Semiconvex compacta

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Abstract. We define and investigate a generalization of the notion of convex compacta. Namely, for semiconvex combination in a semiconvex compactum we allow the existence of non-trivial loops connecting a point with itself. It is proved that any semiconvex compactum contains two non-empty convex compacta, the center and the weak center. The center is the largest compactum such that semiconvex combination induces a convex structure on it. The convex structure on the weak center does not necessarily coincide with the structure induced by semiconvex combination but generates the latter in a special manner. A sufficient condition for a net of semiconvex combinations to converge to the weak center ("the law of large numbers") is established. A semiconvex compactum is called strongly semiconvex if its center and its weak center coincide. Some natural constructions of topology and functional analysis are shown to be (strongly) semiconvex compacta. It is shown that the construction of center is functorial and gives the reflector that is the left adjoint to the embedding of the category of convex compacta into the category of strongly semiconvex compacta. Also the left adjoint to the forgetful functor from the category of strongly semiconvex compacta to the category of compacta is constructed.

Keywords: convexor, convex compactum, (strongly) semiconvex compactum, left adjoint functor

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We use the following notations: $I = [0, 1]$ is a unit segment, $\Delta^n = \{(\lambda_1, \dots, \lambda_{n+1}) \in I^{n+1} \mid \lambda_1 + \dots + \lambda_{n+1} = 1\}$ an n-dimensional simplex. A compactum is a (not necessarily metrizable) compact Hausdorff topological space. A convex compactum is a compactum with a fixed convex structure induced by an affine embedding into a locally convex linear topological space. Let $\mathcal{C}omp$ and $\mathcal{C}onv$ be the categories of compacta and convex compacta resp. and let exp and P be the hyperspace and the probability measure functor resp. [5]. Denote by Cl and conv the closure and the convex hull operators.

1. Semiconvex structure

Let X be a set with ternary operation $c : X \times X \times I \to X$. We usually write $\lambda(x, y)$ instead of $c(x, y, \lambda)$. The pair (X, c) is called *a convexor* ([6]) if the following axioms hold:

(1) for all $x \in X$, $\lambda \in I : \lambda(x, x) = x$;
(2) for all $x, y \in X$, $\lambda \in I : \lambda(x, y) = (1 - \lambda)(y, x)$ (commutative law);

(3) for all $x, y, z \in X$, $\lambda, \mu, \nu \in I$, $\lambda + \mu + \nu = 1$, $\mu \neq 0$:

$$\lambda(x, \frac{\mu}{\mu+\lambda}(y, z)) = (\lambda+\mu)(\frac{\lambda}{\lambda+\mu}(x, y), x)$$

(associative law);

(4) for all $x, y \in X : 1(x, y) = x$.

In this case $\lambda(x, y)$ is called *a convex combination* of x and y. The pair (X, c) is said to be *a semiconvexor* (and respectively $\lambda(x, y)$ *a semiconvex combination*) if axioms (2)–(4) hold.

A semiconvexor (x, c) is said to be *a semiconvex compactum* if X is a compactum, the semiconvex combination is continuous and the topology on X satisfies the condition

(5) there exists a base β of the unique uniformity inducing the topology on X ([3]) such that $B \in \beta$, $(x, y), (z, t) \in B$, $\lambda \in I$ implies $(\lambda(x, z), \lambda(y, t)) \in B$.

The following axiom is equivalent to (5):

(5′) the topology on X is generated by a saturated family of pseudometrics $(d_\alpha)_{\alpha\in\mathcal{A}}$ such that $x, y, z, t \in X$, $\varepsilon > 0$, $\alpha \in \mathcal{A}$, $d_\alpha(x, y) < \varepsilon$, $d_\alpha(z, t) < \varepsilon$, $\lambda \in I$ implies $d_\alpha(\lambda(x, z), \lambda(y, t)) < \varepsilon$.

Axioms (1)–(5) (or (1)–(4), (5′)) are equivalent to the usual definition of convex compactum. Axiom (5) (or (5′)) provides local convexity.

In the sequel (X, c) is a semiconvex compactum with a fixed family of pseudometrics $(d_\alpha)_{\alpha\in\mathcal{A}}$ satisfying (5′).

The notion of semiconvex combination can be extended onto finite and countable number of elements of X. Let $(\lambda_1, \dots, \lambda_n) \in \Delta^{n-1}$ and $x_1, \dots, x_n \in X$,

$$(\lambda_1, \dots, \lambda_n)(x_1, \dots, x_n) = \begin{cases} x_1 & \text{if } \lambda_1 = 1; \\ \lambda_1(x_1, (\frac{\lambda_2}{1-\lambda_1}, \dots, \frac{\lambda_n}{1-\lambda_1})(x_2, \dots, x_n))1 & \text{if } \lambda_1 \neq 1. \end{cases}$$

If $\lambda_1, \lambda_2, \dots \in I$, $\lambda_1 + \lambda_2 + \dots = 1$, $x_1, x_2, \dots \in X$, then define $(\lambda_1, \lambda_2, \dots)(x_1, x_2, \dots) = \lim\limits_{n\to\infty}(\lambda_1, \dots, \lambda_n, 1 - \lambda_1 - \dots - \lambda_n)(x_1, \dots, x_{n+1})$.

Note that we can permute the arguments (simultaneously with the coefficients) of the semiconvex combination without changing its value.

Semiconvex combinations are equicontinuous in the following sense: if $x_1, x_1', x_2, x_2', \dots \in X$ is a finite or infinite sequence with $d_\alpha(x_1, x_1') < \varepsilon$, $d_\alpha(x_2, x_2') < \varepsilon, \dots$ for $\alpha \in \mathcal{A}$ and $\varepsilon > 0$, then for any $\lambda_1, \lambda_2, \dots \in I$ with $\lambda_1 + \lambda_2 + \dots = 1$ we have

$$d_\alpha((\lambda_1, \lambda_2, \dots)(x_1, x_2, \dots), (\lambda_1, \lambda_2, \dots)(x_1', x_2', \dots)) < \varepsilon.$$

Consider the equivalence relation on the simplex Δ^{n-1}, $n \in \mathbb{N}$, defined as follows: $(\lambda_1, \dots, \lambda_n) \sim (\mu_1, \dots, \mu_n)$ if $(\lambda_1, \dots, \lambda_n)$ and $(\mu_1, \dots, \mu_n)$ coincide up to a permutation of indices.

