Sets of determination for solutions of the Helmholtz equation

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Abstract. Let $\alpha > 0$, $\lambda = (2\alpha)^{-1/2}$, S^{n-1} be the $(n-1)$-dimensional unit sphere, σ be the surface measure on S^{n-1} and $h(x) = \int_{S^{n-1}} e^{\lambda\langle x,y\rangle}\, d\sigma(y)$.

We characterize all subsets M of $\mathbb{R}^n$ such that

$$\inf_{x\in\mathbb{R}^n} \frac{u(x)}{h(x)} = \inf_{x\in M} \frac{u(x)}{h(x)}$$

for every positive solution u of the Helmholtz equation on $\mathbb{R}^n$. A closely related problem of representing functions of $L_1(S^{n-1})$ as sums of blocks of the form $e^{\lambda\langle x_k,\cdot\rangle}/h(x_k)$ corresponding to points of M is also considered. The results provide a counterpart to results for classical harmonic functions in a ball, and for parabolic functions on a slab, see References.

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Preliminaries

In this paper the following notation is used: Small letters, such as x, y, will denote points in $\mathbb{R}^n$, S^{n-1} the $(n-1)$-dimensional unit sphere and σ the surface measure on S^{n-1}.

Consider, for $\alpha > 0$ fixed, the Helmholtz equation

$$\Delta u - 2\alpha u = 0 \quad \text{on } \mathbb{R}^n.$$

Theorem A. *A function u on $\mathbb{R}^n$ is a difference of two positive solutions of the Helmholtz equation if and only if there is a signed measure μ_u on S^{n-1} such that for all $x \in \mathbb{R}^n$*

$$\int_{S^{n-1}} e^{\lambda\langle x,y\rangle}\, d|\mu_u|(y) < \infty$$

and

$$u(x) = \int_{S^{n-1}} e^{\lambda\langle x,y\rangle}\, d\mu_u(y),$$

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where $\lambda = (2\alpha)^{-1/2}$.

The solution u *is positive if and only if* μ_u *is a measure.*

Proof: This representation theorem can be proved by means of Martin boundary, see [8]. For a different proof, see [6]. □

The solution corresponding to σ will be denoted by h.

For $\nu \in \mathbb{R}$ the function I_ν is "the Bessel function with an imaginary argument" of the order ν regular at zero. (For details see any book about Bessel functions, for example [14, p. 17].)

Then

$$h(x) = C\lambda^{(2-n)/2}\|x\|^{(2-n)/2} I_{(n-2)/2}(\lambda\|x\|),$$

with C chosen so that $h(0) = \omega_n$, the area of the unit sphere in $\mathbb{R}^n$. (See [6, p. 261].)

For f, g two functions on $\mathbb{R}^n$, $f \sim g$ will mean that $\lim\limits_{\|x\|\to\infty} \frac{f(x)}{g(x)} = 1$.

As $I_\nu(\|x\|) \sim (2\pi\|x\|)^{-1/2} e^{\|x\|}$ (see for example [14, pages 17 and 203]), we have that

$$\lim_{\|x\|\to\infty} \frac{h(x)\|x\|^{(n-1)/2}}{e^{\lambda\|x\|}} = C\lambda^{(2-n)/2}(2\pi)^{-1/2};$$

this constant will be denoted by κ.

A solution u of the Helmholtz equation will be called h-bounded if there exist real constants c_1 and c_2 such that $c_1 h(x) \leqq u(x) \leqq c_2 h(x)$ for all $x \in \mathbb{R}^n$.

Moreover, a solution u of the Helmholtz equation will be called simple if there exists a σ-measurable subset A of S^{n-1} such that $u(x) = \int\limits_A e^{\lambda\langle x,y\rangle}\, d\sigma(y)$ for any $x \in \mathbb{R}^n$.

Definition. *For* $y \in S^{n-1}$, $b \in \mathbb{R}^+$, $k \in \mathbb{R}^+$ *define the admissible region* $A(y,b)$ *to be*

$$\{x \in \mathbb{R}^n; \|x - \|x\|y\| < b\|x\|^{\frac{1}{2}}\}$$

and the truncated admissible region $A^k(y,b)$ *to be*

$$A(y,b) \cap \{x \in \mathbb{R}^n; \|x\| > k\}.$$

Let $M \subset \mathbb{R}^n$ *and* $y \in S^{n-1}$. *The point* y *will be called a* b*-admissible limit point of* M *if for any* $k \in \mathbb{R}^+$ *the set* $M \cap A^k(y,b)$ *is not empty. The point* y *will be called an admissible limit point of* M *if there exists* $b \in \mathbb{R}^+$ *such that* y *is a* b*-admissible limit point of* M.

A function f *on* $\mathbb{R}^n$ *is said to converge admissibly at* y *if, for all* $b > 0$, f *restricted to* $A(y,b)$ *has a limit at* ∞.

We will write $\operatorname*{A-lim}_{x\to y} f(x)$.

The space $\mathbb{R}^n$ endowed with the sheaf of solutions of the Helmholtz equation is a strong harmonic space in the sense of Bauer, see [2, p. 86].

Terms as harmonic functions, superharmonic functions and reduced functions are related to this harmonic space and have a standard meaning.

This harmonic space satisfies conditions (1)–(10) in [13], see [13], and so minimal thinness at points of S^{n-1} is well defined and the Fatou-Naïm-Doob theorem holds. For the reader's convenience the basic facts are presented here.

Definition. *Let $M \subset \mathbb{R}^n$, v positive superharmonic function on D. The reduction of v on M is defined as*

$$R_v^M = \inf\{u;\ u \geqq v \text{ on } M, u \text{ is positive superharmonic function on } \mathbb{R}^n\}.$$

Let $M \subset \mathbb{R}^n$ and $y \in S^{n-1}$. The set M is minimal thin at y if

$$R^M_{e^{\lambda\langle .,y\rangle}} \neq e^{\lambda\langle .,y\rangle}.$$

The minimal fine filter at y is filter: $\mathcal{F}(y) = \{M \subset \mathbb{R}^n; \mathbb{R}^n \backslash M$ is minimal thin at $y\}$.

A function f converging along $\mathcal{F}(y)$ is said to have a minimal fine limit at y. This limit will be denoted mf-lim $f(x)$.

Theorem B (Limit theorems)**.** *Let u be a positive solution and v be a strictly positive solution of the Helmholtz equation defined on all $\mathbb{R}^n$ and μ_u, μ_v be their representing measures on S^{n-1}.*

Then the following equalities hold:

$$\operatorname*{A-lim}_{x\to y} \frac{u(x)}{v(x)} = \frac{d\mu_u}{d\mu_v}(y)$$

for μ_v-almost all points y of S^{n-1} (admissible convergence);

$$\operatorname*{mf-lim}_{x\to y} \frac{u(x)}{v(x)} = \frac{d\mu_u}{d\mu_v}(y)$$

for μ_v-almost all points y of S^{n-1} (the Fatou-Naïm-Doob limit theorem).

