

On non-homogeneous viscous incompressible fluids. Existence of regular solutions

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Abstract. We consider the flow of a non-homogeneous viscous incompressible fluid which is known at an initial time. Our purpose is to prove that, when Ω is smooth enough, there exists a local strong regular solution (which is global for small regular data).

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Introduction

Let Ω be a bounded connected open subset of \mathbb{R}^3 , $T > 0$ and $Q_T = \Omega \times]0, T[$. A non homogeneous fluid is described by its velocity $u = (u_1, u_2, u_3)$, its density ρ , its viscosity $\nu = \nu(\rho)$ and its pressure p . It is modeled by

$$(1) \quad \begin{cases} \rho \partial_t u - \nabla \cdot (\nu(\rho)(\nabla u + {}^t\nabla u)) + \rho(u \cdot \nabla)u + \nabla p = \rho f, \\ \nabla \cdot u = 0, \\ \partial_t \rho + u \cdot \nabla \rho = 0, \end{cases}$$

$$(2) \quad u = 0 \text{ on } \Sigma_T = \partial\Omega \times]0, T[,$$

$$(3) \quad u|_{t=0} = u_0 \text{ and } \rho|_{t=0} = \rho_0 \text{ in } \Omega.$$

The aim of this work is to prove the existence of a local regular solution of (1)–(3) in Q_T , when f and u_0 are regular data and ρ_0 is supposed to be regular and strictly greater than 0, i.e.

$$0 < M_1 \leq \rho_0 \text{ in } \Omega.$$

When the viscosity does not depend on the density, S.A. Antonev and A.V. Kazhikov [1] proved the existence of weak solutions (see also J.L. Lions [7]). O.A. Ladyzhenskaya and V.A. Solonnikov [5] proved the local existence of a strong regular solution and the global existence for small data.

When $\nu = \nu(\rho)$, E. Fernández-Cara and F. Guillén [3] obtained the existence of a weak solution for $u_0 \in L^2(\Omega)^3$, $\nabla \cdot u_0 = 0$ and $u_0 \cdot n = 0$, $\rho_0 \in L^\infty(\Omega)$, $\rho_0 \geq 0$, $f \in L^1(0, T; L^2(\Omega)^3)$ and $\nu \in C(\mathbb{R}_+)$ such that $\nu(s) \geq \beta > 0$ for all $s \in \mathbb{R}_+$ (see also P.L. Lions [8]). According to uniqueness, M. Kabbaj [4] gives a result for a regular strong solution of (1)–(3) when ρ is supposed to be in $C^2(\overline{Q}_T)$.

1. Existence result

In all the paper long, we suppose that

- Ω is a bounded open subset of \mathbb{R}^3 with a \mathcal{C}^2 boundary,
- $\rho_0 \in \mathcal{C}^1(\overline{\Omega})$ satisfies $M_2 \geq \rho_0(x) \geq M_1 > 0$ for all $x \in \Omega$,
- $\nu \in \mathcal{C}^1(]0, +\infty[)$, $\nu(a) \geq \nu_1 > 0$ for all $a > 0$,
- $f \in L^q(Q_T)^3$, $u_0 \in W^{2-2/q,q}(\Omega)^3$, $\nabla \cdot u_0 = 0$, $u_0|_{\partial\Omega} = 0$ with $q > 3$.

Under these hypotheses, one has the following result:

Theorem 1. *There exists $t \leq T$ such that the equations (1)–(3) have a solution $(u, \nabla p, \rho)$ which satisfies*

$$u \in \mathcal{W}_q^{2,1}(Q_t), \quad \nabla p \in L^q(Q_t)^3, \quad \rho \in \mathcal{C}^1(\overline{Q_t}).$$

Moreover, there exists $R > 0$ depending on Ω, ν, T, ρ_0 , such that if

$$\|f\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3} \leq R,$$

then $(u, \nabla p, \rho)$ is a solution of (1)–(3) for $t = T$. □

Outline of the proof. We use a fixed point argument, decoupling the variables u and ρ . More precisely, let us consider $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfying $\nabla \cdot z = 0$, $z(0) = u_0$ in Ω and $z|_{\Sigma_T} = 0$.

In the first part, we prove that there exists a unique regular solution $(u, \nabla p, \rho)$ of the equations

$$(4) \quad \begin{cases} \rho \partial_t u - \nabla \cdot (\nu(\rho)(\nabla u + {}^t\nabla u)) + \rho(z \cdot \nabla)u + \nabla p = \rho f & \text{in } Q_T, \\ \nabla \cdot u = 0 & \text{in } Q_T, \\ \partial_t \rho + z \cdot \nabla \rho = 0 & \text{in } Q_T, \\ u(0) = u_0 \text{ and } \rho(0) = \rho_0 & \text{in } \Omega, \\ u|_{\Sigma_T} = 0. \end{cases}$$

In the second part, we prove that there exists R such that if $\|f\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3} \leq R$ or if T is small enough, then $z \mapsto u$ is a continuous map from a convex closed bounded subset of $\mathcal{W}_q^{2,1}(Q_T)$ with the topology of a Banach space $X_{q,T}$ defined below into itself, where $\mathcal{W}_q^{2,1}(Q_T) \subset X_{q,T}$ with compact imbedding, and by Schauder’s theorem, we infer the existence of a fixed point.

Remark. The proof of Theorem 1 is based on results of O.A. Ladyzenskaya and V.A. Solonnikov [5].

2. Functional spaces and preliminaries

Let $\mathcal{D}(\Omega)$ be the space of \mathcal{C}^∞ functions with compact support in Ω , $\mathcal{D}'(\Omega)$ the space of distributions on Ω and $\langle \cdot, \cdot \rangle_\Omega$ the duality product between $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$.

For $1 \leq r < +\infty$, $L^r(\Omega)$ is the space of distributions f on Ω for which $|f|^r$ is integrable. This space is endowed with the norm

$$\|f\|_r = \left(\int_\Omega |f|^r \right)^{\frac{1}{r}},$$

and $L^\infty(\Omega)$ is the space of distributions f on Ω locally integrable and satisfying

$$\|f\|_\infty = \text{supess } |f| < +\infty.$$

For $1 \leq s \leq +\infty$, the Sobolev spaces are defined by

$$W^{1,s}(\Omega) = \{v \in L^s(\Omega) : \nabla v \in L^s(\Omega)^3\},$$

$$W_0^{1,s}(\Omega) = \text{closure of } \mathcal{D}(\Omega) \text{ in } W^{1,s}(\Omega),$$

$$W^{-1,s}(\Omega) = \left\{ v \in \mathcal{D}'(\Omega) : v = v_0 + \sum_{i=1}^3 \partial_i v_i : v_i \in L^s(\Omega), i = 0, \dots, 3 \right\},$$

and we denote $H^1(\Omega) = W^{1,2}(\Omega)$, $H_0^1(\Omega) = W_0^{1,2}(\Omega)$, $H^{-1}(\Omega) = W^{-1,2}(\Omega)$ and

$$\mathcal{V} = \{v \in \mathcal{D}(\Omega)^3 : \nabla \cdot v = 0\},$$

$$V = \{v \in H_0^1(\Omega)^3 : \nabla \cdot v = 0\}.$$

Let us recall that V coincides with the closure of \mathcal{V} in $H^1(\Omega)^3$ (cf. Temam [12]).

