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Estimators in the location model with gradual changes

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Abstract. A number of papers has been published on the estimation problem in location models with abrupt changes (e.g., Csörgő and Horváth (1996)). In the present paper we focus on estimators in location models with gradual changes. Estimators of the parameters are proposed and studied. It appears that the limit behavior (both the rate of consistency and limit distribution) of the estimators of the change point in location models with abrupt changes and gradual changes differ substantially.

Keywords: gradual changes in location model, estimators, confidence regions

Classification: 62G20, 62E20, 60F17

1. Introduction and main results

We consider here the following location model with gradual changes after an unknown time point m:

$$Y_i = \mu + \delta_n\Big(\frac{i-m}{n}\Big)^+ + e_i, \qquad i = 1, \dots, n, \tag{1.1}$$

where $a^+ = \max\{a, 0\}$, μ, $\delta_n \neq 0$ and m are parameters, $e_1, \dots, e_n$ are i.i.d. random variables with $Ee_i = 0$, $var e_i = \sigma^2$ and $E|e_i|^{2+\Delta} < \infty$, $i = 1, \dots, n$, and some $\Delta > 0$. The model corresponds to the situation when up to unknown m the observations are i.i.d. and then the model changes to a simple regression model with the slope δ_n. The parameter m is the *change point.*

Our main interest is to estimate the parameter m and to study its limit properties. Analogous results for parameters μ, δ_n and σ^2 are also derived.

Similar problems were treated by several authors. Assuming that the error terms e_i have a normal distribution, Hinkley (1971), Feder (1975) and Smith and Cook (1980) considered maximum likelihood type estimators in the model

$$Y_i = \mu + \beta(x_i - \eta)^+ + e_i,\ i = 1, \dots, n,$$

where μ, η are unknown parameters. This model reduces to the model (1.1) with a particular choice of x_i and a particular choice of the distribution of the e_i.

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Siegmund and Zhang (1994) developed a small sample conservative confidence region for parameter θ that works reasonably well even for moderate sample sizes in the model:

$$Y_i = \beta_0 + \beta_1 x_i + \beta_2 (x_i - \theta)^+ + e_i, \quad i = 1, \dots n,$$

where β_0, β_1, β_2 and θ are unknown parameters, $x_1, \dots, x_n$ are known regression constants and $e_1, \dots, e_n$ are i.i.d. with distribution $N(0, \sigma^2)$, $\sigma^2 > 0$ unknown.

Some authors considered the problem in the framework of nonlinear regression (e.g., Ratkowski (1983) p. 122 and Seber, Wild (1989) p. 447).

Jarušková (1996) developed test procedures for testing $H_0 : m = n$ against $H_1 : m < n$ in the model (1.1) and studied their limit behavior under the null hypothesis.

The case of the gradual changes described by model (1.1) can occur, e.g., in meteorogical data or quality control.

In the present paper we derive the limit distribution of least squares type estimators of m, μ, δ_n both for local alternatives ($\delta_n \to 0$ as $n \to \infty$) and fixed ones ($\delta_n = \delta \neq 0$). We also get a consistency result for an estimator of σ^2. It should be pointed out that the limit behavior (both the rate of convergence and the limit distribution) of the estimator of m differs from the case of the abrupt change (see Remark b below).

In the following we shall denote

$$x_{ik} = \Big(\frac{i-k}{n}\Big)^+, \qquad i, k = 1, \dots, n,$$
$$\overline{x}_k = \frac{1}{n} \sum_{i=1}^{n} x_{ik}.$$

In the present paper we study least squares type estimators $\widehat{m}, \widehat{\mu}, \widehat{\delta}_n$ of the parameters m, μ, δ_n, defined as solutions of the minimization problem

$$\min \sum_{i=1}^{n} \big(Y_i - \mu - \delta_n x_{ij}\big)^2,$$
$$\mu \in R^1, \ \delta_n \in R^1, \ j = 1, \dots, n.$$

In other words the estimators minimize the sum of squared deviations. Direct calculations give the explicit expression for the estimators $\widehat{\delta}_n$, $\widehat{\mu}_n$. Namely,

$$\widehat{\delta}_n = \frac{\sum_{i=1}^{n} (x_{i\widehat{m}} - \overline{x}_{\widehat{m}}) Y_i}{\sum_{i=1}^{n} (x_{i\widehat{m}} - \overline{x}_{\widehat{m}})^2}, \tag{1.2}$$

$$\widehat{\mu}_n = \overline{Y}_n - \widehat{\delta}_n \overline{x}_{\widehat{m}}. \tag{1.3}$$