Proof: See [9, p. 85] and [13]. □

Remark. For $v = h$, the admissible convergence follows from the minimal fine convergence (even in a more general situation); see [9, p. 84].

Let $x \in \mathbb{R}^n$, $b, c, k \in \mathbb{R}^+$ and $M \subset \mathbb{R}^n$. In this paper, the following subsets of $\mathbb{R}^n$ will be of special interest:

$B(x,c) = \{z \in \mathbb{R}^n; \|z - x\| \leqq c\}$,
$S(x,b,k) = \{z \in \mathbb{R}^n; \|z\| = k\|x\| \text{ and } \|z - kx\| < k^{\frac{1}{2}} b \|x\|^{\frac{1}{2}}\}$,
$M_{S,b,k} = \cup_{x \in M} S(x,b,k)$,
$S(x,b) = S(x,b,1)$,
$M_{S,b} = \cup_{x \in M} S(x,b)$,
$cM = \{z \in \mathbb{R}^n;$ there exists $x \in M$ such that $z = c.x\}$.

Let $x, y \in \mathbb{R}^n$, $\alpha_{x,y}$ will denote the angle between x and y.

The main results

Theorem. *Let $M \subset \mathbb{R}^n$. Then the following statements are equivalent:*

(i)
$$\inf_{x \in \mathbb{R}^n} \frac{u(x)}{h(x)} = \inf_{x \in M} \frac{u(x)}{h(x)}$$
for all simple solutions u of the Helmholtz equation;

(ii)
$$\inf_{x \in \mathbb{R}^n} \frac{u(x)}{h(x)} = \inf_{x \in M} \frac{u(x)}{h(x)}$$
for all h-bounded solutions u of the Helmholtz equation;

(iii)
$$\inf_{x \in \mathbb{R}^n} \frac{u(x)}{h(x)} = \inf_{x \in M} \frac{u(x)}{h(x)}$$
for all positive solutions u of the Helmholtz equation;

(iv) *the set of points of S^{n-1} which are not admissible limit points of M has σ-measure zero;*

(v) *for any $b \in \mathbb{R}^+$, the set of points of S^{n-1} which are not b-admissible limit points of M has σ-measure zero;*

(vi) *there exist $b, k \in \mathbb{R}^+$, such that the set of points of S^{n-1} at which $M_{S,b,k}$ is minimal thin has σ-measure zero;*

(vii) *for any $b, k \in \mathbb{R}^+$, the set of points of S^{n-1} at which $M_{S,b,k}$ is minimal thin has σ-measure zero;*

(viii) *if ν is a countably finite Borel measure with* $\operatorname{supp}(\nu) = \overline{M}$, *then for every $f \in L_1(S^{n-1})$ there exists $\Phi \in L_1(\nu)$ such that*

$$f(y) = \int_{\mathbb{R}^n} \Phi(x) \frac{e^{\lambda\langle x,y\rangle}}{h(x)}\, d\nu(x) \tag{1}$$

for σ-almost all y and

$$\|f\|_{L_1(S^{n-1})} = \inf\{\|\Phi\|_{L_1(\nu)}; (1) \textit{ holds for some } \Phi \in L_1(\nu)\};$$

(ix) *for every* $f \in L_1(S^{n-1})$, *there is a sequence* $\{x_k\}$, $x_k \in M$ *and* $\{\lambda_k\} \in l_1$ *such that*

$$f(y) = \sum_{k=1}^{\infty} \lambda_k \frac{e^{\lambda\langle x_k, y\rangle}}{h(x_k)} \tag{2}$$

for σ*-almost all* y *and*

$$\|f\|_{L_1(S^{n-1})} = \inf\{\sum_{k=1}^{\infty} |\lambda_k|; (2) \textit{ holds for some } \{x_k\} \textit{ in } M\};$$

(x) *if* ν *is a countably finite Borel measure with* $\operatorname{supp}(\nu) = \overline{M}$, *then for every* $f \in L_1(S^{n-1})$ *there exists* $\Phi \in L_1(\nu)$ *such that*

$$f(y) = \kappa^{-1} \int_{\mathbb{R}^n} \Phi(x) e^{\lambda\|x\|(\cos\alpha_{x,y}-1)} \|x\|^{(n-1)/2}\, d\nu(x) \tag{3}$$

for σ*-almost all* y;

moreover for any $c \in \mathbb{R}^+$ *there exists a function* Φ *satisfying* (3), *such that* $\Phi = 0$ *on* $B(0,c)$, *and*

$$\|f\|_{L_1(S^{n-1})} = \inf\{\|\Phi\|_{L_1(\nu)}; (3) \textit{ holds for some } \Phi \in L_1(\nu), \Phi = 0 \textit{ on } B(0,c)\};$$

(xi) *for every* $f \in L_1(S^{n-1})$ *and for any* $c \in \mathbb{R}^+$, *there is a sequence* $\{x_k\}$, $x_k \in M$, $\|x_k\| > c$ *and* $\{\lambda_k\} \in l_1$ *such that*

$$f(y) = \kappa^{-1} \sum_{k=1}^{\infty} \lambda_k e^{\lambda\|x_k\|(\cos\alpha_{x,y}-1)} \|x\|^{(n-1)/2} \tag{4}$$

for σ*-almost all* y; *such that*

$$\|f\|_{L_1(S^{n-1})} = \inf\{\sum |\lambda_k|; (4) \textit{ holds for some } \{x_k\} \textit{ in } M\backslash B(0,c)\}.$$

Remark. A set satisfying the condition (i) will be called a set of determination.

Proof of Theorem

We will need the following theorem:

Theorem 1. *Let* u *be a positive solution of the Helmholtz equation on* $\mathbb{R}^n$ *and* μ_u *its representing measure on* S^{n-1}. *Then*

$$\inf_{x\in\mathbb{R}^n} \frac{u(x)}{h(x)} = \operatorname*{ess\,inf}_{y\in S^{n-1}} \frac{d\mu_u}{d\sigma}(y).$$

If u is an h-bounded function then

$$\sup_{x\in\mathbb{R}^n}\frac{u(x)}{h(x)}=\operatorname*{ess\,sup}_{y\in S^{n-1}}\frac{d\mu_u}{d\sigma}(y)\qquad \text{and}\qquad \sup_{x\in\mathbb{R}^n}\frac{|u(x)|}{h(x)}=\operatorname*{ess\,sup}_{y\in S^{n-1}}\left|\frac{d\mu_u}{d\sigma}(y)\right|.$$

Proof: By the Lebesgue-Radon-Nikodym theorem the existence of measures μ_a and μ_s, such that $\mu_u=\mu_a+\mu_s$, $\mu_a\leqq\sigma$ and $\mu_s\perp\sigma$, is guaranteed.