Let $\mathcal{W}_q^{2,1}(Q_T)$ be the space of distributions $u \in L^q(0, T; W^{2,q}(\Omega)^3)$ such that $\partial_t u \in L^q(Q_T)^3$. This space, endowed with the norm

$$\|u\|_{q, Q_T}^{(2,1)} = \|\partial_t u\|_{L^q(Q_T)^3} + \|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + \|\nabla u\|_{L^q(Q_T)^9} + \|u\|_{L^q(Q_T)^3}$$

is a Banach space. All functions of $\mathcal{W}_q^{2,1}(Q_T)$ are in $\mathcal{C}_u(0, T; W^{2-2/q, q}(\Omega)^3)$, where $\mathcal{C}_u(0, T) = \mathcal{C}([0, T])$, so we can define $\| \cdot \|_T$ on $\mathcal{W}_q^{2,1}(Q_T)$ by

$$\|u\|_T = \|u\|_{q, Q_T}^{(2,1)} + \sup_{0 \leq t \leq T} \|u\|_{W^{2-2/q, q}(\Omega)^3}.$$

Endowed with this norm, $\mathcal{W}_q^{2,1}(Q_T)$ is a Banach space. Let us recall that for all $u \in \mathcal{W}_q^{2,1}(Q_T)$ and all t , $0 \leq t \leq T$ we have (cf. V.A. Solonnikov [11]):

$$\|u(t)\|_{W^{2-2/q, q}(\Omega)^3} \leq \|u_0\|_{W^{2-2/q, q}(\Omega)^3} + c\|u\|_{q, Q_t}^{(2,1)},$$

where c is independent of $t \in [0, T]$.

We denote by

$$\| (u, \nabla p) \|_T = \| u \|_T + \| \nabla p \|_{L^q(Q_T)^3}.$$

Finally, let $\mathcal{C}^\varepsilon(\overline{\Omega})$, $0 < \varepsilon < 1$, be the set of functions $f \in \mathcal{C}(\overline{\Omega})$ which satisfy $|f(x) - f(y)| \leq c|x - y|^\varepsilon$ for all $x, y \in \overline{\Omega}$ and $\mathcal{C}^{1,\varepsilon}(\overline{\Omega})$ the set of functions $f \in \mathcal{C}^1(\overline{\Omega})$ which satisfy $|\nabla f(x) - \nabla f(y)| \leq c'|x - y|^\varepsilon$ for all $x, y \in \overline{\Omega}$.

Let us now give an evolution case of De Rham's theorem (cf. J. Simon [10, Lemma 2, p. 1096]).

Lemma 2. *Let $h \in \mathcal{D}'(0, T; H^{-1}(\Omega)^3)$ satisfy $\langle h, v \rangle_\Omega = 0$ for all $v \in \mathcal{V}$. Then there exists $g \in \mathcal{D}'(0, T; L^2(\Omega))$ such that $h = \nabla g$. □*

Now one gives a compactness result:

Lemma 3. *There exists $1 > \varepsilon_q > 0$ such that*

$$\mathcal{W}_q^{2,1}(Q_T) \subset \left(L^q(0, T; \mathcal{C}^{1,\varepsilon_q}(\overline{\Omega})^3) \cap \mathcal{C}_u(0, T; \mathcal{C}(\overline{\Omega})^3) \right) =: X_{q,T}$$

with compact imbedding. □

The proof is based on the following result (see J. Simon [9, Corollary 8, p. 90])

Lemma 4. *Let X and Y be two Banach spaces, $X \subset Y$ with corresponding compact imbedding and B a Banach space, $X \subset B \subset Y$, such that there exists C and θ , $0 < \theta < 1$ such that*

$$\| v \|_B \leq C \| v \|_X^{1-\theta} \| v \|_Y^\theta \quad \forall v \in X.$$

Let $1 \leq s_0 \leq +\infty$, $1 \leq s_1 \leq +\infty$ and let \mathcal{F} be a bounded subset of $L^{s_0}(0, T; X)$ such that $\partial_t \mathcal{F}$ is bounded in $L^{s_1}(0, T; Y)$. Then,

(i) if $\theta(1 - 1/s_1) \leq (1 - \theta)/s_0$, \mathcal{F} is relatively compact in $L^s(0, T; B) \forall s < s_*$, where $1/s_* = (1 - \theta)/s_0 - \theta(1 - 1/s_1)$;

(ii) if $\theta(1 - 1/s_1) > (1 - \theta)/s_0$, \mathcal{F} is relatively compact in $\mathcal{C}_u(0, T; B)$. □

PROOF OF LEMMA 3:

(i) One has $\mathcal{W}_q^{2,1}(Q_T) \subset L^q(0, T; \mathcal{C}^{1,\varepsilon_q}(\overline{\Omega})^3)$ with corresponding compact imbedding.

For $X = W^{2,q}(\Omega)^3$ and $Y = L^q(\Omega)^3$, since we have $W^{2,q}(\Omega)^3 \subset L^q(\Omega)^3$ with compact imbedding, using Lemma 4(i), with $s_1 = s_0 = q$, we obtain for all $\theta \leq 1/q$

$$\mathcal{W}_q^{2,1}(Q_T) \subset L^q(0, T; (W^{2,q}(\Omega)^3, L^q(\Omega)^3)_\theta) = L^q(0, T; H_q^{2(1-\theta)}(\Omega)^3)$$

with compact imbedding (cf. H. Triebel [13, Theorem 2, p. 317] and [11, p. 185]).

In addition we have (cf. H. Triebel [13, p. 328]) $H_q^{2(1-\theta)}(\Omega)^3 \subset \mathcal{C}^{1,\alpha}(\overline{\Omega})^3$ for $\alpha = 1 - 2\theta - 3/q > 0$. Therefore we have

$$\mathcal{W}_q^{2,1}(Q_T) \subset L^q(0, T; \mathcal{C}^{1,\varepsilon_q}(\overline{\Omega})^3)$$

with compact imbedding, where $\varepsilon_q = 1 - 2\theta - 3/q$ and $\theta < \inf\{1/q, (q - 3)/2q\}$.

(ii) One has $\mathcal{W}_q^{2,1}(Q_T) \subset \mathcal{C}_u(0, T; \mathcal{C}(\overline{\Omega})^3)$ with corresponding compact imbedding.

Using Lemma 4 (ii) with $s_1 = s_0 = q$, we obtain for all $\theta > 1/q$

$$\mathcal{W}_q^{2,1}(Q_T) \subset \mathcal{C}_u(0, T; H_q^{2(1-\theta)}(\Omega)^3)$$

with compact imbedding.

In addition we have (cf. H. Triebel [13, p.328]) $H_q^{2(1-\theta)}(\Omega)^3 \subset \mathcal{C}(\overline{\Omega})^3$ for all $\theta < 1 - 3/2q$. Since $1/q < 1 - 3/2q$ ($q > 3$), we have

$$\mathcal{W}_q^{2,1}(Q_T) \subset \mathcal{C}_u(0, T; \mathcal{C}(\overline{\Omega})^3)$$

with compact imbedding. □

3. Transport equation

Proposition 5. *Let $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfy $\nabla \cdot z = 0$ and $z|_{\Sigma_T} = 0$. Then for all $\rho_0 \in \mathcal{C}^1(\overline{\Omega})$, there exists a unique solution $\rho \in \mathcal{C}^1(\overline{Q_T})$ of*

$$(5) \quad \begin{cases} \partial_t \rho + z \cdot \nabla \rho = 0 & \text{in } Q_T, \\ \rho|_{t=0} = \rho_0. \end{cases}$$

It satisfies

$$\min_{x \in \overline{\Omega}} \rho_0(x) \leq \rho(y, t) \leq \max_{x \in \overline{\Omega}} \rho_0(x) \quad \forall (y, t) \in Q_T,$$

and the following estimates, for all $t \leq T$:

$$(6) \quad \|\nabla \rho\|_{L^\infty(Q_t)^3} \leq \sqrt{3} \|\nabla \rho_0\|_{L^\infty(\Omega)^3} \exp\{\|\nabla z\|_{L^1(0,t;L^\infty(\Omega)^9)}\},$$

$$(7) \quad \|\partial_t \rho\|_{L^\infty(Q_t)} \leq \sqrt{3} \|\nabla \rho_0\|_{L^\infty(\Omega)^3} \|z\|_{L^\infty(Q_T)^3} \exp\{\|\nabla z\|_{L^1(0,t;L^\infty(\Omega)^9)}\}.$$

Let K be a closed bounded subset of $\mathcal{W}_q^{2,1}(Q_T) \cap L^2(0, T; V)$. Then the map $z \mapsto \rho$ is continuous on K endowed with the topology of $X_{q,T}$ with values in $\mathcal{C}_u(0, T; \mathcal{C}^1(\overline{\Omega}))$.