Let $S = \bigcup_{n=i}^{\infty} (\Delta^{n-1}/\sim)$ and denote by $[\lambda_1, \dots, \lambda_n]$ the equivalence class which contains $(\lambda_1, \dots, \lambda_n)$. Define a commutative semigroup operation on S in the following manner. Let $n, m \in \mathbb{N}$, $k : \{1, \dots, nm\} \to \{1, \dots, n\} \times \{1, \dots, m\}$ be an arbitrary bijection and $k_1 = pr_1 \circ k$, $k_2 = pr_2 \circ k$. Put

$$[\lambda_1, \dots, \lambda_n] \cdot [\mu_1, \dots, \mu_m] = [\lambda_{k_1(1)} \cdot \mu_{k_2(1)}, \dots, \lambda_{k_1(nm)} \cdot \mu_{k_2(nm)}]$$

for $[\lambda_1, \dots, \lambda_n], [\mu_1, \dots, \mu_m] \in S$. Obviously $[1]$ is the unit and thus S is a monoid.

Two partial order relations naturally arise on S:

(1) $[\mu_1, \dots, \mu_m]$ is called to be inscribed in $[\lambda_1, \dots, \lambda_n]$ ($[\lambda_1, \dots, \lambda_n] \leq [\mu_1, \dots, \mu_m]$) if there is a surjection $h : \{1, \dots, m\} \to \{1, \dots, n\}$ such that $\lambda_i = \sum_{j \in h^{-1}(i)} \mu_j$ for all $1 \leq i \leq n$;

(2) $[\lambda_1, \dots, \lambda_n]$ is called to be divisible by $[\mu_1, \dots, \mu_m]$ ($[\lambda_1, \dots, \lambda_n] \prec [\mu_1, \dots, \mu_m]$) if there is $[\nu_1, \dots, \nu_l] \in S$ such that $[\lambda_1, \dots, \lambda_n] \cdot [\nu_1, \dots, \nu_l] = [\mu_1, \dots, \mu_m]$.

Obviously, $[\lambda_1, \dots, \lambda_n] \prec [\mu_1, \dots, \mu_m]$ implies $[\lambda_1, \dots, \lambda_n] \leq [\mu_1, \dots, \mu_m]$. The set S with any of these relations becomes upward directed.

The monoid S naturally acts on X:

$$[\lambda_1, \dots, \lambda_n]x = (\lambda_1, \dots, \lambda_n)(x, \dots, x), \ x \in X,$$

and on $\exp X$:

$$[\lambda_1, \dots, \lambda_n] * A = \{(\lambda_1, \dots, \lambda_n)(a_1, \dots, a_n) \,|\, a_1, \dots, a_n \in A\}, \ A \in \exp X.$$

There is another action of S on $\exp X$ defined by the formula

$$[\lambda_1, \dots, \lambda_n]A = \{[\lambda_1, \dots, \lambda_n]a \,|\, a \in A\}, \ A \in \exp X.$$

The mappings $s(-) : X \to X$, $s \in S$, are equicontinuous, i.e. if $\alpha \in \mathcal{A}$, $\varepsilon > 0$, $x, y \in X$ and $d_\alpha(x, y) < \varepsilon$, then $d_\alpha(sx, sy) < \varepsilon$.

2. Center and weak center of semiconvex compactum. "Law of large numbers"

Call a subset *semiconvex* if it is closed with respect to the semiconvex combination. Denote $\{sa \,|\, s \in S\}$ by Sa.

Lemma 2.1. *For any $a \in X$ the set $\mathrm{Cl}(Sa)$ is the least closed semiconvex subset $A \subset X$ containing a.*

Proof: Obvious. □

Lemma 2.2. *Let $a \in X$, $A = \mathrm{Cl}(Sa)$, then $\bigcap\limits_{s\in S} sA$ is a unique minimal with respect to inclusion closed semiconvex subset $B \subset A$.*

PROOF: By Zorn lemma there exists a minimal subset $B \subset A$ satisfying the conditions of the lemma. It is easy to see that for any $s \in S$ the set sB is closed and semiconvex. Since $sB \subset B$ and B is minimal, we obtain $sB = B$. Then $B \subset A$, $B = sB \subset sA$ implies $B \subset \bigcap\limits_{s\in S} sA$ and the latter set is closed and semiconvex.

Let $\varepsilon > 0$, $\alpha \in \mathcal{A}$. Since $B \subset A$, for any $b \in B$ there exists $s' \in S$ such that $d_\alpha(b, s'a) < \varepsilon/2$. For $x \in \bigcap\limits_{s\in S} sA$ we have $z = s'y$, $y \in A$. There is $s'' \in S$ such that $d_\alpha(s''a, y) < \varepsilon/2$ and therefore $d_\alpha(s's''a, s'y) < \varepsilon/2$. As $d_\alpha(s''b, s's''a) < \varepsilon/2$, the inequality $d_\alpha(s''b, z) < \varepsilon$ holds. Thus $z \in \mathrm{Cl}(SB) = B$ and $B = \bigcap\limits_{s\in S} sA$. $\square$

We are going to investigate the set $B = \bigcap\limits_{s\in S} sA$. Due to the above lemma, for any $s \in S$ the mapping $s(-) : B \to B$ is a non-expanding surjection with respect to all pseudometrics d_α, $\alpha \in \mathcal{A}$. Since any non-expanding surjection of a metric compactum onto itself is an isometric map, for any $x, y \in B$, $s \in S$, $\alpha \in \mathcal{A}$ we have $d_\alpha(sx, sy) = d_\alpha(x, y)$. Putting $\lambda((x, y), (z, t)) = (\lambda(x, z), \lambda(y, t))$ we turn $B \times B$ into a semiconvex compactum.

For $x, y \in B$ put $(x \to y) = \mathrm{Cl}(S(x, y))$. We have $d_\alpha(z, t) = d_\alpha(x, y)$ if $(z, t) \in (x \to y)$, $x, y, z, t \in B$. Since $s(-) : B \to B$ as a surjection, $pr_1((x \to y)) = pr_2((x \to y)) = B$.

Assume that $(z_1, t_1) = s_1(x, y)$, $(z_2, t_2) = s_2(x, y)$, $d_\alpha(z_1, z_2) < \varepsilon$, $\varepsilon > 0$, $x, y, z_1, t_1, z_2, t_2 \in B$, $\alpha \in \mathcal{A}$. As B is minimal there is $s \in S$ such that $d_\alpha(sx, y) < \varepsilon$. Then $d_\alpha(t_1, t_2) = d_\alpha(s_1y, s_2y) \le d_\alpha(s_1y, s_1sx) + d_\alpha(s_1sx, s_2sx) + d_\alpha(s_2sx, s_2y) = d_\alpha(y, sx) + d_\alpha(s_1x, s_2x) + d_\alpha(sx, y) < 3\varepsilon$. Thus $(z, t_1), (z, t_2) \in (x \to y)$ implies $t_1 = t_2$. Therefore $(x \to y)$ is the graph of some isometry. The graph of the inverse isometry is $(y \to x)$.