The estimator $\widehat{m}$ can equivalently be defined as a solution of the maximization problem

$$\max \frac{\Big(\sum_{i=1}^{n}(x_{ij}-\overline{x}_j)Y_i\Big)^2}{\sum_{i=1}^{n}(x_{ij}-\overline{x}_j)^2},\ j=1,\dots,n. \tag{1.4}$$

These estimators coincide with the maximum likelihood estimators if the observations $Y_1,..,Y_n$ have normal distribution. We estimate σ^2 by

$$\widehat{\sigma}_n^2=\frac{1}{n}\sum_{i=1}^{n}(Y_i-\widehat{\mu}_n-\widehat{\delta}_n\overline{x}_{i\widehat{m}})^2. \tag{1.5}$$

Now, we state the main limit properties of these estimators. Theorem A concerns the limit distribution of the estimator $\widehat{m}$ in the model (1.1) with $m<n$ (alternative hypothesis), while limit properties of estimators $\widehat{\mu}_n$, $\widehat{\delta}_n$ and $\widehat{\sigma}_n^2$ for the same situation are formulated in Theorem B. Theorem C then gives the limit behavior of the estimators for $m=n$ (the null hypothesis).

Theorem A. *Let random variables $Y_1,..,Y_n$ be independent and have the property* (1.1). *Let, as $n\to\infty$,*

$$\delta_n=O(1),\quad \frac{\delta_n^2 n}{(\log\log n)^2}\to\infty \tag{1.6}$$

and

$$m=[n\theta] \tag{1.7}$$

for some $\theta\in(0,1)$.

Then, as $n\to\infty$,

$$\frac{\delta_n}{\sigma}\frac{\widehat{m}-m}{\sqrt{n}}\sqrt{\frac{\theta(1-\theta)}{1+3\theta}}\to^D N(0,1). \tag{1.8}$$

Theorem B. *Let assumptions of Theorem A be satisfied. Then, as $n\to\infty$,*

$$\sqrt{n}(\widehat{\delta}_n-\delta_n)\to^D N(0,\frac{12\sigma^2}{(1-\theta)^3(1+3\theta)}), \tag{1.9}$$

$$\sqrt{n}(\widehat{\mu}_n-\mu)\to^D N(0,\frac{4\sigma^2}{1+3\theta}), \tag{1.10}$$

and

$$\widehat{\sigma}_n^2-\sigma^2=o_P((\log\log n)^{-1}). \tag{1.11}$$

Theorem C. *Let $Y_1, \dots, Y_n$ be i.i.d. random variables with $E|X_i|^{2+\Delta} < \infty$ for a positive Δ. Then, as $n \to \infty$,*

$$P(n - \eta_n > \widehat{m} > (1 - \epsilon_n)n) \to 1 \tag{1.12}$$

for arbitrary sequences $\{\epsilon_n\}$ and $\{\eta_n\}$ of positive numbers such that, as $n \to \infty$,

$$\epsilon_n^3 \log\log n \to 0, \ \frac{\eta_n}{\log n} = O(1).$$

Moreover, the assertions (1.10)–(1.11) *remain true and as $n \to \infty$,*

$$\widehat{\delta}_n = o_p((\log n)^{-3/2}). \tag{1.13}$$

Remark a. Theorem A covers both local ($\delta_n \to 0$ as $n \to \infty$) and fixed type ($\delta_n = \delta \neq 0$) of the size of change.

Remark b. Both the rate of consistency and the limit distribution of the estimator $\widehat{m}$ differ from the case of abrupt changes. In case of an abrupt change in a location model we get the rate of consistency δ_n^{-2} while in case of a gradual change (1.1) we received the rate $n^{1/2}\delta_n^{-1}$. The limit distribution of a properly standardized estimator $\widehat{m}$ in case of abrupt changes is the same the *argmax* of a certain Gaussian process with a time dependent drift. For the results for abrupt changes in location models see, e.g., Csörgő and Horváth (1997) or Antoch, Hušková and Veraverbeke (1995).

