On two results of Singhof

Augustin-Liviu Mare

Abstract. For a compact connected semisimple Lie group G we shall prove two results (both related with Singhof's paper [13]) on the Lusternik-Schnirelmann category of the adjoint orbits of G, respectively the 1-dimensional relative category of a maximal torus T in G. The techniques will be classical, but we shall also apply some basic results concerning the so-called $\mathcal{A}$-category (cf. [14]).

Keywords: Lusternik-Schnirelmann category, Lie groups, adjoint orbits
Classification: 55P50, 57T20, 57T15

The following results were proved in [13] by methods which combine in an ingenious manner the classical theories of Lie groups and of Lusternik-Schnirelmann-type categories.

Theorem A. *Let G be a compact connected Lie group and T a maximal torus of G. Then*

$$\operatorname{cat} G/T = \frac{1}{2}\dim G/T + 1.$$

For an arbitrary finitely generated Abelian group π, denote by $\varphi(\pi)$ the smallest number n such that π is the direct sum of n cyclic groups.

Theorem B. *Let G be a compact connected Lie group and T a maximal torus of G. Then*

$$\operatorname{cat}_G T = \varphi(\pi_1 G) + 1.$$

Consider now $\mathfrak{g}$ the Lie algebra of G. Take $X \in \mathfrak{g}$ and denote by G_X the Ad-stabilizer of X (note that X is regular iff G_X is a maximal torus in G). The adjoint orbits $\operatorname{Ad} G.X$ were during the last years frequently considered and studied, both from the topological point of view (mention only the detailed descriptions of the cohomology ring given in [1] or [2]) and from differential perspective (they represent fundamental examples of the so-called theory of isoparametric submanifolds, recently initiated by R. Palais and C.L. Terng). In connection with Theorem A we shall prove:

Theorem 1. *Let G be a compact connected semisimple Lie group and X an element of its Lie algebra. Then*

$$\operatorname{cat}(\operatorname{Ad} G.X) = \frac{1}{2}\dim(\operatorname{Ad} G.X) + 1.$$

In [6] Fox considers for the first time the so-called q-dimensional relative (homotopical) category associated to an inclusion. Many other developments were obtained afterwards; among them, the notion of $\mathcal{A}$-category (cf. [4, Examples 1.2(3)]). The following result concerning the 1-dimensional category will be proved in the second section.

Theorem 2. *Let G be a compact connected semisimple Lie group and T a maximal torus. Then*

$$\pi_1 - \operatorname{cat}_G T = \varphi(\pi_1 G) + 1.$$

1. The Lusternik-Schnirelmann category of G/G_X

Recall that the Lusternik-Schnirelmann category of a topological space M is the number cat M equal to the least number of sets in an open finite covering of M with subsets contractible in M; if such a covering does not exist, take cat $M = \infty$. Both homotopical and differential aspects are concentrated in this notion; on the one hand, it is a homotopical invariant, and on the other hand, when M is a compact differentiable manifold, the number of critical points of a real function on M cannot be less than cat M.

Let us consider G a compact connected Lie group, $T \subseteq G$ a maximal torus and $\mathfrak{t} \subseteq \mathfrak{g}$ their Lie algebras.

Proposition 1. *For any X belonging to $\mathfrak{g}$, the adjoint orbit $\operatorname{Ad} G.X$ is simply connected. Equivalently the stabilizer G_X is connected.*

Proof: Lt $X_0 \in \mathfrak{t}$ be regular. Its orbit $\operatorname{Ad} G.X_0$ is a full isoparametric submanifold of $\mathfrak{g}$, with uniform multiplicity 2. The orbit foliation $\{\operatorname{Ad} G.X \mid X \in \mathfrak{t}\}$ is just the parallel foliation of $\operatorname{Ad} G.X_0$ on $\mathfrak{g}$ (cf. [9, Example 6.5.6]). Since all multiplicities are greater than 1, by Theorem 5.7 of [8], any leaf $\operatorname{Ad} G.X$ is simply connected, and the proof is finished. □

