leq n$. □

Proposition 8. *A function F in a cell A is absolutely continuous if and only if $V^L F$ is absolutely continuous.*

PROOF: Let F be absolutely continuous, and choose an $\eta > 0$ and a negligible set $E \subset A$. There is a $\delta > 0$ such that $\sum_{j=1}^n \big|F(B_j)\big| < \varepsilon$ for each collection $B_1, \dots, B_n$ of nonoverlapping subcells of A with $\sum_{j=1}^n |B_j| < \eta$. Find an open set U containing E so that $|U| < \eta$, and select a positive gage δ on E such that $U\big(x, \delta(x)\big) \subset U$ for each $x \in E$. Now if $\big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ is a δ-fine partition in A anchored in E, then it is a partition in U. If $A_{i,1}, \dots, A_{i,n_i}$ are the connected components of A_i, then

$$\sum_{i=1}^p \big|F(A_i)\big| \le \sum_{i=1}^p \sum_{j=1}^{n_i} \big|F(A_{i,j})\big| < \varepsilon \,,$$

and $V^L F(E) = 0$ by the arbitrariness of ε.

Conversely, assume that $V^L F$ is absolutely continuous, and choose an $\varepsilon > 0$. In view of Lemma 7, there is an $\eta > 0$ such that $V^L F(E) < \varepsilon$ whenever $E \subset A$ and $|E| < \eta$ [14, Theorem 6.11]. If $B \subset A$ is the union of nonoverlapping cells $B_1, \dots, B_n$ and $|B| < \eta$, then Proposition 6 implies

$$\sum_{j=1}^n \big|F(B_j)\big| \le \sum_{j=1}^n VF(B_j) = VF(B) = V^{DP} F(B) \le V^L F(B) < \varepsilon \,,$$

establishing the absolutely continuous of F. □

We shall use the expression "F is the indefinite integral of its derivative," which has the following usual meaning: the function F is differentiable almost everywhere in its domain, and it is the indefinite integral of F' extended arbitrarily to the domain of F.

Theorem 9. *A function F on a cell A is the indefinite Lebesgue integral of its derivative if and only if $V^L F$ is absolutely continuous.*

PROOF: The theorem follows from Proposition 8 and [5, Theorem 4.15]. □

Corollary 10. *A function F on a cell A is the indefinite Lebesgue integral of its derivative whenever $V^{DP} F$ is absolutely continuous and $V^L F$ is σ-finite.*

PROOF: If $V^L F$ is σ-finite, then $V^L F = V^{DP} F$ by Proposition 6, and the corollary follows from Theorem 9.

□

Proposition 11. *Let F be a continuous function on a cell A. If $V^{DP} F$ is absolutely continuous it is σ-finite.*

PROOF: In a roundabout way the proposition was proved in [2, Theorem 5]. We present a direct proof, which is virtually identical to that of [2, Theorem 1].

Suppose $V^{DP}F$ is absolutely continuous but not σ-finite, and denote by $U_\circ$ the union of all open sets U with $V^{DP}F(A \cap U) < +\infty$. Since $U_\circ$ is Lindelöf, the $V^{DP}F$ measure of $A \cap U_\circ$ is σ-finite. The set $K = A - U_\circ$ is compact, and it is easy to verify that $V^{DP}F(K \cap U) = +\infty$ for each open set U which meets K. As $V^{DP}F(\{x\}) = 0$ for every $x \in A$, the set K is perfect.

Claim. If U is an open set which meets K, then $A \cap U$ contains a disjoint collection $A_1, \dots, A_p$ of at least two cells such that the interior of each A_i meets K, and

$$\sum_{i=1}^{p} \big|F(A_i)\big| > 1 \,. \tag{4}$$

PROOF: Select a gage δ on $K \cap U$ so that $U\big(x, \delta(x)\big) \subset U$ for each $x \in K \cap U$. There is a δ-fine cellular partition $\big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ in A anchored in $K \cap U$ such that (4) holds. By the choice of δ, each A_i is contained in $A \cap U$. Since F is continuous and K is perfect, we can modify the cells A_i so that they become disjoint, their interiors meet K, and they are still contained in $A \cap U$ and satisfy (4). If $p = 1$ and $A_1 = [a, b]$, find points c and d so that $a < c < d < b$ and both (a, c) and (d, b) meet K. As F is continuous and

$$1 < \big|F(A_1)\big| \le \big|F\big([a,c]\big)\big| + \big|F\big([c,d]\big)\big| + \big|F\big([d,b]\big)\big| \,,$$

the points c and d can be selected so that $1 < \big|F\big([a,c]\big)\big| + \big|F\big([d,b]\big)\big|$. Thus we may assume $p \ge 2$, and the claim is established.

Using the claim, construct inductively disjoint families $\{A_{k,1}, \dots, A_{k,p_k}\}$ of subcells of A so that the following conditions are satisfied for $k = 1, 2, \dots$.

1. $K \cap \operatorname{int} A_{k,i} \neq \emptyset$ for $i = 1, \dots, p_k$.
2. Each $A_{k+1,j}$ is contained in some $A_{k,i}$.
3. Each $A_{k,i}$ contains at least two cells $A_{k+1,j}$.
4. $\big|\bigcup_{i=1}^{p_k} A_{k,i}\big| < 1/k$.
5. $\sum_{A_{k+1,j} \subset A_{k,i}} \big|F(A_{k+1,j})\big| > 1$ for $i = 1, \dots, p_k$.