Remark c. The assertion of Theorem A remains true if δ_n and σ are replaced by suitable estimators, e.g., given by (1.4) and (1.5), respectively.

2. Proofs

Recall that the estimator $\widehat{m}$ can be equivalently defined as a solution of the maximization problem

$$\max \frac{\Big(\sum_{i=1}^n (x_{ij} - \overline{x}_j)Y_i\Big)^2}{\sum_{i=1}^n (x_{ij} - \overline{x}_j)^2}, \quad j = 1, \dots, n.$$

First we prove several auxiliary lemmas.

Lemma 1. *If* (1.6)–(1.7) *are satisfied, then for each $\epsilon \in (0, \min(\theta, 1-\theta))$, as $n \to \infty$,*

$$\frac{1}{n}\sum_{i=1}^n (x_{im} - \overline{x}_m)^2 = \frac{(1-\theta)^3}{3} - \frac{(1-\theta)^4}{4} + O(n^{-1}), \tag{2.2}$$

and

$$\max_{|k-m|>\epsilon n} \frac{\Big(\sum_{i=1}^n (x_{ik}-\overline{x}_k)x_{im}\Big)^2}{\sum_{i=1}^n (x_{ik}-\overline{x}_k)^2} = \max\Big\{\frac{12(1-\theta)^4(1+2\theta-3(\theta-\epsilon)^2)^2}{(1-\theta+\epsilon)^3(1+3\theta-3\epsilon)},$$

$$\frac{(1-\theta-\epsilon)(1+2(\theta+\epsilon)-3\theta^2)^2}{12(1+3\theta+3\epsilon)}\Big\} + O(n^{-1}) \tag{2.3}$$

$$< \frac{(1-\theta)^3}{3} + O(n^{-1}).$$

Proof: Elementary calculations give, as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^n x_{ik}x_{im} = \int_0^1 (s-\theta)^+(s-k/n)^+\,ds + O\big(\frac{\min(n-k,n-m)}{n^2}\big)$$

$$= (1-\max(\theta,k/n))^2(2+\max(\theta,k/n)-3\min(\theta,k/n))/6$$

$$+O(\frac{\min(n-k,n-m)}{n^2}), \tag{2.4}$$

(2.5)

$$\frac{1}{n}\sum_{i=1}^n x_{ik} = \int_0^1 (s-k/n)^+\,ds + O(\frac{n-k}{n^2}) = (1-k/n)^2/2 + O\big(\frac{n-k}{n^2}\big),$$

(2.6)

$$\frac{1}{n}\sum_{i=1}^n (x_{ik}-\overline{x}_k)^2 = \frac{(1-k/n)^3}{3} - \frac{(1-k/n)^4}{4} + O\big(\frac{n-k}{n^2}\big)$$

uniformly in $1 \le k \le n$.

Hence, as $n \to \infty$,

$$\frac{1}{n}\frac{\Big(\sum_{i=1}^n (x_{ik}-\overline{x}_k)x_{im}\Big)^2}{\sum_{i=1}^n (x_{ik}-\overline{x}_k)^2} = \frac{1}{12}Q(k/n) + O\big(\frac{\min(n-k,n-m)}{n^2}\big)$$

uniformly in $1 \le k < n$, where

$$Q(t) = \frac{(1-\max(\theta,t))^4(1+2\max(\theta,t)-3\min(\theta^2,t^2))^2}{(1-t)^3(1+3t)},\ \ 0<t<1.$$

This immediately implies (2.2). Calculating the derivative of $Q(t)$ we find that

$$Q'(t) > 0 \text{ for } 0 < t < \theta$$

$$Q'(t) < 0 \text{ for } 1 > t > \theta$$

which implies (2.3). □

Lemma 2. *Let the assumptions of Theorem A be satisfied then, as $n \to \infty$,*

$$\max_{1 \le k < n(1-\epsilon_n)} \frac{\big(\sum_{i=1}^n (x_{ik} - \overline{x}_k) e_i\big)^2}{\sum_{i=1}^n (x_{ik} - \overline{x}_k)^2} = O_p(\epsilon_n^{-1}), \tag{2.7}$$