Let $f_u=\frac{d\mu_u}{d\sigma}$. Denote $k_1=\inf\limits_{x\in\mathbb{R}^n}\frac{u(x)}{h(x)}$ and $k_2=\operatorname*{ess\,inf}\limits_{y\in S^{n-1}}f_u(y)$.

Obviously,

$$u(x)=\int\limits_{S^{n-1}}e^{\lambda\langle x,y\rangle}\,d(f_u\sigma+\mu_s)(y)=\int\limits_{S^{n-1}}f_u(y)e^{\lambda\langle x,y\rangle}\,d\sigma(y)+\int\limits_{S^{n-1}}e^{\lambda\langle x,y\rangle}\,d\mu_s(y)$$

and, as the last term is positive,

$$u(x)\geqq\int\limits_{S^{n-1}}f_u(y)e^{\lambda\langle x,y\rangle}\,d\sigma(y)\geqq k_2\int\limits_{S^{n-1}}e^{\lambda\langle x,y\rangle}\,d\sigma(y)=k_2h(x)$$

for all $x\in\mathbb{R}^n$. This gives $k_1\geqq k_2$.

On the other hand $u(x)-k_1h(x)$ is a positive solution of the Helmholtz equation and thus $\mu_u-k_1\sigma$ is a measure, so $(f_u-k_1)\sigma+\mu_s$ is a measure. Since $\mu_s\perp\sigma$, $(f_u-k_1)\sigma$ is a measure and consequently $\operatorname*{ess\,inf}\limits_{y\in S^{n-1}}f_u(y)\geqq k_1$, or $k_2\geqq k_1$.

The proof of the rest of the theorem is analogous. □

Proof of equivalence of (i), (ii), (iii), (iv) and (v).

As the implications (v)⇒(iv), (ii)⇒(i) and (iii)⇒(ii) are trivial (in the last implication just take $u-c_1h$ instead of u), we will prove (iv)⇒(iii) and (i)⇒(v).

Theorem 2. *Let M be a subset of $\mathbb{R}^n$ and σ-almost every point $y\in S^{n-1}$ be an admissible limit point of M. Then*

$$\inf_{x\in\mathbb{R}^n}\frac{u(x)}{h(x)}=\inf_{x\in M}\frac{u(x)}{h(x)}$$

for every positive solution u of the Helmholtz equation on $\mathbb{R}^n$ and

$$\sup_{x\in\mathbb{R}^n}\frac{|u(x)|}{h(x)}=\sup_{x\in M}\frac{|u(x)|}{h(x)}$$

for every h-bounded solution u of the Helmholtz equation on $\mathbb{R}^n$.

Proof: The assertion follows immediately from the previous theorem and the limit theorem. □

Lemma 1. *Let b be a positive number and $x \in \mathbb{R}^n$. Denote $C(x,b)$ the set of all $y \in S^{n-1}$ such that $x \in A(y,b)$. Then*

$$C(x,b) = \{y \in S^{n-1}; \|y - \frac{x}{\|x\|}\| < \frac{b}{\sqrt{\|x\|}}\}$$

and there exists a positive number c such that

$$\int\limits_{C(x,b)} e^{\lambda\langle x,y\rangle}\, d\sigma(y) \geqq c.h(x),$$

whenever $x \in \mathbb{R}^n \setminus \{0\}$.

Proof: See [9, p. 84]. □

Theorem 3. *Let $M \subset \mathbb{R}^n$ and $b \in \mathbb{R}^+$. If*

$$\inf_{x\in\mathbb{R}^n} \frac{u(x)}{h(x)} = \inf_{x\in M} \frac{u(x)}{h(x)}$$

for all simple solutions of the Helmholtz equation, then σ-almost every point $y \in S^{n-1}$ is a b-admissible limit point of M.

Proof: Suppose that it is not true.

Denote the set $M \cap \{x \in \mathbb{R}^n; \|x\| > k\}$ by M^k and the set of all b-admissible limit points of M by M_b. As $M_b = \cap_{k\in\mathbb{N}}(\cup_{x\in M^k} C(x,b))$ is a G_δ set, it is a σ-measurable subset of S^{n-1}. Then its complement M_b' is also measurable and by our assumption $\sigma(M_b') > 0$.

Recall that for $k \in \mathbb{N}$ and $y \in S^{n-1}$, $A^k(y,b)$ denotes the truncated admissible region $A(y,b) \cap \{x \in \mathbb{R}^n; \|x\| > k\}$. Then, for every $y \in M_b'$, there is $k_y \in \mathbb{N}$ such that $A^{k_y}(y,b) \cap M$ is empty. Denote by D_k the set of $y \in M_b'$ for which $A^k(y,b) \cap M$ is empty.

As D_k is a complement of $\cup_{x\in M^k} C(x,b)$, it is a σ-measurable subset of S^{n-1}.

Since $\bigcup_{k=1}^{\infty} D_k = M_b'$, the Lebesgue measure of at least one of the sets D_k, say of D_{k_0}, is strictly positive. Denote this set by D and its complement $(S^{n-1}) \setminus D$ by D'.

It is clear that $C(x,b) \subset D'$, whenever $x \in M^k$.

For any measurable set $A \subset S^{n-1}$ we define

$$u_A(x) = \int\limits_A e^{\lambda\langle x,y\rangle}\, d\sigma(y),\ x \in \mathbb{R}^n.$$

So u_A is a simple solution of the Helmholtz equation. By Theorem 1 we get that if $\sigma(A) > 0$, then $\sup\limits_{x\in\mathbb{R}^n} \frac{u_A(x)}{h(x)} = 1$ and if $\sigma(A') > 0$, then $\inf\limits_{x\in\mathbb{R}^n} \frac{u_A(x)}{h(x)} = 0$.

The set D has a positive measure, so the function $u_{D'}$ is a simple solution of the Helmholtz equation and

$$\inf_{x \in \mathbb{R}^n} \frac{u_{D'}(x)}{h(x)} = 0.$$

But $C(x,b)$ is a subset of D' for every $x \in M^k$. Now from the above lemma there exists a constant c such that

$$\frac{u_{D'}(x)}{h(x)} \geqq \frac{u_{C(x,b)}(x)}{h(x)} \geqq c$$

for every $x \in M^k$.