The following result is mentioned in A. Borel's work [1]: the quotients of two locally isomorphic compact connected Lie groups G and G' by maximal tori T and T' are homeomorphic (see p. 188). We shall generalize it as follows:

Proposition 2. *Let $p : \widetilde{G} \to G$ be the universal group covering of the compact connected Lie group G of Lie algebra $\mathfrak{g}$, X an element of $\mathfrak{g}$, $\widetilde{G}_X$ and G_X the stabilizers of X. Then*

(a) $p(\widetilde{G}_X) = G_X$,
(b) *the induced map $\varrho : \widetilde{G}/\widetilde{G}_X \to G/G_X$ is a homeomorphism.*

Proof: (a) One can easily see that $p(\widetilde{G}_X) \subseteq G_X$. It follows that $p|_{\widetilde{G}_X} : \widetilde{G}_X \to G_X$ is a local isomorphism and because G_X is connected, it is generated by $p(\widetilde{G}_X)$. So $p(\widetilde{G}_X) = G_X$.

(b) By the classical facts: $\ker p \subseteq Z(\widetilde{G})$ (cf. [11, Lemma 6, p. 195]), $Z(\widetilde{G}) \subseteq T$ (cf. [3, Theorem 2.3, Chapter IV]) and $T \subseteq \widetilde{G}_X$, the injectivity of ϱ is clear. So ϱ is a homeomorphism. □

Remark that the homogeneous space G/G_X depends only on $\mathfrak{g}$ and X, but not on the involved connected Lie group G. This fact offers the possibility to deduce informations about the cohomology ring of G/G_X from Theorem III$''$ of [2], even without the hypothesis G simply connected.

Proposition 3. *Let G be a compact connected semisimple Lie group of Lie algebra $\mathfrak{g}$ and X an element of G. Then the ring $H^*(G/G_X, \mathbb{Q})$ is generated by 1 and $H^2(G/G_X, \mathbb{Q})$.*

Notice that the above mentioned orbit G/G_X is of dimension $n = \dim G - \operatorname{rank} G - 2m$, where m is the number of hyperplanes of the infinitesimal diagram containing X; it is also orientable (being simply connected) and so $H^n(G/G_X, \mathbb{Q}) = \mathbb{Q}$. The $\mathbb{Q}$-cohomological length will be then cuplength $(G/G_X) \geq \frac{n}{2}$, and so $\operatorname{cat} G/G_X \geq \frac{n}{2} + 1$.

On the other hand, G/G_X being simply connected, by Corollary 3.3 of [7] one obtains $\operatorname{cat}(G/G_X) \leq \frac{n}{2} + 1$.

In the end of the section, let us take for instance the homogeneous space of the form G/G_X from [12] and calculate their Lusternik-Schnirelmann category ($n, n_1, \ldots, n_k$ will be positive integers, $\sum n_j = n$).

(a) The complex flag manifold $W(n_1, \ldots, n_k) = U(n)/U(n_1) \times \cdots \times U(n_k)$ has the Lusternik-Schnirelmann category equal to $\frac{1}{2}\big(n^2 - \sum_j n_j^2\big) + 1$. Consequently, for the complex Grassmann manifold $G_{k,n} = U(n)/U(k) \times U(n-k)$, we have $\operatorname{cat} G_{k,n} = k(n-k) + 1$.

(b) $\operatorname{cat} \mathrm{SO}(2n)/U(n_1) \times \cdots \times U(n_k) = \frac{1}{2}\big[n(2n-1) - \sum n_j^2\big] + 1$ and so the symmetric space $\mathrm{SO}(2n)/U(n)$ will have $\operatorname{cat} \mathrm{SO}(2n)/U(n) = \frac{1}{2}n(n-1) + 1$.

(c) $\operatorname{cat} \mathrm{SO}(2n+1)/U(n_1) \times \cdots \times U(n_k) \times 1 = \frac{1}{2}\big[n(2n+1) - \sum n_j^2\big] + 1$.

(d) $\operatorname{cat} \mathrm{Sp}(n)/U(n_1) \times \cdots \times U(n_k) = \frac{1}{2}\big[n(2n+1) - \sum n_j^2\big] + 1$.