for every sequence $\{\epsilon_n\}$, $0 < \epsilon_n < 1$ *and*

$$\max_{n-\eta_n \le k < n} \frac{\big(\sum_{i=1}^n (x_{ik} - \overline{x}_k) e_i\big)^2}{\sum_{i=1}^n (x_{ik} - \overline{x}_k)^2} = O_p(\log\log \eta_n), \tag{2.8}$$

for every sequence $\{\eta_n\}$, $\eta_n < n$, $\eta_n \to \infty$. *Moreover,*

$$P\Big(\max_{1 \le k < n} \frac{\big(\sum_{i=1}^n (x_{ik} - \overline{x}_k) e_i\big)^2}{\sigma^2 \sum_{i=1}^n (x_{ik} - \overline{x}_k)^2} > \sqrt{2\log\log n} + \frac{x + \log\frac{\sqrt{3}}{4\pi}}{\sqrt{2\log\log n}}\Big) \to 1 - \exp\{-\exp\{-x\}\}, \qquad x \in R^1. \tag{2.9}$$

Proof: By the Hájek-Renyi inequality (e.g., Theorem 7.4.8 in Chow and Teicher (1987)), as $n \to \infty$,

$$\max_{1 \le k \le n(1-\epsilon_n)} \Big\{\frac{|\sum_{i=k+1}^n e_i|}{n-k}\Big\} = O_p((n\epsilon_n)^{-1/2}), \tag{2.10}$$

which together with standard arguments gives

$$\max_{1 \le k < (1-\epsilon_n)n} \Big\{\frac{\Big(\sum_{i=1}^n (x_{ik} - \overline{x}_k) e_i\Big)^2}{\sum_{i=1}^n (x_{ik} - \overline{x}_k)^2}\Big\}$$
$$= O_p\Big(\max_{1 \le k < (1-\epsilon_n)n} \Big\{\big(\sum_{i=1}^n (i-k)^+ e_i\big)^2 (n-k)^{-3}\Big\} + \big(\sum_{i=1}^n e_i\big)^2 (n\epsilon_n)^{-1}\Big)$$
$$= O_p\Big(\max_{1 \le k < (1-\epsilon_n)n} \Big(\sum_{j=k+1}^n \sum_{i=j+1}^n e_i\Big)^2 (n-k)^{-3}\Big) + O_p(\epsilon_n^{-1}) = O_p(\epsilon_n^{-1}).$$

To prove (2.8) we realize that by the Darling-Erdös theorem (see, e.g., Theorem A.4.2 in Csörgő and Horváth (1997)), as $n \to \infty$

$$\max_{n-\eta_n \le k < n} \Big\{\frac{|\sum_{i=k+1}^n e_i|}{\sqrt{n-k}}\Big\} = O_p(\sqrt{\log\log \eta_n}). \tag{2.11}$$

Now, proceeding analogously as in proving (2.7) and using (2.11) instead of (2.10) we obtain (2.8). Assertion (2.9) follows from Theorem 2 in Jarušková (1996). □

The estimator $\widehat{m}$ can equivalently be defined as a solution of the maximization problem as

$$\max\{A_j + 2\delta_n B_j + \delta_n^2 C_j\}, \qquad j = 1, \dots, n-1, \tag{2.12}$$

where

$$A_k = \frac{\left(\sum_{i=1}^n (x_{ik} - \overline{x}_k)e_i\right)^2}{\sum_{i=1}^n (x_{ik} - \overline{x}_k)^2} - \frac{\left(\sum_{i=1}^n (x_{im} - \overline{x}_m)e_i\right)^2}{\sum_{i=1}^n (x_{im} - \overline{x}_m)^2},$$

$$B_k = \frac{\left(\sum_{i=1}^n (x_{ik} - \overline{x}_k)e_i\right)\left(\sum_{i=1}^n (x_{ik} - \overline{x}_k)x_{im}\right)}{\sum_{i=1}^n (x_{ik} - \overline{x}_k)^2} - \sum_{i=1}^n (x_{im} - \overline{x}_m)e_i,$$

$$C_k = \frac{\left(\sum_{i=1}^n (x_{ik} - \overline{x}_k)x_{im}\right)^2}{\sum_{i=1}^n (x_{ik} - \overline{x}_k)^2} - \sum_{i=1}^n (x_{im} - \overline{x}_m)^2.$$

