Tightness and resolvability

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Abstract. We prove resolvability and maximal resolvability of topological spaces having countable tightness with some additional properties. For this purpose, we introduce some new versions of countable tightness. We also construct a couple of examples of irresolvable spaces.

Keywords: resolvability, maximal resolvability, countable fan-tightness, weakly Fréchet-Urysohn property, empty interior tightness, disjoint tightness

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0. Introduction

The investigation on various aspects of resolvability of topological spaces has been carried out very intensively in the last years.

Recall that in 1943, E. Hewitt [5] called a space resolvable whenever it has two disjoint dense subsets, and irresolvable otherwise. Moreover, a space X is called κ-resolvable, where κ is a cardinal, if X contains κ many disjoint dense subsets and a space is called maximally resolvable if it is $\Delta(X)$-resolvable, where $\Delta(X) = \min\{|V| : V$ is a nonempty open subset of $X\}$. The cardinal $\Delta(X)$ is called the dispersion character of X. For a recent survey on resolvable spaces see [3].

In this note we explore some new relationships between certain variations of the classical notion of tightness and the resolvability of a topological space. Fundamental is the notion of empty interior tightness discussed in Section 1.

Our attention will be primarily concentrated on spaces with countable tightness. The reason for this is in the following theorem of E. G. Pytke′ev.

0.1 Proposition [7]. *A space with countable tightness and uncountable dispersion character is maximally resolvable.*

We will use Pytke′ev's theorem via the following:

0.2 Corollary. *The resolvability or maximal resolvability of a space with countable tightness depends on its countable open subsets.*

Proof: If X is a space with countable tightness then fix a maximal disjoint system $\mathcal{V}$ of nonempty open subsets such that $\Delta(V) = |V|$ for each $V \in \mathcal{V}$. Every

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$V \in \mathcal{V}$ for which $\Delta(V) > \aleph_0$ is maximally resolvable by Proposition 0.1 and therefore it is clear that the final result depends of the behaviour of the countable open subsets of X. $\square$

The starting point of the paper is the observation that resolvable spaces have empty interior tightness and one of the main goal here is to get as much of a converse of this as possible.

In Section 2 we provide some examples to clarify the relationship between resolvability and another quite natural variation of tightness.

Finally in the last section we will exhibit connections with 'older' forms of tightness.

Henceforth all spaces are assumed to be T_1.

Of course, when dealing with resolvability, the spaces under consideration are assumed without isolated points.

1. Characterizing resolvability

As announced in the introduction, a major role in the investigation on resolvability is played by the following:

Definition. A point x of a space X has *empty interior* tightness if whenever $x \in \overline{A}$ there exists a $B \subset A$ such that $x \in \overline{B}$ and B has *empty interior* in X.

The essence of the relevance of the above notion is in the following quite natural observation.

1.1 Theorem. *A resolvable space has empty interior tightness.*

Proof: If the space X is resolvable then let A and B be a partition of X into two disjoint dense subsets. Both two subsets have empty interior and if $x \in \overline{C}$ then $x \in \overline{(C \cap A)}$ or $x \in \overline{(C \cap B)}$ and both subsets $C \cap A$, and $C \cap B$ have empty interior. $\square$

It is remarkable that, at least in the countable case, the previous theorem has a converse.

1.2 Theorem. *A countable space with empty interior tightness is resolvable.*

Proof: We use Hewitt's criterion for irresolvability: *A space is irresolvable iff there exists a nonempty open set in which every dense in itself subset is irresolvable.*

So, assume by contradiction that the considered space is irresolvable and fix in it a nonempty open subset X in which every dense in itself subspace is irresolvable.

Let $X = \{x_n : n \in \omega\}$. As $x_0 \in \overline{X \setminus \{x_0\}}$, we have $x_0 \in \overline{N_0}$ for some set with empty interior $N_0 \subset (X \setminus \{x_0\})$. Even the set $N_0 \cup \{x_0\}$ has empty interior and so $x_0 \in \overline{X \setminus (N_0 \cup \{x_0\})}$. Hence, $x_0 \in \overline{M_0}$ for some set with empty interior $M_0 \subset (X \setminus (N_0 \cup \{x_0\}))$. If $V = \mathrm{Int}(N_0 \cup M_0)$ were not empty, then V would be resolvable because $N_0 \cap V$ and $M_0 \cap V$ are two disjoint dense subsets in V. This contradicts our choice of X and therefore $N_0 \cup M_0$ must have empty interior.