PROOF: The existence and uniqueness of such a solution, and the estimates (6)–(7) are given by O.A. Ladyzenskaya and V.A. Solonnikov [5].

Let us remark that if $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfies $\nabla \cdot z = 0$, $z|_{\Sigma_T} = 0$, there exists a unique $y(\tau, t, x)$ (cf. O.A. Ladyzenskaya and V.A. Solonnikov [5]) solution of

$$(8) \quad y^k(\tau, t, x) = x^k - \int_\tau^t z^k(y(\xi, t, x), \xi) d\xi.$$

In addition, for all $\tau, t, y(\tau, t, \cdot)$ is a one to one map on Ω with Jacobian equal to 1 (cf. V.A. Solomnikov [11]). The solution ρ of (5) satisfies

$$\rho(x, t) = \rho_0(y(0, t, x)).$$

Let us prove the continuity of the map $z \mapsto \rho$. It is well known, (see [5]) that if ρ_1 and ρ_2 are two solutions of (5) associated to z_1 and z_2 belonging to $\mathcal{W}_q^{2,1}(Q_T)$ and satisfying $z_1|_{\Sigma_T} = z_2|_{\Sigma_T} = 0, \nabla \cdot z_1 = \nabla \cdot z_2 = 0$, we have for all $t, 0 < t \leq T$, the following estimate:

$$\|\rho_1 - \rho_2\|_{L^\infty(Q_t)} \leq \|\nabla \rho_2\|_{L^\infty(Q_t)^3} \int_0^t \|z_1 - z_2\|_{L^\infty(\Omega)^3} d\tau.$$

So the map $z \mapsto \rho$ is continuous from K endowed with the topology of $X_{q,T}$ with values in $\mathcal{C}(Q_T)$.

Now, denoting $y_i(\xi) = y_i(\xi, t, x)$, we have:

$$\begin{aligned} |\partial_j \rho_2(x, t) - \partial_j \rho_1(x, t)| &\leq \left| \sum_k (\partial_k \rho_0(y_2(0)) - \partial_k \rho_0(y_1(0))) \partial_j y_2^k(0) \right| \\ &\quad + \left| \sum_k (\partial_k \rho_0)(y_1(0)) (\partial_j y_1^k(0) - \partial_j y_2^k(0)) \right|. \end{aligned}$$

Since $\rho_0 \in \mathcal{C}^1(\bar{\Omega})$ and $y_2 \in \mathcal{C}^1(\bar{Q}_T)^3$, we have:

$$\begin{aligned} &\|\nabla(\rho_2 - \rho_1)\|_{L^\infty(Q_t)^3} \\ &\leq 3\|\nabla y_2(0)\|_{L^\infty(Q_t)^9} \|(\nabla \rho_0)(y_1(0)) - (\nabla \rho_0)(y_2(0))\|_{L^\infty(Q_t)^3} \\ &\quad + 3\|\nabla \rho_0\|_{L^\infty(\Omega)^3} \|\nabla(y_1(0) - y_2(0))\|_{L^\infty(Q_t)^9}. \end{aligned}$$

To prove the continuity of the map $z \mapsto \rho$, since $\nabla \rho_0 \in \mathcal{C}(\bar{\Omega})^3$, it is enough to prove that if $z_1 \rightarrow z_2$ in $X_{q,T}$, then $y_1(0) \rightarrow y_2(0)$ in $\mathcal{C}_u(0, T; \mathcal{C}^1(\bar{\Omega}))$. To prove this property we will estimate $\|y_1(0) - y_2(0)\|_{L^\infty(Q_T)^3}$ and $\|\nabla(y_1(0) - y_2(0))\|_{L^\infty(Q_T)^9}$ in terms of $\|z_1 - z_2\|_{X_{q,T}}$.

Estimate of $\|y_2(0) - y_1(0)\|_{L^\infty(Q_t)^3}$. We have, according to (8):

$$|y_1^k(\tau) - y_2^k(\tau)| \leq \int_\tau^t |z_1^k(y_1(\xi), \xi) - z_1^k(y_2(\xi), \xi)| + |z_1^k(y_2(\xi), \xi) - z_2^k(y_2(\xi), \xi)| d\xi.$$

Since $z_1 \in L^1(0, T; \mathcal{C}^1(\bar{\Omega})^3)$, for all $x, y \in \Omega$ and almost all $t \in]0, T[$ we have:

$$|z_1(x, t) - z_1(y, t)| \leq \|\nabla z_1(t)\|_{L^\infty(\Omega)^9} |x - y|.$$

In addition, taking into account that $y_2(\xi, t, \cdot)$ is a one to one map on Ω and z_i is in $\mathcal{C}(\overline{Q})^3$, we obtain:

$$|y_1^k(\tau) - y_2^k(\tau)| \leq \int_{\tau}^t \|\nabla z_1(\xi)\|_{L^\infty(\Omega)^9} |y_1(\xi) - y_2(\xi)| d\xi + \int_{\tau}^t \|z_1^k(\xi) - z_2^k(\xi)\|_{L^\infty(\Omega)^3} d\xi.$$

So, using Gronwall lemma, we obtain:

$$\|y_2(\tau) - y_1(\tau)\|_{L^\infty(Q_t)^3} \leq ct \|z_1 - z_2\|_{X_{q,t}} \exp \left\{ ct^{1/q'} \|z_1\|_{X_{q,t}} \right\},$$

where q' satisfies $1/q + 1/q' = 1$.

Estimate of $\|\nabla(y_1(0) - y_2(0))\|_{L^\infty(Q_t)^9}$. We have

$$\begin{aligned} |\partial_i y_1^k(\tau) - \partial_i y_2^k(\tau)| &\leq \left| \sum_{\ell} \int_{\tau}^t ((\partial_{\ell} z_1^k)(y_1(\xi), \xi) - (\partial_{\ell} z_1^k)(y_2(\xi), \xi)) \partial_i y_1^{\ell}(\xi) d\xi \right| \\ &\quad + \left| \sum_{\ell} \int_{\tau}^t ((\partial_{\ell} z_1^k)(y_2(\xi), \xi) - (\partial_{\ell} z_2^k)(y_2(\xi), \xi)) \partial_i y_1^{\ell}(\xi) d\xi \right| \\ &\quad + \left| \sum_{\ell} \int_{\tau}^t (\partial_{\ell} z_2^k)(y_2(\xi), \xi) \partial_i (y_1^{\ell}(\xi) - y_2^{\ell}(\xi)) d\xi \right|. \end{aligned}$$

Since $z_1 \in L^1(0, T; C^{1,\varepsilon_q}(\overline{\Omega})^3)$, for all $x, y \in \Omega$ and almost all $t \in]0, T[$ we have:

$$|\nabla z_1(x, t) - \nabla z_1(y, t)| \leq b(t) |x - y|^{\varepsilon_q},$$

where $b(t) = \|\nabla z_1(t)\|_{C^{\varepsilon_q}(\Omega)^9}$ is in $L^1(0, T)$. In addition, $y_2(\xi, t, \cdot)$ is a one to one map on Ω and ∇z_i is in $L^1(0, T; \mathcal{C}(\overline{\Omega})^9)$, so we have, using the estimate of $|y_1(\tau) - y_2(\tau)|$:

$$\begin{aligned} |\nabla(y_1(\tau) - y_2(\tau))| &\leq c \left(\|\nabla y_1\|_{L^\infty(Q_t)^9} \int_{\tau}^t \|\nabla(z_1 - z_2)(\xi)\|_{L^\infty(\Omega)^9} d\xi \right. \\ &\quad + \|\nabla y_1\|_{L^\infty(Q_t)^9} \int_{\tau}^t b(\xi) \left(\int_{\xi}^t \|(z_1 - z_2)(\zeta)\|_{L^\infty(\Omega)^3} d\zeta \right. \\ &\quad \quad \quad \left. \left. \times \exp \left\{ c \int_0^t \|\nabla z_1(\zeta)\|_{L^\infty(\Omega)^9} d\zeta \right\} \right)^{\varepsilon_q} d\xi \right. \\ &\quad \left. + \int_{\tau}^t \|\nabla z_2(\xi)\|_{L^\infty(\Omega)^9} |\nabla(y_1(\xi) - y_2(\xi))| d\xi \right). \end{aligned}$$