Lemma 3. *Let the assumptions of Theorem A be satisfied. Then, as* $n \to \infty$,

$$C_k = -\frac{(m-k)^2}{n}\frac{\theta(1-\theta)}{1+3\theta}(1 + o(\frac{m-k}{n})), \tag{2.13}$$

$$\max_{r_n|\delta_n|^{-1}\sqrt{n} \le |m-k| \le n\epsilon_n} \left\{\frac{A_k + \delta_n B_k}{\delta_n^2 (m-k)^2} n\right\} = o_p(1), \tag{2.14}$$

$$\max_{|m-k| \le r_n|\delta_n|^{-1}\sqrt{n}} \left\{\frac{\sqrt{n}}{(m-k)|\delta_n|}|A_k|\right\} = o_p(1) \tag{2.15}$$

and

$$\max_{|m-k| \le r_n|\delta_n|^{-1}\sqrt{n}} \left\{B_k \frac{\sqrt{n}}{m-k} - Z_n \frac{1}{\sqrt{n}}|\right\} = o_p(1), \tag{2.16}$$

where $\{\epsilon_n\}$ *and* $\{r_n\}$ *satisfy, as* $n \to \infty$,

$$0 < \epsilon_n,\ \epsilon_n \to 0,\ r_n \to \infty,\ \frac{|\delta_n|\sqrt{n}}{r_n\sqrt{\log\log n}} \to \infty \tag{2.17}$$

and where

$$Z_n = \sum_{i=m+1}^n (e_i - \overline{e}_n) - \frac{n\theta(1-\theta)^2}{2\sum_{i=1}^n (x_{im} - \overline{x}_m)^2} \sum_{i=1}^n (e_i - \overline{e}_n)x_{im}. \tag{2.18}$$

Proof: By (2.4)–(2.6) we have, as $n \to \infty$,

$$\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)^2 = n\frac{(1-\theta)^3}{12}(1+3\theta)+O(m-k), \tag{2.19}$$

$$\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)(x_{im}-x_{ik}) = \frac{k-m}{2}(1-\theta)^2\theta(1+O(\frac{m-k}{n})) \tag{2.20}$$

and

$$\sum_{i=1}^{n}(x_{ik}-x_{im}-\overline{x}_k+\overline{x}_m)^2 = \frac{(m-k)^2}{n}(1-\theta)\theta(1+O(\frac{m-k}{n})) \tag{2.21}$$

uniformly in $(m-k)=o(n)$.

Next, the terms A_k, B_k and C_k can be rewritten as

$$\begin{aligned}
A_k = {} & \frac{\Big(\sum_{i=1}^{n}(x_{ik}-x_{im}-\overline{x}_k+\overline{x}_m)e_i\Big)^2}{\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)^2} \\
& + 2\Big(\sum_{i=1}^{n}(x_{im}-\overline{x}_m)e_i\Big)\frac{\sum_{i=1}^{n}(x_{ik}-x_{im}-\overline{x}_k+\overline{x}_m)e_i}{\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)^2} \\
& - \frac{(\sum_{i=1}^{n}(x_{im}-\overline{x}_m)e_i)^2}{\sum_{i=1}^{n}(x_{im}-\overline{x}_m)^2}\frac{\sum_{i=1}^{n}(x_{ik}-x_{im}-\overline{x}_k+\overline{x}_m)^2}{\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)^2} \\
& + 2\frac{(\sum_{i=1}^{n}(x_{im}-\overline{x}_m)e_i)^2}{\sum_{i=1}^{n}(x_{im}-\overline{x}_m)^2}\frac{\sum_{i=1}^{n}(x_{ik}-x_{im})(x_{im}-\overline{x}_m)}{\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)^2},
\end{aligned}$$

$$B_k = \sum_{i=1}^{n}(x_{ik}-x_{im})(e_i-\overline{e}_n) - \sum_{i=1}^{n}(x_{ik}-\overline{x}_k)e_i\frac{\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)(x_{ik}-x_{im})}{\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)^2},$$

$$C_k = \frac{\Big(\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)(x_{im}-x_{ik})\Big)^2}{\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)^2} - \sum_{i=1}^{n}(x_{ik}-x_{im}-\overline{x}_k+\overline{x}_m)^2.$$