It follows from conditions 3 and 4 that $N = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{p_k} A_{k,i}$ is a negligible perfect subset of A. We obtain a contradiction by showing that $V^{DP}F(N) \ge 1$.

To this end, choose a gage δ on N, and for $k = 1, 2, \dots$, let

$$N_k = \big\{x \in N : \delta(x) > 1/k\big\} \,.$$

Since the set $\bigcup_{k=1}^{\infty} N_k = N - N_\delta$ is G_δ, it is completely metrizable [4, Theorem 4.3.23]. By the Baire category theorem some N_s is dense in $(N - N_\delta) \cap U$, where U is an open set which meets $N - N_\delta$. There is an integer $k > s$ such that some $A_{k-1,j}$ is contained in U. Condition 4 implies that $d(A_{k,i}) < 1/s$ for $i = 1, \dots, p_k$. Hence choosing $x_i \in A_{k,i} \cap N_s$, we obtain a δ-fine cellular partition $\big\{(A_{k,1}, x_1), \dots, (A_{k,p_k}, x_{p_k})\big\}$ in A anchored in N. The desired contradiction follows from condition 5.

□

Theorem 12. *A continuous function F on a cell A is the indefinite Denjoy-Perron integral of its derivative if and only if $V^{DP}F$ is absolutely continuous.*

Proof: The theorem follows from Proposition 11 and [1, Theorem 4.4], which asserts that F is the indefinite Denjoy-Perron integral of its derivative if and only if $V^{DP}F$ is absolutely continuous and σ-finite. □

Theorem 13. *A continuous function F on a cell A is the indefinite $\mathcal{F}$-integral of its derivative if and only if $V^{\mathcal{F}}F$ is absolutely continuous.*

Proof: As the converse follows from [3, Theorem 5.3], assume $V^{\mathcal{F}}F$ is absolutely continuous. Then $V^{DP}F$ is absolutely continuous by (1), and Theorem 12 implies that F is differentiable at each $x \in A - N$, where N is a negligible subset of A. We show that F is the indefinite $\mathcal{F}$-integral of the function f defined by the formula

$$f(x) = \begin{cases} F'(x) & \text{if } x \in A - N, \\ 0 & \text{if } x \in N. \end{cases}$$

To this end, choose an $\varepsilon > 0$, and for each $x \in A - N$, find an $\eta_x > 0$ so that

$$\bigl|F'(x)|B| - F(B)\bigr| < \varepsilon^2 d(B)\|B\|$$

for each figure $B \subset A \cap U(x, \eta_x)$; the existence of η_x is a readily verifiable consequence of the differentiability of F at x. By our assumption, there is a gage β on N such that $\sum_{i=1}^{p}\bigl|F(A_i)\bigr| < \varepsilon$ for each β-fine ε-regular partition $\bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ in A anchored in N. Let

$$\delta(x) = \begin{cases} \eta_x & \text{if } x \in A - N, \\ \beta(x) & \text{if } x \in N, \end{cases}$$

and select a δ-fine ε-regular partition $\bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ in A. Then

$$\begin{aligned} \sum_{i=1}^{p}\bigl|f(x)|A_i| - F(A_i)\bigr| &= \sum_{x_i \in N}\bigl|F(A_i)\bigr| + \varepsilon^2 \sum_{x_i \notin N} d(B)\|B\| \\ &< \varepsilon + \varepsilon \sum_{x_i \notin N} |A_i| \le \varepsilon\bigl(1 + |A|\bigr), \end{aligned}$$

and the theorem is proved. □

Corollary 14. *Let F be a continuous function on a cell A. If $V^{\mathcal{F}}F$ is absolutely continuous it is σ-finite.*

Proof: In view of Theorem 13, the function F is the indefinite $\mathcal{F}$-integral of a function f on A. Fix an integer $n \ge 1$ and let $E = \bigl\{x \in A : \bigl|f(x)\bigr| < n\bigr\}$. Since

$$A = \bigcup_{k=1}^{\infty}\bigl\{x \in A : \bigl|f(x)\bigr| < k\bigr\},$$

it suffices to show that $V^{\mathcal{F}}F(E) < +\infty$. To this end, choose a positive $\varepsilon \le 1$, and find a gage δ on A so that

$$\sum_{i=1}^{p} \bigl| f(x)|A_i| - F(A_i) \bigr| < \varepsilon$$

for each δ-fine ε-regular partition in A. If such a partition is anchored in E, then

$$\begin{aligned}\sum_{i=1}^{p} \bigl| F(A_i) \bigr| &\le \sum_{i=1}^{p} \bigl| f(x)|A_i| - F(A_i) \bigr| + \sum_{i=1}^{p} \bigl| f(x) \bigr| \cdot |A_i| \\ &< \varepsilon + n \sum_{i=1}^{p} |A_i| \le 1 + n|A|\,,\end{aligned}$$

and we conclude that $V^{\mathcal{F}}F(E) \le 1 + n|A|$. □

Corollary 15. *A continuous function F on a cell A is the indefinite $\mathcal{F}$-integral of its derivative whenever $V^{DP}F$ is absolutely continuous and $V^{\mathcal{F}}F$ is σ-finite.*

Proof: If $V^{\mathcal{F}}F$ is σ-finite, then $V^{\mathcal{F}}F = V^{DP}F$ by Proposition 6, and the corollary follows from Theorem 13. □

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