A note on Schroeder-Bernstein Property and Primary Property of Orlicz function spaces

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Abstract. It is shown in the note that every reflexive Orlicz function space has the Schroeder-Bernstein Property and the Primary Property.

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Let $G = [0,1]$ and μ be the Lebesgue measure on G. We denote by $M : (-\infty, +\infty) \to [0, +\infty)$ a continuous, convex and even function satisfying $M(u) = 0$ iff $u = 0$ and $M(u)/u \to 0(+\infty)$ as $u \to 0(+\infty)$; by $N(v)$ the complementary function of $M(u)$, i.e., $N(v) := \max_u\{uv - M(u)\}$. We say $M \in \Delta_2$ if for any $u_0 > 0$ there exists $K > 2$ such that $M(2u) \le KM(u)$, $u \ge u_0$. For every μ-measurable function $f : G \to (-\infty, +\infty)$, let $\varrho_M(f) = \int_G M(f(t))\, d\mu$; then the Orlicz space

$$L_M = \{f : \varrho_M(af) < +\infty \ \text{ for some } \ a > 0\}$$

endowed with the Luxemburg norm

$$\|f\| = \inf\{r : \varrho_M(f/r) \le 1\}$$

or Orlicz norm

$$\|f\|_M = \min_{k>0} [1 + \varrho_M(kf)]/k$$

is a Banach space and $\|f\|_M \le \|f\| \le 2\|f\|_M$ for every $f \in L_M$. More details about Orlicz spaces can be found in [2] and [4].

Let Y be a closed subspace of a Banach space X. Y is called a complemented subspace of X if there exists a linear, continuous and surjective projection from X to Y. A Banach space X is said to have the *Schroeder-Bernstein Property* (*SBP*) if for any Banach space Y, X is isomorphic to Y whenever X is isomorphic to a complemented subspace of X. A Banach space X is said to have the *Primary Property* if, for every linear, bounded projection P of X, X is isomorphic to PX or $(I - P)X$. Many spaces, for example, L^p $(1 < p < +\infty)$ and James space J, have *SBP* and *Primary Property* (see [1]).

Without loss of generality, let $M(1) = 1$. Then $(L_M, \|\cdot\|)$ is an r.i. (i.e. rearrangement invariant) function space. More details about this space can be found in [3]. By Proposition 2.b.5 in [3], the Boyd indices for L_M are

$$p_{L_M} = \sup\Big\{p : \inf_{\lambda,t\geq 1} \frac{M(t\lambda)}{M(\lambda)t^p} > 0\Big\}$$

and

$$q_{L_M} = \inf\Big\{q : \sup_{\lambda,t\geq 1} \frac{M(t\lambda)}{M(\lambda)t^p} < +\infty\Big\}.$$

In general, $1 \leq p_{L_M} \leq q_{L_M} \leq +\infty$.

Theorem 1. *For the Orlicz space $(L_M, \|\cdot\|)$, we have*

(1) $q_{L_M} < +\infty$ *if and only if* $M \in \Delta_2$*;*
(2) $p_{L_M} > 1$ *if and only if* $N \in \Delta_2$*.*

PROOF: (1) Necessity. If $q_{L_M} < +\infty$, then there exist constants $K > 1$ and $q_0 \geq 1$ such that $M(t\lambda)/(M(\lambda)t^{q_0}) \leq K$ for all $\lambda, t \geq 1$. Let $t = 2$, then $M(2\lambda) \leq 2^{q_0}KM(\lambda)$ for all $\lambda \geq 1$, i.e., $M \in \Delta_2$.

Sufficiency. Since $M \in \Delta_2$, by [4], there exists a constant $K > 2$ such that $M(2t) \leq KM(t)$ for all $t \geq 1$. Choose an integer $n \geq 0$ such that $2^n \leq t < 2^{n+1}$; then for all $\lambda \geq 1$ and $q > 1$ satisfying $K/2^q \leq 1$, we have

$$\frac{M(t\lambda)}{M(\lambda)t^q} \leq \frac{M(2^{n+1}\lambda)}{2^{nq}M(\lambda)} \leq \frac{K^{n+1}M(\lambda)}{2^{nq}M(\lambda)} = K\left(\frac{K}{2^q}\right)^n \leq K.$$

Thus, by the definition of q_{L_M}, we have $q_{L_M} < +\infty$.

(2) Necessity. If $p_{L_M} > 1$, then there exist $\varepsilon > 0$ and $\delta > 0$ such that $M(t\lambda)/(M(\lambda)t^{1+2\varepsilon}) \geq \delta$ for all $\lambda, t \geq 1$. Choose t_0 satisfying $t_0^{\varepsilon}\delta \geq 1$, then for all $\lambda \geq 1$, we have

$$\frac{M(t_0\lambda)}{M(\lambda)t_0^{1+\varepsilon}} \geq t_0^{\varepsilon}\delta \geq 1.$$

Therefore, $M(t_0\lambda) \geq t_0^{1+\varepsilon}M(\lambda)$ for all $\lambda \geq 1$. So $N \in \Delta_2$ by [4].

Sufficiency. If $N \in \Delta_2$, then, by [4], there exists $\varepsilon > 0$ such that $M(2\lambda) \geq 2^{1+\varepsilon}M(\lambda)$ for all $\lambda \geq 1$. Choose a positive integer k_0 such that $p = (1+\varepsilon)k_0/(1+k_0) > 1$. For all $\lambda \geq 1$ and $t \geq 2^{k_0}$ choose integer k such that $2^{k_0} \leq 2^k \leq t < 2^{k+1}$, then

$$\frac{M(t\lambda)}{M(\lambda)t^p} \geq \frac{M(2^k\lambda)}{M(\lambda)2^{(k+1)p}} \geq \frac{2^{(1+\varepsilon)k}M(\lambda)}{2^{(k+1)p}M(\lambda)} = 2^{(1+\varepsilon)k-(k+1)p} \geq 2^0 = 1.$$

Note the last inequality of the above formula is assured by the monotone increasing of the function $f(x) = (1+\varepsilon)x/(1+x)$ $(x > 0)$.

On the other hand, for all $\lambda \geq 1$ and $t \in [1, 2^{k_0})$, we have

$$\frac{M(t\lambda)}{M(\lambda)t^p} \geq \frac{M(\lambda)}{2^{k_0 p} M(\lambda)} = \frac{1}{2^{k_0 p}}\,.$$

Therefore, for all $\lambda, t \geq 1$, we have

$$\frac{M(t\lambda)}{M(\lambda)t^p} \geq \min\left\{1, \frac{1}{2^{k_0 p}}\right\} > 0.$$

Thus, $p_{L_M} \geq p > 1$. □

By [4] and Theorem 1, we immediately get the following corollary.

Corollary 2. *L_M is reflexive if and only if $p_{L_M} > 1$ and $q_{L_M} < +\infty$.*

Theorem 3. *If L_M is reflexive, then L_M has SBP.*

PROOF: If L_M is reflexive, then by Theorem 1, $p_{L_M} > 1$ and $q_{L_M} < +\infty$. Therefore, if a Banach space X is isomorphic to a complemented subspace of L_M and L_M is also isomorphic to a complemented subspace of X, then Proposition 2.d.5 in [3] implies that L_M is isomorphic to X. So L_M has SBP. □

Theorem 4. *If L_M is reflexive, then L_M has the Primary Property.*

PROOF: If L_M is reflexive, then by [4] and Theorem 1, L_M is separable, $p_{L_M} > 1$ and $q_{L_M} < +\infty$. Since L_M is an r.i. function space, Theorem 2.d.11 in [3] implies that L_M has the Primary Property. □

References

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