Let $x, y_1, y_2, z, t_1, t_2 \in B$, $(z, t_1) \in (x \to y_1)$, $(z, t_2) \in (x \to y_2)$. There exists a net of the form $(s_\beta x)$, $s_\beta \in S$, converging to z. Then we have the convergence $s_\beta y_1 \longrightarrow t_1$, $s_\beta y_2 \longrightarrow t_2$. Since $d_\alpha(s_\beta y_1, s_\beta y_2) = d_\alpha(y_1, y_2)$, the equality $d_\alpha(y_1, y_2) = d_\alpha(t_1, t_2)$ holds.

Fix an arbitrary point $b \in B$. For any $x, y \in B$ there exists a unique $z \in B$ such that $(x \to y) = (b \to z)$. Thus we can properly define an operation on B by the formula: $z = z_1 z_2 \iff (z_2, z) = (b \to z_1)$. With one argument fixed, this operation becomes an isometry with respect to any pseudometric d_α, $\alpha \in \mathcal{A}$, and therefore it is continuous as a mapping $B \times B \to B$.

Assume that $z_1 = s_1 b$, $z_2 = s_2 b$. Then $(b \to z_1) = \{(x, sx) \mid x \in B\}$ and $z_1 z_2 = s_1 s_2 b$. Thus $z_1 z_2 = z_2 z_1$. As the operation is continuous, the commutative law holds for an arbitrary pair of elements in B. In the same way the associative law for the monoid S implies the associative law for the constructed operation.

The inverse for $x \in B$ is a unique $y \in B$ such that $(y, b) \in (b \to x)$. Uniqueness of the inverse and the compactness of B implies the continuity of the inversion.

Consequently B is an Abelian compact contractible topological group. It is known [7], [1] that it is trivial, i.e. B is a singleton. The point $b \in B$ is a unique in A such that $\lambda(b, b) = b$ for any $\lambda \in I$. Let $bX : X \to X$ be the map taking any point $a \in X$ into this point $b \in \mathrm{Cl}(Sa)$. Thus $bX(X)$ is a closed subset consisting of all points $b \in X$ such that $\lambda(b, b) = b$ for any $\lambda \in I$. Call it *the center* of the semiconvex compactum X and denote by $Ctr(X)$.

Theorem 1. *The net sx, $s \in (S, \prec)$ is uniformly convergent to $bX(x)$, $x \in X$ and the mapping bX is a non-expanding with respect to all d_α, $\alpha \in \mathcal{A}$ (and therefore continuous) retraction of the semiconvex compactum X onto its center $Ctr(X)$.*

PROOF: Since X is a compactum and all $s(-) : X \to X$ are non-expanding, it is sufficient to prove the pointwise convergence. Let $a \in X$, $A = \mathrm{Cl}(Sa)$. For any $s, s' \in S$, $s \prec s'$ we have $s'A \subset sA$. As $\{bX(a)\} = \bigcap_{s \in S} sA$, for any open $U \ni bX(a)$ there is $s \in S$ such that $sA \subset U$. Then $s'a \in U$ for any $s' \in S$, $s \prec s'$. □

Since axioms (1)-(5) hold for $Z = Ctr(X)$, define a map $\varphi : PZ \to Z$ as a (unique) continuous extension of the map defined on the subspace of finite measures by the formula: $\varphi(\sum_{n=1}^{n} \lambda_i \delta_{x_i}) = (\lambda_1, \dots, \lambda_n)(x_1, \dots, x_n)$ (by δ_{x_i} we denote the Dirac measure in x_i), $(\lambda_1, \dots, \lambda_n) \in \Delta^{n-1}$, $x_1, \dots, x_n \in X$. Then (Z, φ) is a $\mathbb{P}$-algebra ([9]), and due to Świrszcz, Z is a convex compactum and $c \mid Z$ is a convex combination.

By $\overline{\lim}$ the upper limit ([5]) is denoted.

Lemma 2.3. *For any subset $A \subset X$ the equality $\overline{\lim}_{s \in (S, \leq)} s * A = \overline{\lim}_{|s| \to 0} s * A$ holds.*

PROOF: Let $a \in \overline{\lim}_{s \in (S, \leq)} s * A$, Ua be a neighborhood of a, $\varepsilon > 0$. Take an arbitrary $s_0 \in S$, $|s_0| < \varepsilon$. There exists $s \in S$, $s_0 \leq s$ such that $s * A \cap Ua \neq \emptyset$. Thus $|s| < 0$ implies $a \in \overline{\lim}_{|s| \to 0} s * A$.

Assume that $a \in \overline{\lim}_{|s| \to 0} s * A$, Ua is a neighborhood of a of the form $\{x \in X \mid d_\alpha(x, a) < \varepsilon\}$, $s = [\lambda_1, \dots, \lambda_n] \in S$. Choose $\delta > 0$ such that $d_\alpha(x, \lambda(y, x)) < \varepsilon/4$ for all $x, y \in X$, $0 \leq \lambda < \delta$. There is $s' \in S$, $s' = [p_1, \dots, p_m]$, $|s'| < \delta/n$ such that the inequality $d_\alpha(a, b) < \varepsilon/2$ holds for some $b = (p_1, \dots, p_m)(a_1, \dots, a_m)$, $a_1, \dots, a_n \in A$.

Construct s'' that is inscribed in both s and s'. Put $\tilde{\lambda}_j = \lambda_1 + \dots + \lambda_j$, $1 \leq j < n$, $\tilde{p}_0^0 = 0$, $\tilde{p}_i^0 = p_1 + \dots + p_i$, $1 \leq i \leq m$ (obviously $\tilde{p}_m^0 = 1$). Assume that the segment $[\tilde{p}_i^0, \tilde{p}_{i+1}^0)$, $0 \leq i < m$, contains k_i of points $\tilde{\lambda}_j$. Denote them (in increasing order) $\tilde{p}_i^1, \dots, \tilde{p}_i^{k_i}$. We have a non-decreasing sequence

$$0 = \tilde{p}_0^0 \leq \tilde{p}_0^1 \leq \dots \leq \tilde{p}_0^{k_0} \leq \tilde{p}_1^0 \leq \dots \leq \tilde{p}_{m-1}^0 \leq \tilde{p}_{m-1}^1 \leq \dots \leq \tilde{p}_{m-1}^{k_{m-1}} \leq \tilde{p}_m^0 = 1.$$

Put for $1 \le i \le m$

$$q_i^0 = \tilde{p}_{i-1}^1 - \tilde{p}_{i-1}^0,\ q_i^1 = \tilde{p}_{i-1}^2 - \tilde{p}_{i-1}^1,\ \dots,\ q_i^{k_i-1} = \tilde{p}_{i-1}^{k_i} - \tilde{p}_{i-1}^{k_i-1},\ q_i^{k_i} = \tilde{p}_i^0 - \tilde{p}_{i-1}^{k_i}.$$

Let $c = (q_1^0, \dots, d_1^{k_0}, \dots, d_m^0, \dots, d_m^{k_{m-1}})(\underbrace{x_1, \dots, x_1}_{k_0}, \dots, \underbrace{x_m, \dots, x_m}_{k_{m-1}})$.