The symmetric space $\mathrm{Sp}(n)/U(n)$ will have $\operatorname{cat} \mathrm{Sp}(n)/U(n) = \frac{n(n+1)}{2} + 1$.

2. The 1-dimensional category of T in G

By technical reasons, we prefer to transpose the general definition of $\mathcal{A}$-category and some basic results concerning it (cf. [4]) to the older 1-dimensional category (see [6] or [5]).

Denote by $\mathcal{C}_1$ the class of 1-connected CW-complexes. Define the **$\mathcal{C}_1$-category** of a map $f : N \to M$ to be the number $\mathcal{C}_1 - \operatorname{cat}(f)$, the smallest cardinality k of a finite numerable covering $\{N_1, \ldots, N_k\}$ of N such that for each $j = 1, \ldots, k$ the restriction $f \,|\, N_j : N_j \to M$ factors through some space in $\mathcal{C}_1$ up to homotopy (i.e. there exist $C_j \in \mathcal{C}_1$ and maps $\alpha_j : N_j \to C_j$, $\beta_j : C_j \to M$ such that $\beta_j \alpha_j$

is homotopic to $f\,|\,_{N_j}$). For a subspace N of M, the relative **1-dimensional category** of N in M will be $\pi_1 - \mathrm{cat}_M N = \mathcal{C}_1 - \mathrm{cat}(N \hookrightarrow M)$.

Let G be again a compact connected Lie group and $T \subseteq G$ a maximal torus. Consider the decomposition of $\pi_1 G$ as $\pi_1 G = \mathcal{F} \oplus_q \oplus_{\mathrm{prime}} \mathcal{T}_q$, where $\mathcal{F}$ is the free part and $\mathcal{T}_q$ the subgroup of all order q^m ($m \geq 1$) elements; denote by $r = \mathrm{rank}\,\mathcal{F}$, $r_q = \mathrm{rank}\,\mathcal{T}_q$. A classical result says that the inclusion $i : T \hookrightarrow G$ induces $i_\# : \pi_1 T \to \pi_1 G$ surjective. It then follows that $i^* : H^1(G, \mathbb{Z}_q) \to H^1(T, \mathbb{Z}_q)$ is injective, for any prime q. By the Hurewicz isomorphism, $H^1(G, \mathbb{Z}_q) \cong \mathrm{Hom}(\pi_1 G, \mathbb{Z}_q)$ is isomorphic to a finite direct sum $\oplus \mathbb{Z}_q$ with $r + r_q$ terms. Since $H^*(T, \mathbb{Z}_q)$ is an exterior algebra, there exist in $H^1(G, \mathbb{Z}_q)$ a number of $r + r_q$ elements whose product does not go to zero under i^*. One can now use Proposition 3.1 of [4]: for any $C \in \mathcal{C}_1$ and any $f : C \to G$, the map $f^* : H^1(G, \mathbb{Z}_q) \to H^1(C, \mathbb{Z}_q)$ is identically zero, and so

$$\pi_1 - \mathrm{cat}_G T = \mathcal{C}_1 - \mathrm{cat}(T \hookrightarrow G) \geq r + r_q + 1.$$

But choosing q with r_q maximal, $r + r_q$ will be the minimal number of terms for a decomposition of $\pi_1 G$ in a direct sum of cyclic groups, the number that Singhof denotes by $\varphi(\pi_1 G)$. We have just proved:

Lemma 1. *Let G be a compact connected Lie group and $T \subseteq G$ a maximal torus. Then $\pi_1 - \mathrm{cat}_G T \geq \varphi(\pi_1 G) + 1$.*

It remains to show that:

Lemma 2. *Let G be a compact connected semisimple Lie group and $T \subseteq G$ a maximal torus. If $\pi_1 G$ admits a decomposition as a direct sum of k cyclic groups, then $\pi_1 - \mathrm{cat}_G T \leq k + 1$.*

The proof is based on the relation between the 1-dimensional and sectional categories (see Section 4 of [4] for the definition and basic properties concerning the sectional category).