We arrive at

$$\inf_{x \in M^k} \frac{u_{D'}(x)}{h(x)} \geqq c.$$

Now it will be shown that

$$\inf_{x \in M \setminus M^k} \frac{u_{D'}(x)}{h(x)} > 0.$$

As h is positive and continuous and $B(0,k)$ is compact, there exists $c_1 \in \mathbb{R}^+$ such that $h(x) \leqq c_1$ for all $x \in B(0,k)$.

It follows

$$u_{D'}(x) = \int_{D'} e^{\lambda \langle x,y \rangle} \, d\sigma(y) \geqq \int_{D'} e^{-\lambda \|x\|.\|y\|} \, d\sigma(y) =$$
$$\int_{D'} e^{-\lambda \|x\|} \, d\sigma(y) = \sigma(D').e^{-\lambda \|x\|} \geqq \sigma(D').e^{-\lambda k}.$$

Let us denote this positive constant by c_2.

Thus

$$\inf_{x \in B(0,k)} \frac{u_{D'}(x)}{h(x)} \geqq \frac{c_2}{c_1}.$$

Consequently,

$$\inf_{x \in M} \frac{u_{D'}(x)}{h(x)} \geqq \min(c, \frac{c_2}{c_1}) > 0,$$

contradicting our assumption. $\square$

Proof of (vi) and (vii).

The implication (vii)⇒(vi) is trivial. Now it will be proved, that (vi)⇒(iii) and (v)⇒(vii).

Theorem 4. *Let $n \in \mathbb{N}$, $b \in \mathbb{R}^+$. Then there exists a positive constant c, such that for every $x \in \mathbb{R}^n$, for every $z \in S(x, b, \frac{1}{2})$ and for every positive solution u of the Helmholtz equation on $\mathbb{R}^n$,*

$$\frac{u(z)}{h(z)} \geqq c\frac{u(x)}{h(x)},$$

and for any $M \subset \mathbb{R}^n$

$$\inf_{x \in M_{S,b,\frac{1}{2}}} \frac{u(x)}{h(x)} \geqq c \inf_{x \in M} \frac{u(x)}{h(x)}.$$

Proof: For the first part, see [9, p. 83]. The second part immediately follows. □

Theorem 5. *Let $M \subset \mathbb{R}^n$ such that the set of points of S^{n-1} at which M is minimal thin is of σ-measure zero.*

Then

$$\inf_{x \in \mathbb{R}^n} \frac{u(x)}{h(x)} = \inf_{x \in M} \frac{u(x)}{h(x)}$$

for all positive solutions u of the Helmholtz equation on $\mathbb{R}^n$.

Proof: It follows from Theorem 1 and from the Fatou-Naïm-Doob limit theorem. □

Theorem 6. *Let $M \subset \mathbb{R}^n$ and $b \in \mathbb{R}^+$ such that the set of points of S^{n-1} at which $M_{S,b,\frac{1}{2}}$ is minimal thin is of σ-measure zero.*

Then there exists a constant c depending only on b and n such that

$$\inf_{x \in \mathbb{R}^n} \frac{u(x)}{h(x)} \geqq c \inf_{x \in M} \frac{u(x)}{h(x)}$$

for all positive solutions u of the Helmholtz equation on $\mathbb{R}^n$.

Proof: This theorem is obtained by combining Theorems 4 and 5. □

Theorem 7. *Let $M \subset \mathbb{R}^n$. Then the following statements are equivalent:*

(i)

$$\inf_{x \in \mathbb{R}^n} \frac{u(x)}{h(x)} = \inf_{x \in M} \frac{u(x)}{h(x)}$$

for all positive solutions u of the Helmholtz equation on $\mathbb{R}^n$;

(ii) *there exists $c > 0$ such that*

$$\inf_{x \in \mathbb{R}^n} \frac{u(x)}{h(x)} \geqq c \inf_{x \in M} \frac{u(x)}{h(x)}$$

for all positive solutions u of the Helmholtz equation on $\mathbb{R}^n$.

Proof: (i) $\Rightarrow$ (ii) is clear, put $c = 1$.

(ii) $\Rightarrow$ (i) Let us suppose that there exists a set M satisfying (ii), but not (i). Then c in (ii) belongs to $(0, 1)$.

Let u be a positive solution of the Helmholtz equation for which (i) is not true.

$$\text{Denote } \inf_{x \in \mathbb{R}^n} \frac{u(x)}{h(x)} = c_1 \quad \text{and} \quad \inf_{x \in M} \frac{u(x)}{h(x)} = c_2.$$

Thus by our assumptions, $c_2 > c_1 \geqq c.c_2$.

Let $v(x) = u(x) - c_1 h(x)$ for $x \in \mathbb{R}^n$.

Then v is a positive solution of the Helmholtz equation and

$$\inf_{x \in \mathbb{R}^n} \frac{v(x)}{h(x)} = c_1 - c_1 = 0, \quad \text{and} \quad \inf_{x \in M} \frac{v(x)}{h(x)} = c_2 - c_1 > 0,$$

which is a contradiction with (ii). $\square$

Theorem 8. *Let $M \subset \mathbb{R}^n$ and $b \in \mathbb{R}^+$ such that the set of points of S^{n-1} at which $M_{S,b,\frac{1}{2}}$ is minimal thin has σ-measure zero.*

Then

$$\inf_{x \in \mathbb{R}^n} \frac{u(x)}{h(x)} = \inf_{x \in M} \frac{u(x)}{h(x)}$$

for all positive solution u of the Helmholtz equation on $\mathbb{R}^n$.

Proof: The result is obtained by combining two previous theorems. $\square$

Theorem 9. *Let $M \subset \mathbb{R}^n$, $y \in S^{n-1}$ and $b \in \mathbb{R}^+$. If y is an admissible limit point of M, then $M_{S,b}$ is not minimal thin at y.*

Proof: Let $\{x_k\}$ be a sequence of points of M converging to y admissibly — it means that there exists $b_1 \in \mathbb{R}^+$ such that $\{x_k\}$ converges b_1-admissibly.

Then a straightforward calculation gives that $S(x_k, b) \subset A(y, b_1 + b)$.

Since the Helmholtz equation is invariant with respect to linear isometries of $\mathbb{R}^n$, the harmonic measure μ_0 (for the notion of the harmonic measure, see [2, p. 120]) on $\partial B(0, r)$ corresponding to 0, is invariant with respect to isometries of $\partial B(0, r)$ and hence it is a multiple of the surface measure σ_n on $\partial B(0, r)$.