It is easy to see that $s'' = [q_1^0, \dots, q_m^{k_{m-1}}]$ is inscribed in s and the cardinality of the subset $\mathcal{I} \subset \{1, \dots, m\}$ of indices i such that $k_{i-1} \neq 0$ is not greater than n. Thus $\Delta p = \sum\limits_{i \in \mathcal{I}} p_i < \frac{\delta}{n} \cdot n = \delta$.

Due to the choice of δ, $d_\alpha(b,d) < \varepsilon/4$, $d_\alpha(c,d) < \varepsilon/4$, where d is the convex combination of x_i such that $k_{i-1} = 0$, with coefficients $\frac{p_i}{1-\Delta p}$. Therefore $d_\alpha(b,c) < \varepsilon/2$ and $d_\alpha(a,c) < \varepsilon$, hence $c \in Ua$. Thus $c \in s'' * A$, $s \le s''$. $\square$

Consider the largest semiconvex closed subset $A \subset X$ such that $(\lambda_1, \dots, \lambda_n) : A^n \to A$ is surjective for any $(\lambda_1, \dots, \lambda_n) \in \Delta^{n-1}$, $n \in \mathbb{N}$. It is easy to see that it coincides with $\bigcap\limits_{s \in S} s * X = \lim\limits_{(S,\le)} s * X = \lim\limits_{|s| \to 0} s * X$. Call it *the weak center* of X and denote $WCtr(X)$. Always $Ctr(X) \subset WCtr(X)$.

Lemma 2.4. *Let X be a semiconvex compactum and $(\lambda_1, \dots, \lambda_n) : X^n \to X$ be surjective for some $n \in \mathbb{N}$, $(\lambda_1, \dots, \lambda_n) \in \Delta^{n-1}$, $[\lambda_1, \dots, \lambda_n] \neq [1, 0, \dots, 0]$. Then all mappings $[1/k, \dots, 1/k] : X \to X$, $k \in \mathbb{N}$ are isometries with respect to all d_α, $\alpha \in \mathcal{A}$ and preserve semiconvex combination.*

Proof: Due to the previous lemma, all $(1/k, \dots, 1/k) : X^k \to X$, $k \in \mathbb{N}$ are surjective. Let $\alpha \in \mathcal{A}$, $\varepsilon > 0$ and $\{y_1, \dots, y_m\}$ be a $\varepsilon/2$-net with respect to d_α. There exists $\delta > 0$ such that $0 \le \lambda < \delta$, $x, y \in X$ implies $d_\alpha(x, \lambda(y,x)) < \varepsilon/2$. Take $N \in \mathbb{N}$ such that $\frac{mk}{N} < \delta$. If $x \in X$, then there is $x_1, \dots, x_N \in X$ such that $(1/N, \dots, 1/N)(x_1, \dots, x_N) = x$. For any x_i there exists y_{j_i}, $1 \le j_i \le m$, such that $d_\alpha(x_i, y_{j_i}) < \varepsilon/2$. Let the point y_1 appears l_1 times in the sequence $y_{j_1}, \dots, y_{j_N}$, the point y_2 appears l_2 times, etc. We have $l_1 + \dots + l_m = N$. Assume that $p_1, \dots, p_m \in \{0\} \cup \mathbb{N}$, $0 \le l_i - p_i k < k$, $1 \le i \le m$. Then

$$a = (\frac{1}{N}, \dots, \frac{1}{N})(\underbrace{y_1, \dots, y_1}_{l_1}, \dots, \underbrace{y_m, \dots, y_m}_{l_m})$$

fulfills $d_\alpha(x,a) < \varepsilon/2$.

Let $p = p_1 + \dots + p_m$, then $0 \le N - kp < km$. If

$$b = (\frac{1}{p}, \dots, \frac{1}{p})(\underbrace{y_1, \dots, y_1}_{p_1}, \dots, \underbrace{y_m, \dots, y_m}_{p_m}),$$

$$c = (\frac{1}{N-kp}, \dots, \frac{1}{N-kp})(\underbrace{y_1, \dots, y_1}_{l_1 - kp_1}, \dots, \underbrace{y_m, \dots, y_m}_{l_m - kp_m})$$

(if $N = kp$, then let c be arbitrary), then $a = \frac{N-kp}{N}(c, [1/k, \dots, 1/k]b)$. Since $\frac{N-kp}{N} < \delta$, we have: $d_\alpha(a, [1/k, \dots, 1/k]b) < \varepsilon/2$ and therefore $d_\alpha(x, [1/k, \dots, 1/k]b) < \varepsilon$. Thus $[1/k, \dots, 1/k]X$ is dense and closed. Consequently, it coincides with X.

The preservation of semiconvex combination is obvious. □

Let the conditions of the previous lemma hold. Denote $[\frac{1}{n}, \dots \frac{1}{n}]x$ by $\langle\frac{1}{n}\rangle x$. For any $x \in X$ and $m \in \mathbb{N}$ there exists a unique $y \in X$ such that $\langle\frac{1}{m}\rangle y = x$. Denote it by $\langle m\rangle x$. If $q = \frac{m}{n} \in \mathbb{Q}_+$, then define $\langle q\rangle x = \langle m\rangle\langle\frac{1}{n}\rangle x$ (the result does not depend on the choice of such m, n).

The action of the multiplicative group $\mathbb{Q}_+$ on X is obtained. Prove that this action is equicontinuous (with $x \in X$ as parameter) with respect to the metric on $\mathbb{Q}_+$ induced from $\mathbb{R}$.

It is sufficient to show the continuity in 1 only. Let $\varepsilon > 0$, $\alpha \in \mathcal{A}$ and let $\delta > 0$ be such that $d_\alpha(x, \lambda(y, x)) < \varepsilon$ if $x, y \in X$, $0 \leq \lambda < \delta$. If $m_1, m_2 \in \mathbb{N}$, $m_1 < m_2$, then $\langle\frac{1}{m_2}\rangle x = \frac{m_2-m_1}{m_2}(\langle\frac{1}{m_2-m_1}\rangle x, \langle\frac{1}{m_1}\rangle x)$. Thus for $0 \leq \frac{m_2-m_1}{m_2} < \delta$ we have $d_\alpha(\langle\frac{1}{m_1}\rangle x, \langle\frac{1}{m_2}\rangle x) < \varepsilon$. Since $\langle m_1\rangle$ and $\langle m_2\rangle$ are isometries with respect to d_α, for any $x \in X$ we have

$$d_\alpha(\langle m_1\rangle\langle\frac{1}{m_1}\rangle x, \langle m_1\rangle\langle\frac{1}{m_2}\rangle x) = d_\alpha(x, \langle\frac{m_1}{m_2}\rangle x) =$$
$$d_\alpha(\langle m_2\rangle\langle\frac{1}{m_1}\rangle x, \langle m_2\rangle\langle\frac{1}{m_2}\rangle x) = d_\alpha(x, \langle\frac{m_2}{m_1}\rangle x) < \varepsilon.$$

Therefore it is sufficient to choose a neighborhood U of 1 such that $\frac{m_1}{m_2} \in U$ implies $\frac{\max(m_1,m_2)-\min(m_1,m_2)}{\max(m_1,m_2)} < \delta$.