Using the Gronwall lemma, we deduce the estimate

$$\begin{aligned}
 |\nabla(y_1(0) - y_2(0))| &\leq c\|\nabla y_1\|_{L^\infty(Q_t)^9} \exp\left\{ct^{1/q'}\|z_2\|_{X_{q,t}}\right\} \left(t^{\varepsilon_q}\|z_1 - z_2\|_{X_{q,t}}^{\varepsilon_q}\right. \\
 &\quad \left.\times \|b\|_{L^1(0,t)} \exp\left\{ct^{1/q'}\varepsilon_q\|z_1\|_{X_{q,t}}\right\} + t\|z_1 - z_2\|_{X_{q,t}}\right),
 \end{aligned}$$

which yields

$$\begin{aligned}
 \|\nabla(y_1(0) - y_2(0))\|_{L^\infty(Q_t)^9} &\leq c\|\nabla y_1\|_{L^\infty(Q_t)^9} \exp\left\{ct^{1/q'}\|z_2\|_{X_{q,t}}\right\} \left(t^{\varepsilon_q}\|z_1 - z_2\|_{X_{q,t}}^{\varepsilon_q}\right. \\
 &\quad \left.\times \|b\|_{L^1(0,t)} \exp\left\{ct^{1/q'}\varepsilon_q\|z_1\|_{X_{q,t}}\right\} + t\|z_1 - z_2\|_{X_{q,t}}\right).
 \end{aligned}$$

With all these estimates, we deduce the continuity of the map $z \mapsto y(0)$, and the proof of Proposition 5 is complete. □

4. Existence and uniqueness of a solution of the uncoupled equations (4)

4.1 The result.

Proposition 6. *Let $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfy $\nabla \cdot z = 0$, $z|_{t=0} = u_0$ in Ω , $z|_{\Sigma_T} = 0$. Under the hypothesis of Theorem 1, there exists a unique*

$$u \in \mathcal{W}_q^{2,1}(Q_T), \quad \nabla p \in L^q(Q_T)^3, \quad \rho \in \mathcal{C}^1(\overline{Q}_T)$$

solution of (4).

It satisfies

$$\begin{aligned}
 (9) \quad &\|(u, \nabla p)\|_T \\
 &\leq c\mathcal{M}_1^{\frac{2}{1-\alpha}}(1+Te^{T/2}+Te^{T/2}\|z\|_{L^\infty(Q_T)^3}^{\frac{2}{1-\alpha}})\left(\|f\|_{L^q(Q_T)^3}+\|u_0\|_{W^{2-2/q,q}(\Omega)^3}\right),
 \end{aligned}$$

where $\mathcal{M}_1 = (\overline{M}_3^9 + \overline{M}_4)^3 \overline{M}_3^{12}$, $M_3 = \|\nabla \rho\|_{L^\infty(Q_T)^3}$, $M_4 = \|\partial_t \rho\|_{L^\infty(Q_T)}$, $\overline{M}_i = M_i + 1$, $\alpha = 3(q - 2)[3(q - 2) + 4q]^{-1}$ and c does not depend on T , M_3 and M_4 .

The proof is given in several steps.

4.2 Simplified auxiliary equations. We consider here the following problem: Find a solution $(u, \nabla p)$ of

$$(10) \quad \begin{cases} \rho \partial_t u - \nu(\rho) \Delta u + \nabla p = f & \text{in } Q_T, \\ \nabla \cdot u = 0 & \text{in } Q_T, \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u|_{\Sigma_T} = 0. \end{cases}$$

We have the following result:

Proposition 7. *Let $\rho \in C^1(\overline{Q_T})$, $\rho(x, t) \geq M_1 > 0$ for all $(x, t) \in Q_T$. Under the hypothesis of Theorem 1, there exist*

$$u \in \mathcal{W}_q^{2,1}(Q_T), \quad \nabla p \in L^q(Q_T)^3,$$

solving (10).

In addition, there exists at most one solution of (10) in the space $(\mathcal{C}_u(0, T; L^2(\Omega)^3) \cap L^2(0, T; H^1(\Omega)^3)) \times H^{-1}(Q_T)^3$.

PROOF: Existence. The existence of a solution of (10) in $(L^\infty(0, T; H^1(\Omega)^3) \cap H^1(0, T; L^2(\Omega)^3)) \times L^2(0, T; H^{-1}(\Omega)^3)$ is well known (see for example [6]).

Uniqueness. Let $(u_1, \nabla p_1)$ and $(u_2, \nabla p_2)$ be two solutions of (10) in $(\mathcal{C}_u(0, T; L^2(\Omega)^3) \cap L^2(0, T; H^1(\Omega)^3)) \times H^{-1}(Q_T)^3$. Then $u = u_1 - u_2$, $\nabla p = \nabla(p_1 - p_2)$ is a solution of

$$\begin{cases} \rho \partial_t u - \nu(\rho) \Delta u + \nabla p = 0 & \text{in } Q_T, \\ \nabla \cdot u = 0 & \text{in } Q_T, \\ u|_{t=0} = 0 & \text{in } \Omega, \\ u|_{\Sigma_T} = 0. \end{cases}$$

For all $\varphi \in \mathcal{D}(0, T; \mathcal{V})$, we have in $W^{-1,1}(0, T)$

$$\langle \rho \partial_t u, \varphi \rangle_\Omega - \langle \nu(\rho) \Delta u, \varphi \rangle_\Omega = 0.$$

Since $\langle \nu(\rho) \Delta u, \varphi \rangle_\Omega = - \int_\Omega \nabla(\nu(\rho)) \cdot \nabla u \cdot \varphi - \int_\Omega \nu(\rho) \nabla u \cdot \nabla \varphi$ is in $L^1(0, T)$, we have $\langle \rho \partial_t u, \varphi \rangle_\Omega \in L^1(0, T)$. In addition, $\varphi \mapsto \int_\Omega \nu(\rho) \nabla u \cdot \nabla \varphi$ and $\varphi \mapsto \int_\Omega \nabla(\nu(\rho)) \cdot \nabla u \cdot \varphi$ are continuous in the space $L^2(0, T; V)$ with values in $L^1(0, T)$, so $\varphi \mapsto \int_\Omega \rho \partial_t u \cdot \varphi$ is continuous in $L^2(0, T; V)$ with values in $L^1(0, T)$. Therefore, we deduce that for all $v \in L^2(0, T; V)$, we have in $L^1(0, T)$:

$$\int_\Omega \rho \partial_t u \cdot v + \int_\Omega \nabla(\nu(\rho)) \cdot \nabla u \cdot v + \int_\Omega \nu(\rho) \nabla u \cdot \nabla v = 0.$$

In particular, for $v = u$, we obtain in $L^1(0, T)$:

$$\frac{1}{2} \int_\Omega \rho \partial_t |u|^2 + \nu_1 \int_\Omega |\nabla u|^2 \leq \int_\Omega |\nabla(\nu(\rho)) \cdot \nabla u \cdot u|.$$

In addition, we have $\partial_t(\rho|u|^2) = \rho \partial_t |u|^2 + \partial_t \rho |u|^2$, so we obtain

$$\int_\Omega \partial_t(\rho|u|^2) + \nu_1 \int_\Omega |\nabla u|^2 \leq c \int_\Omega \rho |u|^2,$$

where c depends only on ν, ρ , and we deduce from the Gronwall lemma that $u = 0$. The De Rham theorem implies that $\nabla p = 0$, and the uniqueness follows.