As $\mu_0(\partial B(0, r)) = \frac{h(0)}{h(r.e_1)}$ and $h(0) = \omega_n$ we have that for any σ-measurable subset E of $\partial B(0, r)$

$$\mu_0(E) = \frac{h(0)\sigma_n(E)}{h(r.e_1)\omega_n r^{n-1}} = \frac{\sigma(r^{-1}E)}{h(r.e_1)}.$$

The proof of the theorem is finished in the same way as the proof of Proposition 2.2 in [9, p. 82]; for the reader's convenience it is given here.

Let us denote u_k the solution of the Dirichlet problem on $B(0, \|x_k\|)$ with boundary value 1 on $S(x_k, b)$ and 0 on the rest of the boundary.

Hence $u_k(0) = h(\|x_k\|)^{-1}\sigma(\|x_k\|^{-1}S(x_k, b)) \sim b^{(n-1)/2}\|x_k\|^{-(n-1)/2}h(\|x_k\|)^{-1}$.

As $h(x) \sim \frac{\kappa e^{\lambda\|x\|}}{\|x\|^{(n-1)/2}}$ (see Preliminaries),

$$u_k(0) \sim \kappa\, b^{(n-1)/2}e^{-\lambda\|x_k\|}.$$

Now denote v_k the solution of the Dirichlet problem on $B(0, \|x_k\|)$ with boundary value $e^{\lambda\langle x,y\rangle}$ on $S(x_k, b)$ and 0 on the rest of the boundary.

For any $b_0 \in \mathbb{R}^+$ there is a positive constant c_1 such that for all $x \in A(y, b_0)$ $c_1^{-1}e^{\lambda\|x\|} \leqq e^{\lambda\langle x,y\rangle} \leqq c_1 e^{\lambda\|x\|}$ whenever $x \in A(y, b_0)$.
(Indeed, $0 \leqq \lambda(\|x\| - \langle x, y\rangle) = \lambda\|x\|(1 - \langle x', y\rangle) = \frac{1}{2}\lambda\|x\|\ \|x' - y\|^2 \leqq \frac{1}{2}\lambda\ b_0^2$, where $x' = \frac{x}{\|x\|}$.)

As $S(x_k, b) \subset A(y, b_1 + b)$, for the boundary values of u_k and v_k holds

$$c_1^{-1}e^{\lambda\|x_k\|}u_k(x) \leqq v_k(x) \leqq c_1 e^{\lambda\|x_k\|}u_k(x)$$

for $x \in \partial B(0, \|x_k\|)$ and hence for any $x \in B(0, \|x_k\|)$.

Namely this is true for 0 and so, using the above relation for $u_k(0)$, the existence of a positive constant c_2 such that

$$c_2^{-1} \leqq v_k(0) \leqq c_2$$

for any $k \in \mathbb{N}$ is guaranteed.

Let $S = \cup_{k\in\mathbb{N}}S(x_k, b)$. The Perron-Wiener-Brelot method of solving the Dirichlet problem shows that, for any $k \in \mathbb{N}$, the inequality $v_k \leqq R^S_{e^{\lambda\langle\cdot,y\rangle}}$ holds on $B(0, \|x_k\|)$. As $\{v_k\}$ is bounded in 0, it has by virtue of the Harnack inequality a converging subsequence. Denoting its limit by v, it is easy to see that v is a positive solution of the Helmholtz equation, $v(0) \geqq c_2^{-1}$ and $v \leqq R^S_{e^{\lambda\langle\cdot,y\rangle}}$. Hence its representing measure $\mu_v \leqq \delta_y$ and thus $R^S_{e^{\lambda\langle\cdot,y\rangle}} = e^{\lambda\langle\cdot,y\rangle}$, it means that S is not minimal thin at y and hence $M_{S,b}$ is not minimal thin at y. $\square$

So far we have proved the implication (vi) $\Rightarrow$ (iii) for $k = \frac{1}{2}$ and the implication (v)$\Rightarrow$(vii) for $k = 1$. The conditions for k will be removed using the following lemma.

Lemma 2. *Let $M \subset \mathbb{R}^n$, $c \in \mathbb{R}^+$, $y \in S^{n-1}$. The point y is an admissible limit point of the set M if and only if y is a admissible limit point of cM.*

Let $x \in \mathbb{R}^n$ and $b, k \in \mathbb{R}^+$. Then

$$S(x, b, k) = S(kx, b) = S(2kx, b, \frac{1}{2})$$

and

$$M_{S,b,k} = (kM)_{S,b} = (2kM)_{S,b,\frac{1}{2}}.$$

Proof: A straightforward calculation. □

Now, it si easy to finish the proof of (vi) and (vii).

Let $k \in \mathbb{R}^+$. Using the first part of the lemma and equivalence of (i) and (v) it follows that M is a set of determination if and only if kM is a set of determination.

From that and from $(kM)_{S,b} = M_{S,b,k}$ it immediately follows that (vii) is true for any positive k.

From $M_{S,b,k} = (2kM)_{S,b,\frac{1}{2}}$ it follows that if (vi) holds for some k then $2kM$ is a set of determination, so M is a set of determination.

Proof of (viii) and (ix).

The implication (viii)⇒(ix) is trivial. (Take a countable subset of M and the counting measure on it.) We will prove (v)⇒(viii) and (ix)⇒(ii).

Theorem 10. *Let M be a subset of $\mathbb{R}^n$ and ν be a countably-finite measure on $\mathbb{R}^n$ such that* $\operatorname{supp}(\nu) = \overline{M}$. *Let*

$$\sup_{x\in\mathbb{R}^n} \frac{|u(x)|}{h(x)} = \sup_{x\in M} \frac{|u(x)|}{h(x)}$$

for every h-bounded solution u of the Helmholtz equation on $\mathbb{R}^n$.

Then, for any f in $L_1(S^{n-1})$, there exists Φ in $L_1(\nu)$ such that

$$f = \int\limits_{\mathbb{R}^n} \Phi(x) \frac{e^{\lambda\langle x,.\rangle}}{h(x)}\, d\nu(x) \tag{1}$$

σ-almost everywhere and

$$\|f\|_{L_1(S^{n-1})} = \inf\,\{\|\Phi\|_{L_1(\nu)}; (1) \text{ holds for some } \Phi \in L_1(\nu)\}.$$

We will need the following version of the closed range theorem (see [12, p. 97]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, T a bounded linear mapping of $\mathcal{X}$ into $\mathcal{Y}$. If there exists a constant $c > 0$ such that $\|T^*y^*\| \geqq c\|y^*\|$ for all $y^* \in \mathcal{Y}^*$ then $T\mathcal{X} = \mathcal{Y}$.* In our situation, $\mathcal{X} = L_1(\nu)$, $\mathcal{Y} = L_1(S^{n-1})$ and for $\Phi \in L_1(\nu)$ we define

$$T_\nu\Phi = \int\limits_{S^{n-1}} \Phi(x) \frac{e^{\lambda\langle x,.\rangle}}{h(x)}\, d\nu(x).$$