Thus an action $\langle-\rangle : \mathbb{Q}_+ \times X \to X$ can be uniquely extended to a continuous action $\langle-\rangle : \mathbb{R}_+ \times X \to X$ that is consistent with semiconvex combination.

Define a new operation $\diamond : X \times X \times I \to X$ by $\lambda \diamond (x, y) = \lambda(\langle\frac{1}{\lambda}\rangle x, \langle\frac{1}{1-\lambda}\rangle y)$, $x, y \in X$, $0 < \lambda < 1$, $1 \diamond (x, y) = x$, $0 \diamond (x, y) = y$. The continuity (and even equicontinuity in the above mentioned sense) of this operation is obvious. It is easy to check that for $(X, \diamond), (d_\alpha)_{\alpha\in\mathcal{A}}$, the axioms (1)–(4), (5′) hold. Thus $(X, \diamond)$ is a convex compactum, and $\langle-\rangle : \mathbb{R}_+ \times X \to X$ is an action that is consistent with convex combination.

The following result describes the nature of the weak center of a semiconvex compactum.

Theorem 2. *A semiconvex compactum X is the weak center of some semiconvex compactum if and only if there exists a continuous operation $\diamond : X \times X \times I \to X$ and a continuous action $\langle-\rangle : \mathbb{R}_+ \times X \to X$ of the multiplicative group $\mathbb{R}_+$ such that $(X, \diamond)$ is a convex compactum, $\langle-\rangle : \mathbb{R}_+ \times X \to X$ is an action that is consistent with convex combination and for any $x, y \in X$, $\lambda \in I$, the equality $\lambda(x, y) = \lambda \diamond (\langle\lambda\rangle x, \langle 1-\lambda\rangle y)$ holds.*

PROOF: It is easy to check that for such $(X, \diamond)$ and $\langle - \rangle : \mathbb{R}_+ \times X \to X$, the formula $\lambda(x, y) = \lambda \diamond (\langle \lambda \rangle x, \langle 1-\lambda \rangle y)$ defines properly a semiconvex combination on X, for which any $(\lambda_1, \ldots, \lambda_n) : X^n \to X$, $(\lambda_1, \ldots, \lambda_n) \in \Delta^{n-1}$, is surjective. □

A semiconvex compactum is said to be *strongly semiconvex* if $Ctr(X) = WCtr(X)$. If this holds, then any closed semiconvex subset $A \subset X$ such that any $(\lambda_1, \ldots, \lambda_n) : A^n \to A$, $(\lambda_1, \ldots, \lambda_n) \in \Delta^{n-1}$, is surjective, is contained in $Ctr(X)$. Here are equivalent axioms:

(6) the intersection $\bigcap\limits_{n \in \mathbb{N}} [1/n, \ldots, 1/n]X$ lies in $Ctr(X)$;

(6′) for any $x \in X$, $\alpha \in \mathcal{A}$ we have: $\lim\limits_{n \to \infty} d_\alpha([1/n, \ldots, 1/n]x, Ctr(X)) = 0$.

Thus for a semiconvex compactum, "averages" of elements converge to the weak center if coefficients become "small" ("law of large numbers"). If all the elements are taken equal to x and X is a strongly semiconvex compactum, then the "averages" converge uniformly to $bX(x)$: $\lim\limits_{|s| \to 0} sx \to bX(x)$.

Examples.

(a) Let a convex compactum X be affinely embedded into a locally convex topological vector space L. Then the requirements of (5) are satisfied for the base of the uniformity

$$\beta = \big\{ \{(x, y) \in X^2 \mid x - y \in U\} \mid U \text{ is a balanced convex neighbourhood of zero in } L \big\}.$$

The center of X is X itself (and therefore X is a strongly semiconvex compactum).

(b) Let $X = I$, $\lambda(x, y) = \max(\lambda x, (1-\lambda)y)$, $\lambda, x.y \in I$. The usual metric on I meets the definition. The set $\{0\}$ is the (weak) center.

(c) Let a convex compactum Y be affinely embedded into a locally convex topological vector space L, $X = \exp Y$, $\lambda(A, B) = \lambda A + (1-\lambda)B$, $\lambda \in I$,

$$\beta = \big\{ \{(A, B) \in X^2 \mid A \subset b + U \text{ for any } b \in B,\ B \subset a + U \text{ for any } a \in A\} \mid U \text{ is a balanced convex neighbourhood of zero in } L \big\}.$$

The center (and the weak center) of X is the hyperspace of convex closed subsets $\mathrm{cc}\, X \subset \exp X$.

(d) Let Y, L be as in (c), $y \in Y$ a fixed point, $X = \{A \in \mathrm{cc}\, Y \mid A \ni y\}$, $\lambda(A, B) = \mathrm{conv}((\lambda A + (1-\lambda)y) \cup (\lambda y + (1-\lambda)B))$, $A, B \in X$, $\lambda \in I$. A base that meets (5) is defined as in (c). The (weak) center is a singleton $\{\{y\}\}$ and X is also a strongly semiconvex compactum.

(e) Let Y be a semiconvex compactum with a base β_0 of the unique uniformity that satisfies (5). Put $X = \exp Y$ and $\lambda(A, B) = \{\lambda(a, b) \mid a \in A, b \in B\}$.

The suitable base is of the form

$$\beta = \big\{\{(A,B) \in X^2 \mid \text{for any } a \in A \text{ there exists } b \in B \text{ such that } d(a,b) < F;\ \text{for any } b \in B \text{ there exists } a \in A \text{ such that } d(a,b) < F\} \bigm| F \in \beta_0\big\}.$$

The center consists of all semiconvex closed subsets of the weak center. This construction preserves the class of strongly semiconvex compacta.