Regularity of such a solution. Choose p such that $\int_{\Omega} p/\nu(\rho) = 0$. Dividing by $\nu(\rho) \geq \nu_1 > 0$ and denoting $P = p/\nu(\rho)$, $\lambda = \rho/\nu(\rho)$, the equation (10) can be rewritten in the following form

$$\begin{cases} \lambda \partial_t u - \Delta u + \nabla P = \frac{f}{\nu(\rho)} - \frac{\nu'(\rho) \nabla \rho}{\nu(\rho)} P, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0, \\ u|_{\Sigma_T} = 0, \end{cases}$$

with $\lambda \in C^1(\overline{Q_T})$ and $\lambda \geq \lambda_1 > 0$.

Let us consider the following equation:

$$(11) \quad \begin{cases} \lambda \partial_t u' - \Delta u' + \nabla P' = \frac{f}{\nu(\rho)} - \frac{\nu'(\rho) \nabla \rho}{\nu(\rho)} P, \\ \nabla \cdot u' = 0, \\ u'|_{t=0} = u_0, \\ u'|_{\Sigma_T} = 0, \end{cases}$$

where $P \in L^2(Q_T)$ is defined above. Since $f \in L^2(Q_T)^3$, there exists a unique solution $(u', \nabla P')$ of (11) in $(C_u(0, T; L^2(\Omega)^3) \cap L^2(0, T; H^1(\Omega)^3)) \times H^{-1}(Q_T)^3$.

In addition (cf. O.A. Ladyzenskaya and V.A. Solonnikov [5]), $u' \in \mathcal{W}_2^{2,1}(Q_T)$, $\nabla P' \in L^2(Q_T)^3$. Now, since $(u, \nabla P)$ is solution of (11) we deduce that $u \in \mathcal{W}_2^{2,1}(Q_T)$, $\nabla P \in L^2(Q_T)^3$, and therefore $\nabla p \in L^2(Q_T)^3$. Then, from Lemma 9 (in appendix) we deduce that $p \in L^{\sigma_0}(Q_T)$, where $\sigma_0 = \min(q, 8/3)$. Therefore, since $f \in L^q(Q_T)^3$, we deduce from the equation (11) that $u \in \mathcal{W}_{\sigma_0}^{2,1}(Q_T)$ and $\nabla p \in L^{\sigma_0}(Q_T)^3$ (see O.A. Ladyzenskaya, V.A. Solonnikov [5]). Repeating this process a finite number of times, we obtain Proposition 7. \square

4.3 Auxiliary equations. We consider now the following problem: Find a solution $(u, \nabla p)$ of

$$(12) \quad \begin{cases} \rho \partial_t u - \nabla \cdot (\nu(\rho)(\nabla u + {}^t \nabla u)) + \nabla p = f \text{ in } Q_T, \\ \nabla \cdot u = 0 \text{ in } Q_T, \\ u|_{t=0} = u_0 \text{ in } \Omega, \\ u|_{\Sigma_T} = 0. \end{cases}$$

We have the following result:

Proposition 8. *Under the hypothesis of Proposition 7, there exist*

$$u \in \mathcal{W}_q^{2,1}(Q_T), \nabla p \in L^q(Q_T)^3,$$

solving (12).

It satisfies

$$(13) \quad \|(u, \nabla p)\|_T \leq \mathcal{M}_1 (\|f\|_{L^q(Q_T)^3} + \|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}),$$

where $\mathcal{M}_1 = c(\overline{M}_3^9 + M_4)\overline{M}_3^{12}$, $\overline{M}_i = (M_i + 1)$, $1 \leq i \leq 4$,

$$(14) \quad \|(u, \nabla p)\|_t \leq c\mathcal{M}_1^2 (\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}) \exp\{c\mathcal{M}_1 t\},$$

for all t , $0 \leq t \leq T$, where c depends only on ν , M_1 , M_2 , M_3 and M_4 .

In addition, there exists at most one solution of (12) in the space

$$\left(\mathcal{C}_u(0, T; L^2(\Omega)^3) \cap L^2(0, T; H^1(\Omega)^3) \right) \times H^{-1}(Q_T)^3.$$

PROOF: Existence. The existence of a solution of (12) in $\left(L^\infty(0, T; H^1(\Omega)^3) \cap H^1(0, T; L^2(\Omega)^3) \right) \times L^2(0, T; H^{-1}(\Omega)^3)$ is known (see for example [6]). As in Proposition 7, we can prove that there exists at most one solution of (12) in $\left(\mathcal{C}_u(0, T; L^2(\Omega)^3) \cap L^2(0, T; H^1(\Omega)^3) \right) \times H^{-1}(Q_T)^3$.

Regularity of this solution. The first equation of (12) can be written in the form

$$\rho \partial_t u - \nu(\rho) \Delta u + \nabla p = f + \nabla(\nu(\rho))(\nabla u + {}^t \nabla u).$$

Since $f + \nabla(\nu(\rho))(\nabla u + {}^t \nabla u) \in L^2(Q_T)^3$, there exists (cf. Proposition 7) one solution $u' \in \mathcal{W}_2^{2,1}(Q_T)$, $\nabla p' \in L^2(Q_T)^3$ of

$$(15) \quad \begin{cases} \rho \partial_t u' - \nu(\rho) \Delta u' + \nabla p' = f + \nabla(\nu(\rho))(\nabla u + {}^t \nabla u), \\ \nabla \cdot u' = 0, \\ u'|_{t=0} = u_0, \\ u'|_{\Sigma_T} = 0, \end{cases}$$

where $(u, \nabla p)$ is the solution of (12). In addition, this solution is unique in the space $\left(\mathcal{C}_u(0, T; L^2(\Omega)^3) \cap L^2(0, T; H^1(\Omega)^3) \right) \times H^{-1}(Q_T)^3$. Since the solution $(u, \nabla p)$ of (12) is a solution of (15), we deduce that the solution of (12) verifies $u \in \mathcal{W}_2^{2,1}(Q_T)$, $\nabla p \in L^2(Q_T)^3$. Therefore (cf. Lemma 9), there exists σ_0 , $2 < \sigma_0 \leq q$ such that $u \in L^{\sigma_0}(0, T; W^{1,\sigma_0}(\Omega)^3)$. So we deduce from (15), since $f \in L^q(Q_T)^3$, that $u \in \mathcal{W}_{\sigma_0}^{2,1}(Q_T)$ and $\nabla p \in L^{\sigma_0}(Q_T)^3$. Repeating this process a finite number of times (until $\sigma_m = q$), we obtain the regularity.

Estimates. Choose p such that $\int_{\Omega} p = 0$. Then setting $P = p/\nu(\rho)$, (12) can be rewritten in the following way:

$$\begin{cases} \frac{\rho}{\nu(\rho)} \partial_t u - \Delta u + \nabla P = \frac{f}{\nu(\rho)} - \frac{\nu'(\rho) \nabla \rho}{\nu(\rho)} P + \frac{\nu'(\rho) \nabla \rho}{\nu(\rho)} \cdot (\nabla u + {}^t \nabla u), \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0, \\ u|_{\Sigma_T} = 0. \end{cases}$$

Since $\nu_2 \geq \nu(\rho) \geq \nu_1 > 0$ and $\nu'_2 \geq \nu'(\rho)$, we have the following estimate (cf. O.A. Ladyzenskaya, V.A. Solonnikov [5]),

$$\begin{aligned} \|(u, \nabla P)\|_T \leq c(M_4 + \overline{M}_3^9) & \left(\|f\|_{L^q(Q_T)^3} + M_3 \|P\|_{L^q(Q_T)} + M_3 \|\nabla u\|_{L^q(Q_T)^9} \right. \\ & \left. + \|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3} \right), \end{aligned}$$

where c depends on ν , M_1 and M_2 only. Then we obtain

$$\begin{aligned} \|(u, \nabla p)\|_T \leq c(M_4 + \overline{M}_3^9) & \left(\|f\|_{L^q(Q_T)^3} + M_3 \|p\|_{L^q(Q_T)} + M_3 \|\nabla u\|_{L^q(Q_T)^9} \right. \\ & \left. + \|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3} \right), \end{aligned}$$

where c depends on ν , M_1 and M_2 only.