Since even the set $C = N_0 \cup M_0 \cup \{x_1\}$ has empty interior, we have $x_1 \in \overline{X \setminus C}$ and $x_1 \in \overline{N_1}$ for some set with empty interior $N_1 \subset X \setminus C$. Arguing as before, we see that $N_0 \cup M_0 \cup N_1$ has empty interior and so we may find a set with empty interior $M_1 \subset X \setminus (N_0 \cup M_0 \cup N_1 \cup \{x_1\})$ such that $x_1 \in \overline{M_1}$. It is now clear that this process can be continued for every integer i. If at the end we put $A = \cup\{N_i : i \in \omega\}$ and $B = \cup\{M_i : i \in \omega\}$ then we obtain two disjoint dense subsets of X. This is a contradiction and so the result follows. □

Theorem 1.2 together with Proposition 0.2 give:

1.3 Theorem. *A space with countable empty interior tightness is resolvable.*

1.4 *Remark.* As Van Douwen's countable resolvable but not 3-resolvable space from [4] shows, Theorem 1.2 is optimal.

We can get $\aleph_0$-resolvability by a strengthening of the notion of empty interior tightness.

Definition. A point x of a space X has *nowhere dense* tightness if whenever $x \in \overline{A}$ there exists a $B \subset A$ such that $x \in \overline{B}$ and B is *nowhere dense* in X.

1.5 Theorem. *A space with countable nowhere dense tightness is maximally resolvable.*

Proof: Observe first that, being the notion of nowhere dense set hereditary with respect to dense subspaces, every dense subspace of a space with nowhere dense tightness has still nowhere dense tightness and so even empty interior tightness. Furthermore, by applying Proposition 0.2, it is enough to show that a countable space X with nowhere dense tightness is $\aleph_0$-resolvable.

By Theorem 1.3, the space X can be divided into two disjoint dense subspaces A_0 and A'_0. Since even the space A'_0 has countable empty interior tightness, applying again Theorem 1.3, we can divide A'_0 into disjoint dense subspaces A_1 and A'_1. Continuing this process, we get a family $\{A_i : i \in \omega\}$ of pairwise disjoint dense subsets of X. □

Of course, the material presented in this section leaves open the question whether the empty interior tightness can fully characterize resolvability (here we have given a positive answer for spaces with countable tightness).

2. Disjoint tightness and resolvability

A resolvable space could be described as a space splittable into two disjoint parts "arbitrarily near" to each point. Taking this picture in mind, it seems reasonable that even the following variation of tightness might serve to guarantee the resolvability of a space.

Definition. A point x of a space X has *disjoint* tightness if whenever $x \in \overline{A}$ there exist *two disjoint* subsets $B_1, B_2 \subset A$ such that $x \in \overline{B_1}$ and $x \in \overline{B_2}$.

In spite to what we imagined at first glance, it turned out that the conjecture that a space with disjoint tightness is resolvable is not true.

2.1 Lemma. *If (X, τ) is a dense in itself Hausdorff space then there exists a finer topology ρ such that (X, ρ) is dense in itself, every set with empty interior of (X, ρ) is closed and (X, τ) and (X, ρ) have the same regular closed sets. In particular, (X, ρ) is irresolvable.*

PROOF: Let $\{N_\alpha : \alpha < \delta\}$ be an enumeration of all subsets of X. Define a chain of topologies $\{\tau_\alpha : \alpha < \delta\}$ on the set X by setting $\tau_{-1} = \tau$, $\tau_{\alpha+1}$ is the enlargement of τ_α obtained by declaring the set N_α closed if it has empty interior with respect to τ_α or $\tau_{\alpha+1} = \tau_\alpha$ otherwise and $\tau_\alpha = \sup\{\tau_\beta : \beta < \alpha\}$ if α is a limit ordinal. Then let $\rho = \tau_\delta$. It is clear that in (X, ρ) every set with empty interior is closed. We need only to check that (X, τ) and (X, ρ) have the same regular closed sets.