Let $\widetilde{G}$ be the universal covering group of G. One can consider $G = \widetilde{G}/C$, with $C \subseteq Z(\widetilde{G})$ a finite central subgroup; moreover $\pi_1 G \cong C$ (cf. [3, Chapter V, Remark 7.13]). Any maximal torus of G is of the form $\widetilde{T}/C$, $\widetilde{T}$ maximal torus in $\widetilde{G}$.

The map $p : \widetilde{G} \to G$ is $\mathcal{C}_1$-universal (in the sense of [4]). Consequently $\pi_1 - \mathrm{cat}_G \widetilde{T}/C = \mathrm{secat}(p')$, where $p' : U' \to \widetilde{T}/C$ is the pullback over $i : \widetilde{T}/C \hookrightarrow G$ of the Hurewicz fibration associated to p. Here $U' = \{(g, \alpha, tC) \in \widetilde{G} \times \mathrm{Top}(I, G) \times \widetilde{T}/C \,|\, \alpha(0) \text{ and } \alpha(1) = tC\}$ and $p'(g, \alpha, tC) = tC$. But considering $h : \widetilde{T} \to U'$, $h(t) = (t, e_{tC}, tC)$, where e_{tC} is the constant loop in G, we have $\mathrm{secat}(p') \leq \mathrm{secat}(p'h)$ (notice that $g = p'h : \widetilde{T} \to \widetilde{T}/C$ is the natural map). Because C is a direct sum of k cyclic subgroups of $\widetilde{T}$, one can find a torus $\widetilde{T}_C$, embedded as a subgroup of $\widetilde{T}$, $\dim \widetilde{T}_C \leq k$. There also exist an another toral subgroup $\widetilde{T}' \subseteq \widetilde{T}$, $\widetilde{T} = \widetilde{T}_C \times \widetilde{T}'$. It follows that $\widetilde{T}/C = \widetilde{T}_C/C \times \widetilde{T}'$ and $g' \times 1_{\widetilde{T}'} : \widetilde{T}_C \times \widetilde{T}' \to \widetilde{T}_C/C \times \widetilde{T}'$,

$g' : \widetilde{T}_C \to \widetilde{T}_C/C$ the natural map. Conclude by $\mathrm{secat}(g' \times 1_{\widetilde{T}'}) = \mathrm{secat}(g') \leq 1 + \dim \widetilde{T}_C/C \leq k+1$ (cf. [4, Corollary 4.7]).

References

[1] Borel A., *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. Math. **57** (1953), 115–207.

[2] Bott R., Samelson H., *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029.

[3] Bröcker Th., tom Dieck T., *Representations of Compact Lie Groups*, Springer Verlag, 1985.

[4] Clapp M., Puppe D., *Invariants of the Lusternik-Schnirelmann type and the topology of critical sets*, Trans. Amer. Math. Soc. **298** (1986), 604–620.

[5] Eilenberg S., Ganea T., *On the Lusternik-Schnirelmann category of abstract groups*, Ann. of Math. **65** (1957), 517–518.

[6] Fox R.H., *On the Lusternik-Schnirelmann category*, Ann. of Math. **42** (1941), 333–370.

[7] Ganea T., *Lusternik-Schnirelmann category and strong category*, Illinois J. Math. **11** (1967), 417–427.

[8] Hsiang W.Y., Palais R.S., Terng C.L., *The topology of isoparametric submanifolds*, J. Diff. Geom. **27** (1988), 423–461.

[9] Palais R.S., Terng C.L., *Critical Point Theory and Submanifolds Geometry*, Springer Verlag, 1988.

[10] Pop I., *Topologie Algebraică*, Ed. Ştiinţifică, Bucureşti, 1990. (Romanian)

[11] Postnikov M., *Lie Groups and Lie Algebras*, Mir Publishers, Moscow, 1986.

[12] Ramanujam S., *Applications of Morse theory to some homogeneous spaces*, Tohoku Math. J. **21** (1969), 343–354.

[13] Singhof W., *On the Lusternik-Schnirelmann category of Lie groups*, Math. Z. **145** (1975), 111–116.

Faculty of Mathematics, Babeş-Bolyai University, Cluj-Napoca, Romania

(*Received* April 4, 1996)