Using (15) we have (cf. Lemma 9):

$$\|p\|_{L^q(Q_t)} \leq c \overline{M}_3^3 \left(\|f\|_{L^q(Q_t)^3} + \overline{M}_3 \|\nabla u\|_{L^q(Q_t)^9} + \|\nabla u\|_{L^q(\Sigma_t)^9} \right),$$

so we obtain

$$(16) \quad \begin{aligned} \|(u, \nabla p)\|_T \leq A_2 & \left(\|f\|_{L^q(Q_T)^3} + \|\nabla u\|_{L^q(\Sigma_T)^3} \right) + A_1 \|\nabla u\|_{L^q(Q_T)^9} \\ & + c(M_4 + \overline{M}_3^9) \left(\|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3} \right) \end{aligned}$$

with

$$\begin{aligned} A_1 &= c(\overline{M}_4 + \overline{M}_3^9) \overline{M}_3^5 \\ A_2 &= c(\overline{M}_4 + \overline{M}_3^9) \overline{M}_3^4, \end{aligned}$$

where c depends on ν , M_1 and M_2 only.

Using the following interpolation inequalities (cf. O.A. Ladyzenskaya and V.A. Solonnikov [5]), since $(a^q + b^q) \leq (a + b)^q$ we have

$$\begin{aligned} \|\nabla u\|_{L^q(Q_T)^9} &\leq \alpha_1 \|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + c\alpha_1^{-1} \|u\|_{L^q(Q_T)^3}, \\ \|\nabla u\|_{L^q(\Sigma_T)^9} &\leq \alpha_2 \|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + c\alpha_2^{-\frac{q+1}{q-1}} \|u\|_{L^q(Q_T)^3}, \end{aligned}$$

for all $\alpha_i \in]0, 1]$, and taking $\alpha_1 = (4A_1)^{-1}$ and $\alpha_2 = (4A_2)^{-1}$, we obtain:

$$\begin{aligned} A_1 \|\nabla u\|_{L^q(Q_T)^9} &\leq \frac{1}{4} \|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + cA_1^2 \|u\|_{L^q(Q_T)^3}, \\ A_2 \|\nabla u\|_{L^q(\Sigma_T)^9} &\leq \frac{1}{4} \|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + cA_2^{2q/(q-1)} \|u\|_{L^q(Q_T)^3}, \end{aligned}$$

where c depends on Ω and q only. With these estimates, (16) gives:

$$\begin{aligned} \|(u, \nabla p)\|_T &\leq c \left(A_2 \|f\|_{L^q(Q_T)^3} + A_1^2 \|u\|_{L^q(Q_T)^3} + A_2^{2q/(q-1)} \|u\|_{L^q(Q_T)^3} \right. \\ &\quad \left. + (M_4 + \overline{M}_3^9) (\|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}) \right). \end{aligned}$$

Now, since A_1^2 , A_2 and $A_2^{2q/(q-1)}$ are smaller than $c(\overline{M}_3^9 + \overline{M}_4)^3 \overline{M}_3^{12}$, we deduce from the previous inequality the estimate (13).

To prove the estimate (14), let

$$y(t) = \int_0^t \|u(\tau)\|_{L^q(\Omega)^3}^q d\tau = \|u\|_{L^q(Q_t)^3}^q.$$

We have $y \in W^{1,1}(0, T)$, $y(0) = 0$ and $y'(t) = \|u(t)\|_q^q$. In addition, for all $t' \leq t$ we have:

$$\begin{aligned} y'(t') &= \int_0^{t'} \frac{d}{d\tau} \|u(\tau)\|_q^q d\tau + \|u_0\|_q^q = \int_0^{t'} \int_\Omega \frac{d}{d\tau} (|u(\tau)|^2)^{\frac{q}{2}} d\tau + \|u_0\|_q^q \\ &\leq q \|\partial_t u\|_{L^q(Q_{t'})^3} \|u\|_{L^q(Q_{t'})^3}^{q-1} + \|u_0\|_q^q \\ &\leq q\mathcal{M}_1 \|u\|_{L^q(Q_{t'})^3}^q + q\mathcal{M}_1 (\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}) \|u\|_{L^q(Q_{t'})^3}^{q-1} \\ &\quad + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}^q. \end{aligned}$$

Since $\mathcal{M}_1 > 1$, using Young's inequality we obtain:

$$y'(t') \leq (2q - 1)\mathcal{M}_1 y(t') + q^2 \mathcal{M}_1 (\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3})^q.$$

Integrating this equation from 0 to t we have:

$$y(t) \leq q^2 \mathcal{M}_1 (\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3})^q \exp\{(2q - 1)\mathcal{M}_1 t\}.$$

Now, taking into account that $\mathcal{M}_1 > 1$ we obtain:

$$\|u\|_{L^q(Q_t)^3} \leq c\mathcal{M}_1 (\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}) \exp\{c\mathcal{M}_1 t\}.$$

Using this estimate, (13) gives

$$\|(u, \nabla p)\|_t \leq c\mathcal{M}_1^2 (\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}) \exp\{c\mathcal{M}_1 t\},$$

and the proof is complete. □

4.4 Proof of Proposition 6.

Existence of a regular solution. We prove the existence by successive approximations. Let $u^0 = 0$ and for all $m \geq 1$:

$$(17) \quad \begin{cases} \rho \partial_t u^m - \nabla \cdot (\nu(\rho)(\nabla u^m + {}^t \nabla u^m)) + \nabla p^m = \rho f - \rho(z \cdot \nabla)u^{m-1} & \text{in } Q_T, \\ \partial_t \rho + z \cdot \nabla \rho = 0 & \text{in } Q_T, \\ \nabla \cdot u^m = 0 & \text{in } Q_T, \\ u^m|_{t=0} = u_0 & \text{in } \Omega, \\ u^m|_{\Sigma_T} = 0, \\ \rho|_{t=0} = \rho_0 & \text{in } \Omega. \end{cases}$$

It is known (cf. Proposition 8) that there exists a unique solution of (17). Denoting $w^m = u^m - u^{m-1}$, $\nabla P^m = \nabla(p^m - p^{m-1})$ and $\mathcal{W}_m(t) = \|(w^m, \nabla p^m)\|_t$, we deduce from (14) the following estimate

$$\begin{aligned} \mathcal{W}_m^q(t) &\leq c \|\nabla w^{m-1}\|_{L^q(Q_t)^3}^q \leq c \int_0^t \|w^{m-1}\|_{W^{2,q}(\Omega)^3}^q d\tau \\ &\leq c \int_0^t \mathcal{W}_{m-1}^q(\tau) d\tau \leq c^{m-1} \frac{t^{m-1}}{(m-1)!} \mathcal{W}_1^q(t), \end{aligned}$$

which implies the convergence of the series $\sum \mathcal{W}_m(t)$ for all $t \leq T$. From this, it follows the convergence of u^m in $\mathcal{W}_q^{2,1}(Q_T)$ and ∇p^m in $L^q(Q_T)^3$.

The uniqueness of a such solution is obvious.

Estimation. We have the following estimate of $\|(u, \nabla p)\|_T$ given in Proposition 8

$$(18) \quad \|(u, \nabla p)\|_T \leq \mathcal{M}_1(F + \|u\|_{L^q(Q_T)^3} + \|(z \cdot \nabla)u\|_{L^q(Q_T)^3}),$$

where $F = \|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}$.