We proceed by induction. If $\alpha = \beta + 1$ and N_β has empty interior then any element of τ_α has the form $V \setminus N_\beta$ for some $V \in \tau_\beta$. So the τ_α-closure of any element of τ_β coincides with the τ_β-closure. On the other hand, $V \setminus N_\beta$ is dense in V with respect to both topologies and consequently we have $\overline{V \setminus N_\beta}^{\tau_\alpha} \subset \overline{V \setminus N_\beta}^{\tau_\beta} = \overline{V}^{\tau_\beta} = \overline{V}^{\tau_\alpha} = \overline{V \setminus N_\beta}^{\tau_\alpha}$. Next, assume α to be a limit ordinal. Since $\overline{A}^{\tau_\alpha} = \cap\{\overline{A}^{\tau_\beta} : \beta < \alpha\}$, by the inductive hypothesis we have, for any $V \in \tau_\beta$, $\overline{V}^{\tau_\alpha} = \overline{V}^{\tau_\beta} = \overline{W}^{\tau}$ for some $W \in \tau$. Now, let $V \in \tau_\alpha$ and write it as $\bigcup\{V_\beta : V_\beta \in \tau_\beta \text{ and } \beta < \alpha\}$ and find $W_\beta \in \tau$ for V_β as before. Then the set $W = \cup\{W_\beta : \beta < \alpha\}$ is dense in $\overline{V}^{\tau_\alpha}$ and so $\overline{V}^{\tau_\alpha} = \overline{W}^{\tau}$. □

2.2 Example. *A countable Hausdorff irresolvable space with disjoint tightness.*

Construction: Apply Lemma 2.1, starting with the space (X, τ) of rational numbers with the usual topology. The space (X, ρ) is irresolvable and we need only to check that it has disjoint tightness. Observe that if $x \in \overline{A}$ and $x \notin A$ then we must have $x \in \overline{\operatorname{Int}_\rho A}^{\rho}$, otherwise $A \setminus \overline{\operatorname{Int}_\rho A}^{\rho}$ would be a non closed set with empty interior. But (X, ρ) has the same regular closed sets of (X, τ) and so there exists a set $V \subset A$ such that $V \in \tau$ and $x \in \overline{V}^{\tau}$. In the (X, τ) it is easy to split V into two disjoint open set V_1, V_2 such that $x \in \overline{V_i}^{\tau}$ and therefore also $x \in \overline{V_i}^{\rho}$ for $i = 1, 2$. □

With the help of CH, we can now construct a regular example.

2.3 Lemma. *Let α be a countable infinite ordinal and X be a countable dense subspace of D^α (D is the two-point discrete space $\{0, 1\}$). If N is a subset with empty interior of X then there exists a continuation X' of X on $\alpha + \omega$ such that N' is closed in X' and X' is still dense in $D^{\alpha+\omega}$.*

PROOF: Let ν be the topology on $X \subset D^\alpha$. Let $M = X \setminus N = \{m_i : i \in \omega\}$. M is dense in X. We can easily find a system $\mathcal{M} = \{M_i : i \in \omega\}$ of subsets of M such that $m_i \in M_i$ and both M_{i+1} and $M \setminus M_{i+1}$ for $i \in \omega$ are dense in the topology ν_i, generated by the family $\nu \cup \{M_k, M \setminus M_k : 0 \leq k \leq i\}$ ($\nu_{-1} = \nu$). Now for every $x \in X$ we set $x(\alpha + i) = 1$ if and only if $x \in M_i$. In this manner, we have continued X on the ordinal $\alpha + \omega$ and in the resulting $X' \subset D^{\alpha+\omega}$ the subset N' is closed, because $x \in N$ if and only if x takes the value 0 on $[\alpha, \alpha + \omega[$.

Finally, observe that the independentness of the family $\mathcal{M}$ guarantees that every nonempty clopen subset of $D^{\alpha+\omega}$ meets X' in an infinite set. Thus X' is dense in $D^{\alpha+\omega}$. □

2.4 Example [CH]**.** *There exists a countable dense irresolvable subspace of* $D^{\mathfrak{c}}$ *with disjoint tightness. Moreover, in this space each subset with empty interior is closed.*

Construction: Let X be any countable dense subset of the Cantor set D^{ω} and let $\{N_\alpha : \alpha < \omega_1\}$ be an enumeration of all subsets of X with empty interior. We use Lemma 2.3 to continue X to a set $X' \subset D^{\omega_1}$ by a transfinite induction in the following way: if α is a limit ordinal then simply take the union of all the continuations of X previously defined, if $\alpha = \beta + 1$ and $Y \subset D^{\omega\beta}$ is the continuation of X defined at step β then apply Lemma 2.3 taking as N the continuation of N_α on $\omega\beta$ (notice that this $N \subset Y$ has still empty interior in Y).