Lemma 3. *The mapping T_ν is a bounded linear mapping of $L_1(\nu)$ into $L_1(S^{n-1})$, $\|T_\nu\| = 1$; T_ν^* is the bounded mapping $L_\infty(S^{n-1})$ into $L_\infty(\nu)$ such that*

$$T_\nu^* g(x) = \frac{1}{h(x)} \int\limits_{S^{n-1}} e^{\lambda\langle x,y\rangle} g(y)\, d\sigma(y).$$

Proof: Using the Fubini theorem we arrive at

$$\|T_\nu \Phi\|_{L_1(S^{n-1})} = \int\limits_{S^{n-1}} |T_\nu \Phi|\, d\sigma = \int\limits_{S^{n-1}} |\int\limits_{\mathbb{R}^n} \Phi(x) \frac{e^{\lambda\langle x,y\rangle}}{h(x)}\, d\nu(x)|\, d\sigma(y) \leqq$$
$$\int\limits_{S^{n-1}} (\int\limits_{\mathbb{R}^n} |\Phi(x)| \frac{e^{\lambda\langle x,y\rangle}}{h(x)}\, d\nu(x))\, d\sigma(y) = \int\limits_{\mathbb{R}^n} (\int\limits_{S^{n-1}} |\Phi(x)| \frac{e^{\lambda\langle x,y\rangle}}{h(x)}\, d\sigma(y))\, d\nu(x) =$$
$$\int\limits_{\mathbb{R}^n} \frac{|\Phi(x)|}{h(x)} (\int\limits_{S^{n-1}} e^{\lambda\langle x,y\rangle}\, d\sigma(y))\, d\nu(x) = \int\limits_{\mathbb{R}^n} \frac{|\Phi(x)|}{h(x)} h(x)\, d\nu(x) = \|\Phi\|_{L_1(\nu)}.$$

So the first part of Lemma is proved.

Let $g \in L_\infty(S^{n-1})$ and $\Phi \in L_1(\nu)$. Using again the Fubini theorem we have

$$[\Phi, T_\nu^* g] = [T_\nu \Phi, g] = \int\limits_{S^{n-1}} g.T_\nu \Phi\, d\sigma = \int\limits_{S^{n-1}} g(y) (\int\limits_{\mathbb{R}^n} \Phi(x) \frac{e^{\lambda\langle x,y\rangle}}{h(x)}\, d\nu(x))\, d\sigma(y) =$$
$$\int\limits_{\mathbb{R}^n} \frac{\Phi(x)}{h(x)} (\int\limits_{S^{n-1}} g(y) e^{\lambda\langle x,y\rangle}\, d\sigma(y))\, d\nu(x) = [\Phi, \frac{1}{h} \int\limits_{S^{n-1}} e^{\lambda\langle .,y\rangle} g(y)\, d\sigma(y)].$$

□

Proof of Theorem. We shall prove the existence of a constant $c > 0$ such that $\|T_\nu^* g\|_{L_\infty(\nu)} \geqq c\|g\|_{L_\infty(S^{n-1})}$ for all $g \in L_\infty(S^{n-1})$ and the first part of the theorem will be proved.

The function $h.(T_\nu^* g)$ is an h-bounded solution of the Helmholtz equation on $\mathbb{R}^n$. Then, by hypothesis,

$$\sup_{x\in M} |(T_\nu^* g)(x)| = \sup_{x\in\mathbb{R}^n} |(T_\nu^* g)(x)| = \|g\|_{L_\infty(S^{n-1})}.$$

Since $T_\nu^* g$ is a continuous function on $\mathbb{R}^n$ and $\operatorname{supp}(\nu) = \overline{M}$,

$$\|T_\nu^* g\|_{L_\infty(\nu)} = \sup_{x\in M} |(T_\nu^* g)(x)|.$$

Consequently,

$$\|T_\nu^* g\|_{L_\infty(\nu)} = \|g\|_{L_\infty(S^{n-1})}.$$

So we can take $c = 1$. The first part of Theorem is proved.

To prove the other part define the space

$$\mathcal{Z} = L_1(\nu)/\ker T_\nu.$$

For $z \in \mathcal{Z}$ and $\Phi \in z$ put $Sz = T_\nu \Phi$.

Then S is an invertible bounded linear mapping of $\mathcal{Z}$ into $L_1(S^{n-1})$ and so its adjoint S^* is an invertible bounded linear mapping of $L_\infty(S^{n-1})$ into $\mathcal{Z}^*$ (see [12, p. 94]).

Let $z \in \mathcal{Z}$, $\Phi \in z$ and $g \in L_\infty(S^{n-1})$. Then we have

$$(S^* g)(z) = [Sz, g] = [T_\nu \Phi, g] = [\Phi, T_\nu^* g].$$

If $\varepsilon > 0$, there exists $\Phi_0 \in L_1(\nu)$ with $\|\Phi_0\|_{L_1(\nu)} = 1$ and

$$|[\Phi_0, T_\nu^* g]| > \|T_\nu^* g\|_{L_\infty(\nu)} - \varepsilon.$$

Let z_0 denote the coset of Φ_0 in $\mathcal{Z}$. Then

$$|(S^* g)(z_0)| > \|T_\nu^* g\|_{L_\infty(\nu)} - \varepsilon$$

and

$$\|z_0\|_{\mathcal{Z}} \leqq \|\Phi_0\|_{L_1(\nu)} = 1.$$

Therefore, the norm of the functional $S^* g$ satisfies

$$\|S^* g\|_{\mathcal{Z}^*} > \|T_\nu^* g\|_{L_\infty(\nu)} - \varepsilon = \|g\|_{L_\infty(S^{n-1})} - \varepsilon.$$

Since ε was arbitrary, we proved that

$$\|S^* g\|_{\mathcal{Z}^*} \geqq \|g\|_{L_\infty(S^{n-1})}$$

for any $g \in L_\infty(S^{n-1})$, and so, using the fact that the norm of any operator is the same as the norm of its adjoint (see [1, p. 93]) and the obvious fact that $(S^*)^{-1} = (S^{-1})^*$, we have

$$\|S^{-1}\| = \|(S^*)^{-1}\| \leqq 1.$$

Fix $f \in L_1(S^{n-1})$ and put $z = S^{-1} f$. Then

$$\|z\|_{\mathcal{Z}} \leqq \|f\|_{L_1(S^{n-1})},$$

that is

$$\inf\{\|\Phi\|_{L_1(\nu)}; T_\nu \Phi = f\} \leqq \|f\|_{L_1(S^{n-1})}.$$

By Lemma we have

$$\|f\|_{L_1(S^{n-1})} = \|T_\nu \Phi\|_{L_1(S^{n-1})} \leqq \|T_\nu\| . \|\Phi\|_{L_1(\nu)} = \|\Phi\|_{L_1(\nu)}.$$