Now, let us estimate each term of the right hand side of this inequality. Multiplying the first equation of (4) by u and integrating on Ω , we obtain

$$\int_{\Omega} \rho \left[\frac{1}{2} \partial_t (u^2) + (z \cdot \nabla)u \cdot u \right] + \int_{\Omega} \nu(\rho)(\nabla u + {}^t \nabla u) \cdot \nabla u = \int_{\Omega} \rho f \cdot u.$$

Since $\nu(\rho) \geq \nu_1 > 0$, we obtain, summing this equation with the transport equation multiplied by $(1/2)|u|^2$:

$$\frac{d}{dt} \int_{\Omega} \rho |u|^2 + 2\nu_1 \int_{\Omega} |\nabla u|^2 \leq 2 \int_{\Omega} \rho f \cdot u.$$

So we deduce the following estimate:

$$\left(\int_{\Omega} |u|^2 \right)(t) \leq ce^t \left(\int_0^t \|f\|_2^2 d\tau + \|u_0\|_2^2 \right),$$

where c depends on M_1 and M_2 only.

Now, using Hölder inequality, we have:

$$\begin{aligned} \left(\int_{\Omega} |u|^2\right)(t) &\leq ce^t \left(\int_0^t \|f\|_q^2 d\tau + \|u_0\|_q^2\right) \\ &\leq ce^t \left(\int_0^t \|f\|_q^q d\tau\right)^{2/q} + ce^t \|u_0\|_q^2, \end{aligned}$$

and therefore

$$(19) \quad \|u\|_2 \leq ce^{t/2} (\|f\|_{L^q(Q_T)^3} + \|u_0\|_q).$$

Using the fact that

$$\|u\|_q \leq c(\|u\|_{W^{2,q}(\Omega)^3})^\alpha \|u\|_2^{1-\alpha}$$

with $\alpha = 3(q - 2)[3(q - 2) + 4q]^{-1}$ (cf. O.A. Ladyzenskaya and V.A. Solonnikov [5]), the previous estimate gives:

$$(20) \quad \|u\|_{L^q(Q_T)^3} \leq c\|(u, \nabla p)\|_T^\alpha \left(\int_0^T \|u\|_2^q dt\right)^{(1-\alpha)/q}.$$

Since (19) gives

$$\left(\int_0^T \|u\|_2^q\right)^{1/q} \leq cTe^{T/2}F,$$

we obtain from (20):

$$(21) \quad \|u\|_{L^q(Q_T)^3} \leq c\|(u, \nabla p)\|_T^\alpha (Te^{T/2}F)^{1-\alpha}.$$

Now, to estimate the last term of the right hand of (18), we remark that

$$\|(z \cdot \nabla)u\|_{L^q(Q_T)^3} \leq \|z\|_{L^\infty(Q_T)^3} \|\nabla u\|_{L^q(Q_T)^9}.$$

Since

$$\|\nabla u\|_{L^q(Q_T)^9} \leq c(\|u\|_{L^q(0,T;W^{2,q}(\Omega)^3)})^{1/2} \|u\|_{L^q(Q_T)^3}^{1/2},$$

we obtain

$$\begin{aligned} \|(z \cdot \nabla)u\|_{L^q(Q_T)^3} &\leq c\|z\|_{L^\infty(Q_T)^3} \|(u, \nabla p)\|_T^{1/2} \|u\|_{L^q(Q_T)^3}^{1/2} \\ &\leq c\|z\|_{L^\infty(Q_T)^3} \|(u, \nabla p)\|_T^{\frac{\alpha+1}{2}} (Te^{T/2}F)^{\frac{1-\alpha}{2}}. \end{aligned}$$

We deduce from this estimate and from the estimates (18) and (21)

$$(22) \quad \begin{aligned} \|(u, \nabla p)\|_T &\leq c\mathcal{M}_1(F + \|(u, \nabla p)\|_T^\alpha (Te^{T/2}F)^{1-\alpha} \\ &\quad + \|z\|_{L^\infty(Q_T)^3} \|(u, \nabla p)\|_T^{\frac{\alpha+1}{2}} (Te^{T/2}F)^{\frac{1-\alpha}{2}}), \end{aligned}$$

where $\mathcal{M}_1 = c(\overline{M}_3^9 + \overline{M}_4)^3 \overline{M}_3^{12}$. Using the following Young's inequalities:

$$\mathcal{M}_1 \|(u, \nabla p)\|_T^\alpha (Te^{T/2}F)^{1-\alpha} \leq \alpha \varepsilon \|(u, \nabla p)\|_T + (1-\alpha)\varepsilon^{-\frac{\alpha}{1-\alpha}} Te^{T/2}F \mathcal{M}_1^{\frac{1}{1-\alpha}},$$

and

$$\begin{aligned} \mathcal{M}_1 \|z\|_{L^\infty(Q_T)^3} \|(u, \nabla p)\|_T^{\frac{\alpha+1}{2}} (TFe^{T/2})^{\frac{1-\alpha}{2}} \\ \leq \frac{1+\alpha}{2} \varepsilon \|(u, \nabla p)\|_T + \frac{1-\alpha}{2} \varepsilon^{-\frac{1+\alpha}{1-\alpha}} (\mathcal{M}_1 \|z\|_{L^\infty(Q_T)^3})^{\frac{2}{1-\alpha}} TFe^{T/2}, \end{aligned}$$

with ε small enough, we deduce from (22) the following estimate, since $\mathcal{M}_1 \geq 1$:

$$\|(u, \nabla p)\|_T \leq cF \mathcal{M}_1^{\frac{2}{1-\alpha}} (1 + Te^{T/2} + Te^{T/2} \|z\|_{L^\infty(Q_T)^3}^{\frac{2}{1-\alpha}}),$$

and the proof is complete. □

5. Proof of Theorem 1

As we have seen (cf. Proposition 6), for all $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfying $z(0) = u_0$, $z|_{\Sigma_T} = 0$ and $\nabla \cdot z = 0$, there exists a unique solution $u \in \mathcal{W}_q^{2,1}(Q_T)$, $\nabla p \in L^q(Q_T)^3$, $\rho \in C^1(\overline{Q}_T)$ of (4).

Local existence. This proof is based on the Schauder theorem that can be found for example in N. Dunford, J.T. Schwartz [2, Theorem 5, p. 456].

In the first step, let us prove that there exist $T_M > 0$ and a convex compact subset K of X_{q,T_M} such that $z \mapsto u$ maps K into K .

For all $t \leq T$, we have:

$$\|(u, \nabla p)\|_t \leq cF \mathcal{M}_1^{\frac{2}{1-\alpha}} (1 + te^{t/2} + te^{t/2} \|z\|_t^{\frac{2}{1-\alpha}})$$

with $\mathcal{M}_1 = c(\overline{M}_3^9 + \overline{M}_4)^3 \overline{M}_3^{12}$ and

$$\begin{aligned} \overline{M}_3 &= 1 + M_3 \leq 1 + \sqrt{3} \|\nabla \rho_0\|_{L^\infty(\Omega)^3} \exp\{ct^{1/q'} \|z\|_t\}, \\ \overline{M}_4 &= 1 + M_4 \leq 1 + \sqrt{3} \|\nabla \rho_0\|_{L^\infty(\Omega)^3} \|z\|_{L^\infty(Q_t)^3} \exp\{ct^{1/q'} \|z\|_t\}, \end{aligned}$$

where q' satisfies $1/q + 1/q' = 1$. Let q_1 be a real, $3 < q_1 < q$. Then

$$\|z\|_{L^\infty(Q_t)^3} \leq c \sup_{0 \leq \tau \leq t} \|z\|_{W^{2-2/q_1, q_1}(\Omega)^3} \leq c(\|u_0\|_{W^{2-2/q_1, q_1}(\Omega)^3} + \|z\|_{q_1, Q_t}^{(2,1)}),$$

where c is a constant which does not depend on t (see V.A. Solonnikov [11]). Moreover

$$\|z\|_{q_1, Q_t}^{(2,1)} \leq ct^{(q-q_1)/qq_1} \|z\|_{q, Q_t}^{(2,1)},$$

so, since $\|u_0\|_{W^{2-2/q_1, q_1}(\Omega)^3} \leq c\|u_0\|_{W^{2-2/q, q}(\Omega)^3}$,

$$\overline{M_4} \leq 1 + c\sqrt{3}\|\nabla\rho_0\|_{L^\infty(\Omega)^3} (\|u_0\|_{W^{2-2/q, q}(\Omega)^3} + t^{(q-q_1)/qq_1}\|z\|_{q, Q_t}^{(2,1)}) \exp\{ct^{1/q'}\|z\|_t\}.$$