(f) Let Y be a semiconvex compactum with a family of pseudometrics $(\rho_\alpha)_{\alpha\in\mathcal{A}}$ that satisfies $(5')$, $X = PY$ and $\lambda \in I$, $m_1, m_2 \in X$. If $m = m_1 \otimes m_2$ is a tensor product ([5]) and $h : Y \times Y \to Y$ is defined by $h(y_1, y_2) = \lambda(y_1, y_2)$, then define $\lambda(m_1, m_2)$ as $Ph(m)$. Construct a family $(d_\alpha)_{\alpha\in\mathcal{A}}$ of pseudometrics on X by a method of Kantorovich and Rubinstein [8]

$$d_\alpha(m_1, m_2) = \inf\{m_3(\rho_\alpha) \mid m_3 \in P(Y \times Y), P\,pr_1(m_3) = m_1,\ P\,pr_2(m_3) = m_2\}.$$

If Y is a strongly semiconvex compactum, then the (weak) center is the set of Dirac measures with supports lying in the center of Y.

3. Categorial properties of strongly semiconvex compacta

Denote by $\mathcal{S}s\mathcal{C}onv$ the category of all strongly semiconvex compacta and their continuous maps preserving semiconvex combinations. We have three forgetful functors ([2]): $U : \mathcal{C}onv \to \mathcal{C}omp$, $U_s : \mathcal{C}onv \to \mathcal{S}s\mathcal{C}onv$, $U^s : \mathcal{S}s\mathcal{C}onv \to \mathcal{C}omp$. The left adjoint to the first one is known (the probability measure functor) [9] and thoroughly investigated [4]. We are going to introduce and investigate the left adjoints to the two other functors.

The construction of the center of a strongly semiconvex compactum determines a functor $Ctr : \mathcal{S}s\mathcal{C}onv \to \mathcal{C}onv$. Indeed, if $f : X \to Y$ is an arrow in $\mathcal{S}s\mathcal{C}onv$, then $f(Ctr(X)) \subset Ctr(Y)$ and $f \mid Ctr(X)$ is an affine mapping, and we can define $Ctr(f)$ as $f \mid Ctr(X)$.

Theorem 3. *The functor Ctr is the left adjoint to the embedding of the categories $U_s : \mathcal{C}onv \to \mathcal{S}s\mathcal{C}onv$, bX is a component of a natural transformation $b : \mathbf{1}_{\mathcal{S}s\mathcal{C}onv} \to U_s \circ Ctr$ that is the unit of the adjunction (i.e. $\mathcal{C}onv \subset \mathcal{S}s\mathcal{C}onv$ is a reflective subcategory and Ctr is the reflector).*

Proof: Since bX preserves semiconvex combinations, it is sufficient to prove that for any strongly semiconvex compactum X, a convex compactum Y and a map $f : X \to Y$ that preserves semiconvex combination, there exists a unique representation of the form $f = \tilde{f} \circ bX$ such that $\tilde{f} : Ctr(X) \to Y$ is an affine continuous map.

For arbitrary $x \in X$, $s \in S$ we have $f(sx) = sf(x) = f(x)$. As $\lim\limits_{|s|\to 0} sx = bX(x)$, the equality $f(x) = f \circ bX(x)$ holds. Denote $f \mid Ctr(X)$ by $\tilde{f}$, then

$f = \tilde{f} \circ bX$. Since $Ctr(X)$ is a convex compactum, the restriction of f to it is affine. Uniqueness of such an $\tilde{f}$ is a consequence of the surjectivity of bX. □

We shall construct the left adjoint to $U^s : \mathcal{S}s\mathcal{C}onv \to \mathcal{C}omp$. Let Y be a compactum, $Sc(Y) \subset P(Y \times I)$ be the subspace of all measures m satisfying the following properties:

(a) for any $a \in (0;1]$ the set $\operatorname{supp} m \cap (Y \times [a;1])$ is finite;
(b) if $a > 0$ and $(y,a) \in \operatorname{supp} m$, then $m(\{(y,a)\}) = ka$ for some $k \in \mathbb{N}$.

Obviously, $Sc(Y)$ is closed in $P(Y \times I)$. Let $h_\lambda((y,a)) = (y, \lambda a)$. Define the combination $\lambda(m_1, m_2)$ for $m_1, m_2 \in Sc(Y)$, $\lambda \in I$ by the formula $\lambda P h_\lambda(m_1) + (1-\lambda) P h_{1-\lambda}(m_2)$.

Theorem 4.

(a) *$Sc(Y)$ with the above defined operation is a strongly semiconvex compactum;*
(b) *if $f : Y_1 \to Y_2$ is an arrow in $\mathcal{C}omp$, then $P(f \times \mathbf{1}_I)(Sc(Y_1)) \subset Sc(Y_2)$, and $Sc(f) : Sc(Y_1) \to Sc(Y_2)$ defined as the restriction $P(f \times \mathbf{1}_I) \bigm| Sc(Y_1)$ is an arrow in $\mathcal{S}s\mathcal{C}onv$;*
(c) *this defines a functor $Sc : \mathcal{C}omp \to \mathcal{S}s\mathcal{C}onv$ that is the left adjoint to the forgetful functor $U^s : \mathcal{S}s\mathcal{C}onv \to \mathcal{C}omp$.*

Proof: (a) Axioms (2)-(4) are easily checked. Show that (5′) holds. Let $(\rho_\alpha)_{\alpha \in \mathcal{A}}$ be a family of pseudometrics generating the topology on Y. Putting $\tilde{\rho}_\alpha((y_1,t_1),(y_2,t_2)) = \rho_\alpha(y_1,y_2) + |t_1 - t_2|$ we obtain a respective family for $Y \times I$. Now define the pseudometric d_α on $P(Y \times I)$ by

$$d_\alpha(m_1, m_2) = \inf\{m(\tilde{\rho}_\alpha) \mid m \in P((Y \times I)^2), Ppr_1(m) = m_1,\ Ppr_2(m) = m_2\}.$$

The restrictions of these pseudometrics to $Sc(Y)$ are the required family that meets (5′).

Since for any $(\lambda_1, \ldots, \lambda_n) \in \Delta^{n-1}$, $n \in \mathbb{N}$, $m_1, \ldots, m_n \in Sc(Y)$ we have $(\lambda_1, \ldots, \lambda_n)(m_1, \ldots, m_n) \in P(Y \times [0, |(\lambda_1, \ldots, \lambda_n)|\,])$, the (weak) center of $Sc(Y)$ is $P(Y \times \{0\})$ and (6) holds.

(b) Since the functor P preserves supports, for $m \in P(Y_1 \times I)$ we have: $\operatorname{supp} P(f \times \mathbf{1}_I)(m) \cap (Y_2 \times [a;1]) = (f \times \mathbf{1}_I)(\operatorname{supp} m \cap (Y_2 \times [a;1]))$. Consequently, it preserves the subsets $Sc(-) \subset P(- \times I)$. The preservation of semiconvex combinations can be checked directly.