So the opposite inequality holds as well. $\square$

Theorem 11. *Let ν be a countably finite measure on $\mathbb{R}^n$ and* $\operatorname{supp}(\nu) = \overline{M}$. *Assume that for every function $f \in L_1(S^{n-1})$ there exists Φ in $L_1(\mathbb{R}^n)$ such that*

$$f = \int_{\mathbb{R}^n} \Phi(x) \frac{e^{\lambda\langle x,.\rangle}}{h(x)}\, d\nu(x) \tag{1}$$

σ-almost everywhere and

$$\|f\|_{L_1(S^{n-1})} = \inf\,\{\|\Phi\|_{L_1(\nu)}; (1) \text{ holds for some } \Phi \text{ in } L_1(\nu)\}.$$

Then

$$\sup_{x\in\mathbb{R}^n} \frac{u(x)}{h(x)} = \sup_{x\in M} \frac{u(x)}{h(x)}$$

for any h-bounded positive solution u of the Helmholtz equation on $\mathbb{R}^n$.

Proof: Put $c = \sup\limits_{x\in M} \frac{u(x)}{h(x)}$. We have $c < \infty$.

Let $\varepsilon > 0$. If we fix $x_0 \in \mathbb{R}^n$, then $e^{\lambda\langle x_0,.\rangle} \in L_1(S^{n-1})$ and $\|e^{\lambda\langle x_0,.\rangle}\|_{L_1(S^{n-1})} = h(x_0)$. By our assumptions there is a function $\Phi \in L_1(\nu)$ such that

$$e^{\lambda\langle x_0,.\rangle} = \int_{\mathbb{R}^n} \Phi(x)\frac{e^{\lambda\langle x,.\rangle}}{h(x)}\, d\nu(x) \leqq \int_{\mathbb{R}^n} |\Phi(x)|\frac{e^{\lambda\langle x,.\rangle}}{h(x)}\, d\nu(x)$$

and

$$\|\Phi\|_{L_1(\nu)} < h(x_0) + \varepsilon.$$

As u is an h-bounded positive solution of the Helmholtz equation, we can integrate the first inequality with respect to $f_u d\sigma$. Using the Fubini theorem and the fact that $u \leqq ch$ on $\operatorname{supp}(\nu)$, we have

$$u(x_0) = \int_{S^{n-1}} e^{\lambda\langle x_0,y\rangle} f_u(y)\, d\sigma(y) \leqq \int_{S^{n-1}} (\int_{\mathbb{R}^n} |\Phi(x)| e^{\lambda\langle x,y\rangle}\, d\nu(x)) f_u(y)\, d\sigma(y) =$$
$$\int_{\mathbb{R}^n} |\Phi(x)| (\int_{S^{n-1}} e^{\lambda\langle x,y\rangle} f_u(y)\, d\sigma(y))\, d\nu(x) = \int_{S^{n-1}} |\Phi(x)| u(x)\, d\nu(x) \leqq$$
$$\int_{S^{n-1}} c.|\Phi(x)|\, d\nu(x) = c\|\Phi\|_{L_1(\nu)} \leqq c(h(x_0) + \varepsilon).$$

Since x_0 and ε were arbitrary, we have $\sup\limits_{x\in\mathbb{R}^n} \frac{u(x)}{h(x)} = c$. □

Of course, the following special form of Theorem 11 holds:

Theorem 12. *Let M be a subset of $\mathbb{R}^n$. Assume that for every function $f \in L_1(S^{n-1})$ there exist $\{\lambda_k\}_{k=1}^{\infty} \in l_1$ and a sequence $\{x_k\}_{k=1}^{\infty}$ of points in M such that*

$$f = \sum_{k=1}^{\infty} \lambda_k \frac{e^{\lambda\langle x_k, . \rangle}}{h(x_k)} \tag{2}$$

$\sigma-$almost everywhere and

$$\|f\|_{L_1(S^{n-1})} = \inf \{ \sum_{k=1}^{\infty} |\lambda_k|; (2) \text{ holds for some } \{x_k\} \text{ in } M\}.$$

Then

$$\sup_{x\in\mathbb{R}^n} \frac{u(x)}{h(x)} = \sup_{x\in M} \frac{u(x)}{h(x)}$$

for any bounded positive solution u of the Helmholtz equation.

Proof of the conditions (x) and (xi).

We will prove the equivalence of (viii) and (x). The equivalence of (ix) and (xi) is just a special form of it.

Proof of (viii)⇒(x)

Let us denote

$$K_1(x,y) = \frac{e^{\lambda\langle x,y\rangle}}{h(x)} \qquad \text{and} \qquad K_2(x,y) = \frac{e^{\lambda\langle x,y\rangle}\|x\|^{(n-1)/2}}{\kappa e^{\lambda\|x\|}}.$$

Then we have

$$\|K_1(x,.)\|_{L_1(S^{n-1})} = \int_{S^{n-1}} |\frac{e^{\lambda\langle x,y\rangle}}{h(x)}|\, d\sigma(y) = 1$$

and

$$\|K_1(x,.) - K_2(x,.)\|_{L_1(S^{n-1})} = \int_{S^{n-1}} |\frac{e^{\lambda\langle x,y\rangle}}{h(x)} - \frac{e^{\lambda\langle x,y\rangle}\|x\|^{(n-1)/2}}{\kappa e^{\lambda\|x\|}}|\, d\sigma(y) =$$

$$\int_{S^{n-1}} e^{\lambda\langle x,y\rangle}|\frac{1}{h(x)} - \frac{\|x\|^{(n-1)/2}}{\kappa e^{\lambda\|x\|}}|\, d\sigma(y) =$$

$$|\frac{1}{h(x)} - \frac{\|x\|^{(n-1)/2}}{\kappa e^{\lambda\|x\|}}| \int_{S^{n-1}} e^{\lambda\langle x,y\rangle}\, d\sigma(y) = |1 - \frac{h(x)\|x\|^{(n-1)/2}}{\kappa e^{\lambda\|x\|}}|,$$

from the asymptotic behaviour of the function h (see Preliminaries) it follows, that to every positive ε, there exists a positive number c_ε such that

$$\|K_1(x,.) - K_2(x,.)\|_{L_1(S^{n-1})} < \varepsilon$$

and

$$\|K_2(x,.)\|_{L_1(S^{n-1})} < 1+\varepsilon,$$

whenever $\|x\| > c_\varepsilon$.