Therefore we have

$$\|(u, \nabla p)\|_t \leq cH(t, \|z\|_t),$$

where $H(t, a)$ is continuous function in (t, a) defined by

$$H(t, a) = F\left(1 + \sqrt{3}\|\nabla\rho_0\|_{L^\infty(\Omega)^3}(1 + \|u_0\|_{W^{2-2/q, q}(\Omega)^3} + t^{(q-q_1)/qq_1}a)\right)^{\frac{42}{1-\alpha}} \times \exp\left\{\frac{42}{1-\alpha}ct^{\frac{1}{q'}}a\right\}(1 + te^{t/2} + te^{t/2}a^{\frac{2}{1-\alpha}}).$$

Since $H(0, a) = H(0, a)$ for all $a \geq 0$, for $M = H(0, 0)$, there exists T_M such that, if $\|z\|_{T_M} \leq M$, then $\|(u, \nabla p)\|_{T_M} \leq M$.

Let us denote

$$K = \{u \in \mathcal{W}_q^{2,1}(Q_{T_M}), u(0) = u_0, u|_{\Sigma_{T_M}} = 0, \nabla \cdot u = 0, \|u\|_{T_M} \leq M\}.$$

Then K is a convex compact subset of X_{q, T_M} , and $z \mapsto u$ maps K into K .

In the second step, let us prove that $z \mapsto u$ is continuous from K endowed with the topology of X_{q, T_M} into itself. Let z_1 and z_2 be two elements of K . Then we obtain, setting $z = z_1 - z_2$, $u = u_1 - u_2$, $\nabla p = \nabla(p_1 - p_2)$ and $\rho = \rho_1 - \rho_2$:

$$\begin{cases} \rho_1 \partial_t u - \nabla \cdot (\nu(\rho_1)(\nabla u + {}^t\nabla u)) + \rho_1(z_1 \cdot \nabla)u + \nabla p = G & \text{in } Q_T, \\ \nabla \cdot u = 0 & \text{in } Q_T, \\ \partial_t \rho + z_1 \cdot \nabla \rho = -z \cdot \nabla \rho_2 & \text{in } Q_T, \\ u|_{t=0} = 0 & \text{in } \Omega, \\ u|_{\Sigma_T} = 0, \\ \rho|_{t=0} = 0 & \text{in } \Omega, \end{cases}$$

where

$$G = \rho f - \rho \partial_t u_2 + \nabla \cdot ((\nu(\rho_1) - \nu(\rho_2))(\nabla u_2 + {}^t\nabla u_2)) - \rho_1(z \cdot \nabla)u_2 + \rho(z_1 \cdot \nabla)u_2.$$

We deduce from (9) the following estimate:

$$\|(u, \nabla p)\|_{T_M} \leq c\|G\|_{L^q(Q_{T_M})^3} \mathcal{M}_1^{\frac{2}{1-\alpha}} (1 + Te^{T/2} + Te^{T/2}\|z_1\|_{T_M}^{\frac{2}{1-\alpha}}),$$

where $\|G\|_{L^q(Q_{T_M})^3}$ verifies

$$\begin{aligned} \|G\|_{L^q(Q_{T_M})^3} &\leq \|\rho\|_{L^\infty(Q_{T_M})} \left(\|f\|_{L^q(Q_{T_M})^3} + \|\partial_t u_2\|_{L^q(Q_{T_M})^3} \right. \\ &\quad \left. + \|z_1\|_{L^\infty(Q_{T_M})^3} \|\nabla u_2\|_{L^q(Q_{T_M})^9} \right) \\ &\quad + \|\nabla \cdot ((\nu(\rho_1) - \nu(\rho_2))(\nabla u_2 + {}^t\nabla u_2))\|_{L^q(Q_{T_M})^3} \\ &\quad + \|\rho_1\|_{L^\infty(Q_{T_M})} \|z\|_{L^\infty(Q_{T_M})^3} \|\nabla u_2\|_{L^q(Q_{T_M})^9}. \end{aligned}$$

As we have proved in Proposition 5, if $z_2 \rightarrow z_1$ in X_{q,T_M} , then $\rho_2 \rightarrow \rho_1$ in the space $\mathcal{C}_u(0, T_M; \mathcal{C}^1(\overline{\Omega}))$. So, $\nu(\rho_2) \rightarrow \nu(\rho_1)$ in $\mathcal{C}_u(0, T_M; \mathcal{C}^1(\overline{\Omega}))$. From this, we obtain that if $z_2 \rightarrow z_1$ in X_{q,T_M} , then $\|G\|_{L^q(Q_{T_M})^3} \rightarrow 0$ and therefore $\| (u, \nabla p) \|_{T_M} \rightarrow 0$. This proves that the map $z \mapsto u$ is continuous. Using the Schauder fixed point theorem, we obtain that there exist $u \in K$ and $\nabla p \in L^q(Q_{T_M})^3$ solving (1)–(3).

Global existence. We have

$$\| (u, \nabla p) \|_T \leq cF\mathcal{M}_1^{\frac{2}{1-\alpha}} (1 + Te^{T/2} + Te^{T/2} \|z\|_{L^\infty(Q_T)^3}^{\frac{2}{1-\alpha}}).$$

Let $M > 0$ and suppose $\|z\|_T \leq M$. Then there exists $R > 0$ such that if $F = \|f\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3} \leq R$, then $\|u\|_T \leq M$. As in the proof of the local existence,

$$K = \{u \in \mathcal{W}_q^{2,1}(Q_T), u(0) = u_0, u|_{\Sigma_T} = 0, \nabla \cdot u = 0, \|u\|_T \leq M\}$$

is a convex compact subset of $X_{q,T}$, and $z \mapsto u$ maps continuously K into K . Therefore we deduce the existence of $u \in K$ and $\nabla p \in L^q(Q_T)^3$ solving (1)–(3). □

Appendix

Lemma 9. *Let $f \in L^q(Q_T)^3$ and let $(u, \nabla p)$ be the unique solution of (10) satisfying*

$$u \in L^2(0, T; H^2(\Omega)^3), \partial_t u \in L^2(Q_T)^3, \nabla p \in L^2(Q_T)^3.$$

Suppose that $u \in L^s(0, T; W^{2,s}(\Omega)^3) \cap W^{1,s}(0, T; L^s(\Omega)^3)$ and $\nabla p \in L^s(Q_T)^3$ with $2 \leq s < q$. Choosing p such that $\int_\Omega p = 0$, there exists $\sigma, s < \sigma \leq q$ defined by

$$\sigma = \begin{cases} q & \text{if } s \geq 5, \\ \min(q, \frac{4s}{5-s}) & \text{if } 2 \leq s < 5, \end{cases}$$

such that

$$u \in L^\sigma(0, T; W^{1,\sigma}(\Omega)^3), \nabla u|_{\Sigma_T} \in L^\sigma(\Sigma_T)^9, p \in L^\sigma(Q_T).$$

Moreover we have

$$\|p\|_{L^\sigma(Q_t)} \leq c_3 (\|f\|_{L^\sigma(Q_t)}^3 + \|\nabla u\|_{L^\sigma(Q_t)}^9 + \|\nabla u\|_{L^\sigma(\Sigma_t)}^9),$$

where

$$c_3 = c [M_1^{-2}(M_2 + 1)M_3 + M_1^{-1}] M_2 (1 + M_1^{-4} M_2^2 M_3^2).$$

□

A proof of this lemma is given by M. Kabbaj [4].

□

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