Let us denote X with the topology of D^{α}, where $\alpha \leq \omega_1$, by X_α. So, $X' = X_{\omega_1}$. If a subset $N' \subset X'$ has empty interior then the corresponding N has also empty interior in every X_α for every $\alpha \leq \omega_1$ and therefore it is closed in X'. This in particular gives the irresolvability of X'.

To check that X' has disjoint tightness, it is essential the "speciality" of the construction made in Lemma 2.3. Let $x' \in \overline{A}^{X'}$. Since every set with empty interior in X' is closed, we must have $x' \in \overline{V}^{X'}$ for some open set $V \subset A$. Then, we may find some countable ordinal α and an open $W \subset X_\alpha$ such that $W' \subset V$ and $\overline{W'}^{X'} \supset V$ (because our X' has countable Suslin's number). But the topology of D^{α} allows us to find two disjoint open sets $W_1, W_2 \subset W$ such that $x \in \overline{W_1}^{X\alpha}$ and $x \in \overline{W_2}^{X\alpha}$. Finally, it is easy to prove that we have $x' \in \overline{W_1'}^{X'}$ and $x' \in \overline{W_2'}^{X'}$. This obviously shows that X' has disjoint tightness. □

The space $N_p = \omega \cup \{p\}$, where p is a point of ω^*, clearly does not have disjoint tightness. Thus, by multiplying the space N_p above with the space of rational numbers, we obtain a space without isolated points with nowhere dense tightness and empty interior tightness, but without disjoint tightness.

3. Older forms of tightness and resolvability

In this section we compare the new notions of tightness discussed in the paper with some older forms of tightness which are relevant when dealing with resolvability.

Definition. A point x of a space X has *discrete* tightness if whenever $x \in \overline{A}$ there exists a *discrete* set $B \subset A$ such that $x \in \overline{B}$.

To understand the role of this notion for the theory of resolvability, see for example [8].

A much older notion is that of countable fan-tightness, introduced in a natural manner by A. V. Arhangel'skiĭ ([1]) during his investigation on the topological properties of a function space in the topology of pointwise convergence. In fact, the main result of [1] says that the space $C_p(X)$ has countable fan-tightness if and only if X^n is a Hurewicz space for any integer n.

Definition. A space X has *countable fan-tightness* if for any countable family $\{A_n : n \in \omega\}$ of subsets of X satisfying $x \in \cap_{n\in\omega}\overline{A_n}$ it is possible to select finite sets $K_n \subset A_n$ in such a way that $x \in \overline{\cup_{n\in\omega} K_n}$.

Finally, another relevant form of countable tightness was recently introduced by E.G. Reznichenko.

Definition. A point $x \in X$ is weakly Fréchet-Urysohn if and only if whenever $x \in \overline{A} \subset X$ there exists a countable disjoint family $\mathcal{F}$ of finite subsets of A such that for every neighbourhood V of x the subfamily $\{F \in \mathcal{F} : F \cap V = \emptyset\}$ is finite.

A space whose points are all weakly Fréchet-Urysohn is said to be a wFU-space.

In [6] the authors gave the following reformulation of Reznichenko's definition:

A point $x \in X$ is a wFU-point if whenever $x \in \overline{A}$ there exists a countable disjoint family $\mathcal{F}$ of finite subsets of A such that $x \in \overline{\cup\mathcal{F}'}$ for every infinite subfamily $\mathcal{F}' \subset \mathcal{F}$.