Let $f \in L_1(S^{n-1})$ and $c > 1$. Then there exists $\Phi_0 \in L_1(\nu)$, such that

$$f = \int\limits_{\mathbb{R}^n} \Phi_0(x)K_1(x,.)\,d\nu(x), \quad \text{and} \quad \|f\|_{L_1(S^{n-1})} \leqq \|\Phi_0\|_{L_1(\nu)} \leqq c\|f\|_{L_1(S^{n-1})},$$

and moreover, as (viii) is equivalent to (v) and (v) holds for M, if and only if it holds for $M\backslash B(0,c_\varepsilon)$, Φ_0 can be chosen to be zero on $B(0,c_\varepsilon)$.

Put $f_0 = f$. Now, functions $f_k \in L_1(S^{n-1})$ and $\Phi_k \in L_1(\nu)$ for any $k = 1,2,\dots,$ will be defined.

$$f_{k+1} = f_k - \int\limits_{\mathbb{R}^n} \Phi_k(x)K_2(x,.)\,d\nu(x), \text{ for } k = 0,1,\dots;$$

Φ_{k+1} is, for $k = 0,1,\dots,$ a function for which

$$f_{k+1} = \int\limits_{\mathbb{R}^n} \Phi_{k+1}(x)K_1(x,.)\,d\nu(x),$$

$$\|f_{k+1}\|_{L_1(S^{n-1})} \leqq \|\Phi_{k+1}\|_{L_1(\nu)} \leqq c\|f_{k+1}\|_{L_1(S^{n-1})}$$

and Φ_{k+1} is zero on $B(0,c_\varepsilon)$.

We have $f_0 \in L_1(S^{n-1})$ and $\Phi_0 \in L_1(\nu)$ and above relations are satisfied. Suppose, it is true for $0,1,\dots,k$, and prove it for $k+1$:

$$\|f_{k+1}\|_{L_1(S^{n-1})} = \|f_k - \int\limits_{\mathbb{R}^n} \Phi_k(x)K_2(x,.)\,d\nu(x)\|_{L_1(S^{n-1})} =$$

$$\|\int\limits_{\mathbb{R}^n} \Phi_k(x)K_1(x,y)\,d\nu(x) - \int\limits_{\mathbb{R}^n} \Phi_k(x)K_2(x,y)\,d\nu(x)\|_{L_1(S^{n-1})} \leqq$$

$$\int\limits_{S^{n-1}} \int\limits_{\mathbb{R}^n} |\Phi_k(x)(K_1(x,y) - K_2(x,y))|\,d\nu(x)\,d\sigma(y)$$

using Fubini theorem

$$= \int_{\mathbb{R}^n} |\Phi_k(x)| \int_{S^{n-1}} |K_1(x,y) - K_2(x,y)|\, d\sigma(y) \leqq \varepsilon \|\Phi_k\|_{L_1(\nu)}.$$

So $f_{k+1} \in L_1(S^{n-1})$ and by this fact and (v) and (viii) the existence of a function Φ_{k+1} with required properties is guaranteed.

Combining the above estimates for $\|\Phi_k\|_{L_1(\nu)}$ and $\|f_{k+1}\|_{S^{n-1}}$ we obtain

$$\|f_{k+1}\|_{S^{n-1}} \leqq c\varepsilon \|f_k\|_{L_1(S^{n-1})} \text{ for all } k = 0, 1, 2, \dots,$$

and from that

$$\|f_k\|_{S^{n-1}} \leqq (c\varepsilon)^k \|f_0\|_{L_1(S^{n-1})} \text{ for all } k = 1, 2, \dots.$$

Put $\Phi = \sum\limits_{k=0}^{\infty} \Phi_k$. From the previous estimates it follows

$$\|\Phi\|_{L_1(\nu)} \leqq \sum_{k=0}^{\infty} \|\Phi_k\|_{L_1(\nu)} \leqq \sum_{k=0}^{\infty} c\|f_k\|_{L_1(S^{n-1})} \leqq$$

$$c\|f_0\|_{L_1(S^{n-1})} + \sum_{k=1}^{\infty} (c\varepsilon)^k \|f_0\|_{L_1(S^{n-1})} = (c + \frac{c\varepsilon}{1 - c\varepsilon}) \|f_0\|_{L_1(S^{n-1})}.$$

The constant $(c + \frac{c\varepsilon}{1-c\varepsilon})$ can be chosen arbitrarily close to 1.

We have proved that $\Phi \in L_1(\nu)$ and the required relation between $\|f\|_{L_1(S^{n-1})}$ and $\|\Phi\|_{L_1(\nu)}$, and we have proved as well that $\sum_{k=1}^{\infty} |\Phi_k| \in L_1(\nu)$.

As $\Phi_k = 0$ on $B(0, c_\varepsilon)$ for any $k = 0, 1, \dots$, the same is true for Φ (what was to be proved) and $\sum_{k=1}^{\infty} |\Phi_k|$.

From these facts and the fact that $\|K_2(x,.)\|_{L_1(S^{n-1})} < 1 + \varepsilon$ whenever $\|x\| > c_\varepsilon$ we get (using the Fubini theorem) that

$$\int_{\mathbb{R}^n} (\sum_{k=0}^{\infty} |\Phi_k(x)|) K_2(x,.)\, d\nu(x) \in L_1(S^{n-1}).$$

From here it follows that for σ-almost all y

$$\sum_{k=0}^{\infty} |\Phi_k(.)| K_2(., y) \in L_1(\nu).$$

Using the Lebesgue Dominated Convergence Theorem with the above sum as dominating function we arrive to

$$\int_{\mathbb{R}^n} \Phi(x) K_2(x,y)\, d\nu(x) = \int_{\mathbb{R}^n} (\sum_{k=0}^{\infty} \Phi_k(x)).K_2(x,y)\, d\nu(x) =$$

$$\sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \Phi_k(x) K_2(x,y)\, d\nu(x) = \sum_{k=0}^{\infty} (f_k(y) - f_{k+1}(y)) = f_0(y) = f(y)$$

for σ-almost all $y \in S^{n-1}$.

So

$$f = \int_{\mathbb{R}^n} \Phi(x) K_2(x,.)\, d\nu(x)$$

and the proof is finished. □

The implication (x)⇒(viii) can be proved in the same way.

Remark

Similar problems have been recently investigated for classical harmonic functions on a ball in [3], [4], [5], [7] and for more general domains in [1], and for parabolic functions on a slab in [10] and [11]. In the present paper methods of proofs adopted in [7] and [5] turned out to be useful.

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