(c) Define an embedding $i : Y \to Sc(Y)$ as $i(y) = \delta_{(y,1)}$. If $m \in Sc(Y)$, then $m = \sum\limits_{i=1}^{N} a_i \delta(y_i, a_i) + a_0 m_0$, where $N \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, $y_i \in Y$, $m_0 \in P(Y \times \{0\})$, $a_0, a_i \in I$, $a_0 + \sum\limits_{i=1}^{N} a_i = 1$.

Let X be a strongly semiconvex compactum and $f : Y \to X$ be continuous. Since PY is a free convex compactum over Y, there exists a unique affine continuous extension $\hat{f} : PY \to Ctr(X)$ of the map $bX \circ f : Y \to Ctr(X)$. Assume that $b = \hat{f} \circ Ppr_1(m_0)$ and $\tilde{f}(m) = (a_0, a_1, a_2, \dots)(b, f(y_1), f(y_2), \dots)$. One can check that $\tilde{f} \circ i = f$ and $\tilde{f}(\lambda(m_1, m_2)) = \lambda(\tilde{f}(m_1), \tilde{f}(m_2))$.

Let us prove the continuity of $\tilde{f}$. It is sufficient to show that for any $\varepsilon > 0$ and a pseudometric d_α satisfying $(5')$, the map $\tilde{f}$ is (d_α, ε)-continuous, i.e. for any $m \in Sc(Y)$ there exists a neighborhood Om, $m \in Om$ such that $d_\alpha(\tilde{f}(m), \tilde{f}(m')) < \varepsilon$ whenever $m' \in Om$.

Recall that $h_0((y, a)) = (y, 0)$. There is an $\delta_1 > 0$ such that for all $m \in Sc(Y) \cap P(Y \times [0; \delta_1])$, the inequality $d_\alpha(\tilde{f}(m), \tilde{f}(Ph_0(m))) < \varepsilon/3$ holds. We can take any $\delta_1 > 0$ such that $|(\lambda_1, \dots, \lambda_n)| < 2\delta_1$, $(\lambda_1, \dots, \lambda_n) \in \Delta^{n-1}$, $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$ implies $d_\alpha(x, bX(x)) < \varepsilon/3$ where $x = (\lambda_1, \dots, \lambda_n)(x_1, \dots, x_n)$. Let $m = \sum\limits_{i=1}^{N} a_i \delta_{(y_i, a_i)} + a_0 m_0$ (see above), $a_i \le \delta_1$. Choose a $k \in \mathbb{N}$ such that $a_0/k \le \delta_1$. We have

$$\tilde{f}(m) = (\underbrace{\frac{a_0}{k}, \dots, \frac{a_0}{k}}_{k}, a_1, a_2, \dots)(\underbrace{b, \dots, b}_{k}, f(y_1), f(y_2), \dots),$$

because $b = \hat{f} \circ Ppr_1(m_0)$ belongs to the convex compactum $Ctr(X)$.

Since $|(\underbrace{\frac{a_0}{k}, \dots, \frac{a_0}{k}}_{k}, a_1, a_2, \dots)| \le \delta_1 < 2\delta_1$, we have $d_\alpha(\tilde{f}(m), bX \circ \tilde{f}(m)) < \varepsilon/3$. We have

$$bX \circ \tilde{f}(m) = (\underbrace{\frac{a_0}{k}, \dots, \frac{a_0}{k}}_{k}, a_1, a_2, \dots)(\underbrace{b, \dots, b}_{k}, bX \circ f(y_1), bX \circ f(y_2), \dots) =$$

$$(\underbrace{\frac{a_0}{k}, \dots, \frac{a_0}{k}}_{k}, a_1, a_2, \dots)(\underbrace{b, \dots, b}_{k}, \tilde{f}(\delta_{(y_1,0)}), \tilde{f}(\delta_{(y_2,0)}), \dots) =$$

$$\tilde{f}(\sum_{i=1}^{N} a_i \delta_{(y_i,0)} + a_0 m_0) = \tilde{f}(Ph_0(m)),$$

and $d_\alpha(\tilde{f}(m), \tilde{f}(Ph_0(m))) < \varepsilon/3$.

There exists $\delta_2 > 0$ such that for all $x, y \in X$, $0 \le \lambda \le \delta_2$, the inequality $d_\alpha(x, \lambda(y, x)) < \varepsilon/3$ holds. Take a natural $M > \frac{1}{\delta_1 \delta_2}$ and define a map $G : \Delta^{M-1} \times Y^M \times Sc(Y) \times I \to Sc(Y)$ by the formula

$$G((\lambda_1, \dots, \lambda_M), (y_1, \dots, y_M), m_0, \lambda) = \lambda(\sum_{i=1}^{M} \lambda_i \delta_{(y_i, \lambda_i)}, m_0).$$

Let

$$B_1 = \Delta^{M-1} \times Y^M \times Sc(Y) \times [1 - \delta_2, 1],$$
$$B_2 = \Delta^{M-1} \times Y^M \times Sc(Y) \cap P(Y \times [0, \delta_1]) \times I.$$

The map G is a surjection as well as the restriction of G to $B_1 \cup B_2$.

Since G is a continuous map of compacta, for proving (d_α, ε)-continuity of $\tilde{f}$ it is sufficient to show the (d_α, ε)-continuity of $\tilde{f} \circ G \bigm| B_1$ and $\tilde{f} \circ G \bigm| B_2$.

We have

$$\tilde{f} \circ G((\lambda_1, \ldots, \lambda_M), (y_1, \ldots, y_M), m_0, \lambda) = \\ \lambda((\lambda_1, \ldots, \lambda_M)(f(y_1), \ldots, f(y_M)), \tilde{f}(m_0)).$$

If $\lambda \in [1 - \delta_2]$, then for any $(\lambda_1, \ldots, \lambda_M) \in \Delta^{M-1}$, $y_1, \ldots, y_M \in Y$, $m_0 \in Sc(Y)$ the following holds

$$d_\alpha(\tilde{f} \circ G((\lambda_1, \ldots, \lambda_M), (y_1, \ldots, y_M), m_0, \lambda), \\ (\lambda_1, \ldots, \lambda_M)(f(y_1), \ldots, f(y_M))) < \varepsilon/3.$$