Inserting (2.19)–(2.21) into these expressions for A_k, B_k and C_k and applying standard arguments we obtain (2.13) and, as $n \to \infty$,

$$\begin{aligned}
A_k = O_p\Big(& (\sum_{i=1}^{n}(x_{ik}-x_{im})(e_i-\overline{e}_n))^2/n \\
& + \Big((\sum_{i=1}^{n}(x_{ik}-x_{im})(e_i-\overline{e}_n))^2/n\Big)^{1/2} + |k-m|/n\Big)
\end{aligned} \tag{2.22}$$

and

(2.23)

$$B_k = \sum_{i=1}^{n}(x_{ik} - x_{im})(e_i - \overline{e}_n) - \sum_{i=1}^{n}(e_i - \overline{e}_n)x_{ik}\frac{6(m-k)\theta}{(1-\theta)(1+3\theta)n}(1 + O((m-k)/n))$$

uniformly for $(k-m) = o(n)$. Moreover, we find that, as $n \to \infty$,

$$\begin{aligned}
&\Big|B_k\frac{\sqrt{n}}{m-k} - Z_n\frac{1}{\sqrt{n}}\Big| \\
&\le \Big|\frac{\sqrt{n}}{m-k}\sum_{i=1}^{n}(x_{ik} - x_{im})(e_i - \overline{e}_n) - \frac{1}{\sqrt{n}}\sum_{i=m+1}^{n}(e_i - \overline{e}_n)\Big| \\
&\quad + \frac{6\theta}{(1-\theta)(1+3\theta)}\frac{1}{\sqrt{n}}\Big|\sum_{i=1}^{n}(x_{ik} - x_{im})(e_i - \overline{e}_n)\Big| + o_p(1)
\end{aligned} \tag{2.24}$$

uniformly for $(k-m) = o_p(n)$. Hence to establish (2.15) and (2.16) it suffices to prove that, as $n \to \infty$,

$$\Big|\frac{\sqrt{n}}{m-k}\sum_{i=1}^{n}(x_{ik} - x_{im})(e_i - \overline{e}_n) - \frac{1}{\sqrt{n}}\sum_{i=m+1}^{n}(e_i - \overline{e}_n)\Big| = o_p(1) \tag{2.25}$$

and

$$\Big|\sum_{i=1}^{n}(x_{ik} - x_{im})(e_i - \overline{e}_n)\Big|/\sqrt{n} = o_p(1) \tag{2.26}$$

uniformly for $(k-m) = o_p(n)$. We have

(2.27)

$$\begin{aligned}
&\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(x_{ik} - x_{im})(e_i - \overline{e}_n) - \frac{1}{\sqrt{n}}\frac{m-k}{n}\sum_{i=m+1}^{n}(e_i - \overline{e}_n)\Big| \\
&\le \frac{1}{n^{3/2}}\Big(I\{k > m\}\Big|\sum_{i=m+1}^{k}(k-i)(e_i - \overline{e}_n)\Big| + I\{k \le m\}\Big|\sum_{i=k+1}^{m}(i-k)(e_i - \overline{e}_n)\Big|\Big) \\
&= \frac{1}{n^{3/2}}\Big(I\{k > m\}\Big|\sum_{j=m+2}^{k}\sum_{i=m+1}^{j-1}(e_i - \overline{e}_n)\Big| + I\{k \le m\}\Big|\sum_{j=k+1}^{m}\sum_{i=j}^{m}(e_i - \overline{e}_n)\Big|\Big).
\end{aligned}$$

Since by the law of iterated logarithm, as $n \to \infty$,

$$\max_{1 \le k \le r_n|\delta_n|^{-1}\sqrt{n}}\Big\{\Big|\sum_{i=m+1}^{m+k}e_i\Big|k^{-1/2} + \Big|\sum_{i=m-k}^{m}e_i\Big|k^{-1/2}\Big\} = O_p(\sqrt{\log\log n})$$

we also have

$$\max_{1\le k\le r_n|\delta_n|^{-1}\sqrt{n}}\Big(|\sum_{j=m+2}^{m+k}\sum_{i=m+1}^{j-1}(e_i-\overline{e}_n)|+|\sum_{j=m-k}^{m}\sum_{i=j}^{m}(e_i-\overline{e}_n)|\Big)$$
$$=O_p((r_n|\delta_n|^{-1/2}\sqrt{n})^{3/2}\sqrt{\log\log n}).$$