3.1 Proposition. *If a dense in itself Hausdorff space has countable fan-tightness then it has countable discrete tightness.*

PROOF: Of course, it is enough to consider the case of a countable space X. Let $A \subset X$ and $x \in \overline{A}$. As X is countable and Hausdorff, there is a countable decreasing family of closed neighbourhoods $\{V_n : n \in \omega\}$ of x such that the intersection of all of them is $\{x\}$. As X has countable fan-tightness, there is a family $\{F_n : n \in \omega\}$ of finite subsets such that $F_n \subset (V_n \cap A) \setminus \{x\}$ and $x \in \overline{\cup\{F_n : n \in \omega\}}$. It is easy to realize that the fact that the set $\cup\{F_n : n \in \omega\} \setminus V_n$ is finite for every n implies that the set $\cup\{F_n : n \in \omega\}$ is discrete. □

Taking into account the previous proposition and Theorem 1.5, we immediately have a proof of the following fact:

3.2 Theorem. *If X^n is a Hurewicz space for every integer n, in particular X is σ-compact, then every dense in itself subspace of $C_p(X)$ is maximally resolvable.*

It is known [2] that a regular countably compact space with countable tightness has countable fan-tightness. So we have

3.3 Corollary. *Every dense in itself subspace of a regular countably compact space with countable tightness is maximally resolvable.*

Notice that, although it is known that any regular countably compact space is $\aleph_0$-resolvable, it is still unknown if every regular countably compact space is maximally resolvable ([3, Problem 8.10]).

A well-known still open problem concerning resolvable spaces, which we may attribute to W.W. Comfort (see [3, Problem 8.12]), asks whether a Tychonoff pseudocompact space is resolvable in ZFC.

In this case we cannot argue as in Corollary 3.3 because it is unknown (see [2]) whether every Tychonoff pseudocompact space with countable tightness has also countable fan-tightness. However, taking into account Proposition 0.1 and the fact that every nonempty open subset of a pseudocompact Tychonoff space without isolated points is uncountable, we have:

3.4 Proposition. *Every pseudocompact Tychonoff space with countable tightness is maximally resolvable.*

For σ-compact spaces a result similar to Proposition 3.2 (but without maximality) was previously obtained by Reznichenko, who proved the following two things:

3.5 Proposition. *$C_p(X)$ over a σ-compact space X is a wFU-space.*

3.6 Proposition. *A wFU-space is $\aleph_0$-resolvable.*

3.7 Proposition. *If x is a wFU-point of the space X and $x \in \overline{A}$ then there exists an uncountable almost disjoint family $\mathcal{E}$ of subsets of A such that $x \in \overline{E}$ for each $E \in \mathcal{E}$.*

Proof: According to the definition, if $x \in \overline{A} \subset X$ then there exists a countable disjoint family $\mathcal{F}$ of finite subsets of A such that for every neighbourhood V of x the subfamily $\{F \in \mathcal{F} : F \cap V = \emptyset\}$ is finite. Let $\mathcal{Q}$ be an uncountable almost disjoint family of infinite subsets of $\mathcal{F}$. Then $\mathcal{E} = \{\cup Q : Q \in \mathcal{Q}\}$ is the desired family. □

3.8 Theorem. *A wFU-point has countable empty interior tightness.*

Proof: Let x be a wFU-point in the space X and let $x \in \overline{A} \subset X$. Since a wFU-point has countable tightness, we may assume that A is countable. By Proposition 3.7, there exists an uncountable almost disjoint family $\mathcal{E}$ of subsets of A such that $x \in \overline{E}$ for each $E \in \mathcal{E}$. Now, the family $\{\mathrm{Int}(E) : E \in \mathcal{E}\}$ consists of pairwise disjoint subsets of A and therefore there exists some $E \in \mathcal{E}$ with empty interior. □

The previous theorem provides an alternative proof that a wFU- space is maximally resolvable.

We can distinguish countable nowhere dense tightness and wFU-property. Indeed, in [6] the authors constructed a countable Hausdorff wFU-space in which each nowhere dense subset is closed (and discrete). So, this space has not nowhere dense tightness. On the other hand, the space $N_p = \omega \cup \{p\}$ is not a wFU-space, as it was proved in [6], and therefore multiplying it with the space of rational numbers we get a space with countable nowhere dense tightness which is not a wFU-space.

Moreover, it is clear that the wFU-property is stronger than the property to have countable disjoint tightness. Examples 2.2 and 2.4 show that a countable space can have disjoint tightness, but not the weak Fréchet-Urysohn property.

3.9 Problem. *Find in ZFC a countable regular irresolvable space with disjoint tightness.*

Finally, we would like to draw the reader's attention to the problem whether every countable topological group with disjoint tightness is resolvable.

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