Since the value of $(\lambda_1, \ldots, \lambda_M)(f(y_1), \ldots, f(y_M))$ is a continuous function of λ_i, y_i, there exist neighborhoods $O(\lambda_1, \ldots, \lambda_M) \ni (\lambda_1, \ldots, \lambda_M)$, $Oy_1 \ni y_1$, … , $Oy_M \ni y_M$ such that

$$d_\alpha((\lambda_1, \ldots, \lambda_M)(f(y_1), \ldots, f(y_M)), (\lambda'_1, \ldots, \lambda'_M)(f(y'_1), \ldots, f(y'_M))) < \varepsilon/3$$

whenever $(\lambda'_1, \ldots, \lambda'_M) \in O(\lambda_1, \ldots, \lambda_M)$, $y_1 \in Oy_1$, … , $y_M \in Oy_M$. Taking into consideration

$$d_\alpha(\tilde{f} \circ G((\lambda'_1, \ldots, \lambda'_M), (y'_1, \ldots, y'_M), m'_0, \lambda'), \\ (\lambda'_1, \ldots, \lambda'_M)(f(y'_1), \ldots, f(y'_M))) < \varepsilon/3$$

for any $m'_0 \in Sc(Y)$, $\lambda' \in [1 - \delta_2, 1]$, we obtain

$$d_\alpha(\tilde{f} \circ G((\lambda'_1, \ldots, \lambda'_M), (y'_1, \ldots, y'_M), m'_0, \lambda'), \\ \tilde{f} \circ G((\lambda_1, \ldots, \lambda_M), (y_1, \ldots, y_M), m_0, \lambda)) < \varepsilon$$

for $(\lambda'_1, \ldots, \lambda'_M) \in O(\lambda_1, \ldots, \lambda_M)$, $y_1 \in Oy_1$, … , $y_M \in Oy_M$, $m'_0 \in Sc(Y)$, $\lambda' \in [1 - \delta_2, 1]$. Thus $\tilde{f} \circ G \bigm| B_1$ is (d_α, ε)-continuous.

Let $m_0 \in Sc(Y) \cap P(Y \times [0, \delta_1])$. Then by the property of δ_1, we have

$$d_\alpha(\tilde{f} \circ G((\lambda_1, \ldots, \lambda_M), (y_1, \ldots, y_M), m_0, \lambda), \\ \tilde{f} \circ G((\lambda_1, \ldots, \lambda_M), (y_1, \ldots, y_M), Ph_0(m_0), \lambda)) < \varepsilon/3$$

for arbitrary $(\lambda_1, \ldots, \lambda_M) \in \Delta^{M-1}$, $y_1, \ldots, y_M \in Y$, $\lambda \in I$. Since

$$\tilde{f} \circ G((\lambda_1, \ldots, \lambda_M), (y_1, \ldots, y_M), Ph_0(m_0), \lambda) = \\ \lambda((\lambda_1, \ldots, \lambda_m)(f(y_1), \ldots, f(y_m)), \hat{f} \circ Ppr_1(m_0))$$

depends continuously on λ, λ_i, y_i, m_0, there exist neighborhoods $O(\lambda_1,\dots,\lambda_M)\ni(\lambda_1,\dots,\lambda_M)$, $Oy_1\ni y_1$, …, $Oy_M\ni y_M$, $O\lambda\ni\lambda$, $Om_0\ni m_0$, $Om_0\subset P(Y\times[0,\delta_1])$ such that $(\lambda'_1,\dots,\lambda'_M)\in O(\lambda_1,\dots,\lambda_M)$, $y_1\in Oy_1$, …, $y_M\in Oy_M$, $\lambda'\in O\lambda$, $m'_0\in Om_0$ implies

$$d_\alpha(\tilde f\circ G((\lambda'_1,\dots,\lambda'_M),(y'_1,\dots,y'_M),Ph_0(m'_0),\lambda'),$$
$$\tilde f\circ G((\lambda_1,\dots,\lambda_M),(y_1,\dots,y_M),Ph_0(m_0),\lambda))<\varepsilon/3.$$

As a consequence we get

$$d_\alpha(\tilde f\circ G((\lambda'_1,\dots,\lambda'_M),(y'_1,\dots,y'_M),m'_0,\lambda'),$$
$$\tilde f\circ G((\lambda_1,\dots,\lambda_M),(y_1,\dots,y_M),m_0,\lambda))<\varepsilon.$$

Thus $\tilde f\circ G\mid B_1$ and therefore also $\tilde f$, is (d_α,ε)-continuous. This completes the proof of continuity.

Let us prove that $\tilde f$ is a unique continuous extension of f onto $Sc(Y)$ that preserves semiconvex combinations. It is sufficient to prove that for any continuous extension $\tilde g$ of f onto $Sc(Y)$ that preserves semiconvex combinations, and $m\in P(Y\times\{0\})\subset Sc(Y)$ we have $\tilde g(m)=\hat f\circ Ppr_1(m)$. It is known ([5]) that finite measures form a dense subset in the space of probability measures. Thus there is a net $(\sum\limits_{i=1}^{k_\beta}a_i^\beta\delta_{y_i^\beta})_{\beta\in\mathcal B}$ that converges to $Ppr_1(m)$. Without loss of generality we can assume that $a_i^\beta>0$, $\max\limits_{1\le i\le k_\beta}a_i^\beta\underset{\beta}{\to}0$. Then $(\sum\limits_{i=1}^{k_\beta}a_i^\beta\delta_{(y_i^\beta,a_i^\beta)})\underset{\beta}{\to}m$ in $Sc(Y)$, and by preservation of semiconvex combinations we have

$$\tilde g(\sum_{i=1}^{k_\beta}a_i^\beta\delta_{(y_i^\beta,a_i^\beta)})=(a_1^\beta,\dots,a_{k_\beta}^\beta)(f(y_1^\beta),\dots,f(y_{k_\beta}^\beta))\underset{\beta}{\to}\tilde g(m).$$

Since $|(a_1^\beta,\dots,a_{k_\beta}^\beta)|\underset{\beta}{\to}0$, we have

$$\lim_{\beta\in\mathcal B}(a_1^\beta,\dots,a_{k_\beta}^\beta)(f(y_1^\beta),\dots,f(y_{k_\beta}^\beta))=\lim_{\beta\in\mathcal B}bX((a_1^\beta,\dots,a_{k_\beta}^\beta)(f(y_1^\beta),\dots,f(y_{k_\beta}^\beta)))=$$
$$\lim_{\beta\in\mathcal B}(a_1^\beta,\dots,a_{k_\beta}^\beta)(bX\circ f(y_1^\beta),\dots,bX\circ f(y_{k_\beta}^\beta))=\lim_{\beta\in\mathcal B}\hat f(\sum_{i=1}^{k_\beta}a_i^\beta\delta_{y_i^\beta})=$$
$$\hat f(\lim_{\beta\in\mathcal B}(\sum_{i=1}^{k_\beta}a_i^\beta\delta_{y_i^\beta}))=\hat f\circ Ppr_1(m).$$

Thus $Sc(Y)$ is a free strongly semiconvex compactum over the compactum Y. The isomorphism $\mathcal{S}s\mathcal{C}onv(Sc(-),-)\cong\mathcal{C}omp(-,U^s(-),)$ is checked by a direct substitution. Thus Sc is the left adjoint. $\square$

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