The last relation together with (2.27) and assumption (2.17) then imply (2.26). Relation (2.25) follows from (2.26) and $\sum_{i=1+m}^{n} e_i = O_p(\sqrt{n})$. Our lemma is proved. □

Proof of Theorem A: Lemma 1, Lemma 2 and Lemma 3 imply that, as $n\to\infty$,

$$P\Big(\max_{1\le k<n}\frac{\Big(\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)Y_i\Big)^2}{\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)^2}$$
$$=\max_{|k-m|\le r_n|\delta_n|^{-1}\sqrt{n}}\frac{\Big(\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)Y_i\Big)^2}{\sum_{i=1}^{n}(x_{ik}-\overline{x}_k)^2}\Big)\to 1.$$

Next, Lemma 3 ((2.12), (2.14), (2.15)) implies that

$$A_k+2\delta_nB_k+\delta_n^2C_k$$
$$=\delta_n\frac{m-k}{\sqrt{n}}\Big(-\delta_n\frac{m-k}{\sqrt{n}}\frac{\theta(1-\theta)}{1+3\theta}+2\frac{Z_n}{\sqrt{n}}+o_p(1)\Big),$$

uniformly for $|k-m|\le r_n|\delta_n|^{-1}\sqrt{n}$, where r_n satisfies (2.16). Then regarding the definition of $\widehat{m}$ we can infer that $\delta_n\frac{m-\widehat{m}}{\sqrt{n}}\frac{\theta(1-\theta)}{1+3\theta}$ has the same limit distribution as $2Z_nn^{-1/2}$. The random variable Z_n is the sum of independent random variables, its variance fulfills, as $n\to\infty$,

$$varZ_n=\sigma^2\Big(\sum_{i=1}^{n}(c_i-\overline{c}_n)^2+\frac{n^2\theta^2(1-\theta)^4}{4\sum_{i=1}^{n}(x_{im}-\overline{x}_m)^2}\Big)$$
$$=\sigma^2n\frac{\theta(1-\theta)}{1+3\theta}(1+o(1))$$

and it can be easily checked that the assumptions of CLT are fulfilled and therefore, as $n\to\infty$,

$$n^{-1/2}Z_n\to^D N(0,\sigma^2\frac{\theta(1-\theta)}{1+3\theta}).$$

This together with the above arguments imply the assertion (1.8). □

Proof of Theorem B: Since Theorem A implies that $\widehat{m} - m = O_p(\sqrt{n}\delta_n^{-1}) = o_p(n)$, then by (2.4)–(2.6) we have

$$\sum_{i=1}^{n}(x_{i\widehat{m}} - \overline{x}_{\widehat{m}})^2 = \sum_{i=1}^{n}(x_{im} - \overline{x}_m)^2 + O_p(\sqrt{n}\delta_n^{-1})$$

$$\sum_{i=1}^{n}(x_{im} - x_{i\widehat{m}})e_i = O_p((m - \widehat{m})n^{-1/2}) = O_p(\delta_n^{-1}).$$

This together with (2.6) and (2.23) further implies that $\sqrt{n}(\widehat{m} - m)$ has the same limit distribution as

$$\sqrt{n}\frac{\sum_{i=1}^{n}(x_{im} - \overline{x}_m)e_i}{\sum_{i=1}^{n}(x_{im} - \overline{x}_m)^2}\,.$$

This is the sum of independent random variables and it can be easily checked that the assumptions of CLT are satisfied and hence (1.9) holds true.

The limit distribution of $\widehat{\mu}$ can be obtained in a very similar way and hence the proof is omitted.

Concerning (1.11) we notice that by (1.9)–(1.10) $\widehat{\delta}_n - \delta_n = O_p(n^{-1/2})$ and $\widehat{\mu}_n - \mu = O_p(n^{-1/2})$ which after few standard steps leads to the desired assertion. □

Proof of Theorem C: By (2.9) we have, as $n \to \infty$,

$$P\Big(\max_{1\le k<n}\frac{\big(\sum_{i=1}^{n}(x_{ik} - \overline{x}_k)e_i\big)^2}{\sigma^2\sum_{i=1}^{n}(x_{ik} - \overline{x}_k)^2} > \sqrt{\log\log n}\Big) \to 1,$$

which together with (2.8)–(2.9) yields the assertion of the theorem. □

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