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On the Hölder continuity of weak solutions to nonlinear parabolic systems in two space dimensions

J. Naumann, J. Wolf, M. Wolff

Abstract. We prove the interior Hölder continuity of weak solutions to parabolic systems

$$\frac{\partial u^j}{\partial t} - D_\alpha a_j^\alpha(x,t,u,\nabla u) = 0 \ \text{ in } \ Q \quad (j = 1,\dots,N)$$

$(Q = \Omega\times(0,T), \Omega \subset \mathbb{R}^2)$, where the coefficients $a_j^\alpha(x,t,u,\xi)$ are measurable in x, Hölder continuous in t and Lipschitz continuous in u and ξ.

Keywords: nonlinear parabolic systems, Hölder continuity, Fourier transform

Classification: 35B65, 35K55

1. Introduction. Statement of the main result

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a domain, let $0 < T < +\infty$ and set $Q = \Omega \times (0,T)$. We consider the following system of nonlinear PDE's:

$$\frac{\partial u^j}{\partial t} - D_\alpha a_j^\alpha(x,t,u,\nabla u) = 0 \text{ [1]} \ \text{ in } \ Q \quad (j = 1,\dots,N), \tag{1.1}$$

where

$$u = \{u^1,\dots,u^N\} \quad (N \geq 2)$$

$$D_\alpha = \frac{\partial}{\partial x_\alpha} \ (\alpha = 1,\dots,n), \quad \nabla u = \{D_\alpha u^j\} \ (= \text{matrix of spatial derivatives}).$$

In this paper we study the interior Hölder continuity of weak solutions to (1.1) under the following assumptions on the functions a_j^α:

$$\begin{cases} x \mapsto a_j^\alpha(x,t,u,\xi) \ \textit{is measurable on} \ \Omega \ \forall\,(t,u,\xi) \in (0,T)\times\mathbb{R}^N\times\mathbb{R}^{nN} \\ a_j^\alpha(\cdot,0,0,0) \in L^\sigma(\Omega) \ (\sigma > 2); \end{cases} \tag{1.2}$$

[1]Throughout the paper, a repeated Greek (resp. Latin) index stands for the summation over $1,\dots,n$ (resp. $1,\dots,N$).

$$(1.3)\quad \begin{cases} |a_j^\alpha(x,s,u,\eta) - a_j^\alpha(x,t,v,\xi)| \le \\ \le c_0\Big\{|s-t|^\mu(1+|u|^{(n+2)/n} + |v|^{(n+2)/n} + |\eta| + |\xi|) \\ \qquad\qquad\qquad + |u-v| + |\eta-\xi|\Big\} \\ \forall\, x\in\Omega,\ \forall\,(s,u,\eta),(t,v,\xi)\in(0,T)\times\mathbb{R}^N\times\mathbb{R}^{nN} \\ (c_0 = \text{const},\ 0<\mu\le 1); \end{cases}$$

$$(1.4)\quad \begin{cases} (a_j^\alpha(x,t,u,\eta) - a_j^\alpha(x,t,u,\xi))(\eta_\alpha^j - \xi_\alpha^j) \ge \nu_0|\eta-\xi|^2 \\ \forall\,(x,t,u)\in\Omega\times(0,T)\times\mathbb{R}^N,\ \forall\,\eta,\xi\in\mathbb{R}^{nN}\ \ (\nu_0 = \text{const} > 0) \end{cases}$$

$(\alpha = 1,\dots,n;\ j = 1,\dots,N)$.

By (1.2), (1.3) and (1.4),

$$|a_j^\alpha(x,t,u,\xi)| \le c_1(1+|u|+|\xi|) + |a_j^\alpha(x,0,0,0)|,$$

$$a_j^\alpha(x,t,u,\xi)\xi_\alpha^j \ge \frac{\nu_0}{2}|\xi|^2 - c_2\Big(1+|u|^2 + \sum_{\beta=1}^n\sum_{k=1}^N (a_k^\beta(x,0,0,0))^2\Big)$$

for all $(x,t,u,\xi) \in \Omega\times(0,T)\times\mathbb{R}^N\times\mathbb{R}^{nN}$ $(\alpha = 1,\dots,n;\ j = 1,\dots,N;$ $c_1, c_2 = \text{const})$.

By $W_p^1(\Omega)$ $(1\le p\le+\infty)$ we denote the usual Sobolev space. If Ω is a bounded domain with smooth boundary $\partial\Omega$ we denote

$$\overset{\circ}{W}{}_p^1(\Omega) = \{\varphi\in W_p^1(\Omega) \mid \varphi = 0 \ \text{ a.e. on } \ \partial\Omega\}.$$

Next, define

$$W_2^{1,0}(Q) = \{\varphi\in L^2(Q) \mid D_\alpha\varphi\in L^2(Q) \quad (\alpha = 1,\dots,n)\},$$

$$V_2^{1,0}(Q) = \Big\{\varphi\in W_2^{1,0}(Q) \mid \operatorname*{ess\,sup}_{(0,T)}\int_\Omega \varphi^2(x,t)\,\mathrm{d}x < +\infty\Big\},$$

$$W_2^{1,1}(Q) = \Big\{\varphi\in W_2^{1,0}(Q) \mid \frac{\partial\varphi}{\partial t}\in L^2(Q)\Big\} \quad (= W_2^1(Q)).$$

The following imbedding theorem is well-known (cf. e.g. [9]):

$$(1.5)\quad \begin{cases} \textit{Let } \Omega_0\subset\mathbb{R}^n \textit{ be a bounded domain with smooth boundary } \partial\Omega_0. \\ \textit{Then:} \\ \|\varphi\|_{L^{2(n+2)/n}(\Omega_0\times(0,T))} \le c_0\Big(\operatorname*{ess\,sup}_{(0,T)}\int_{\Omega_0}\varphi^2(x,t)\,\mathrm{d}x + \int_0^T\int_{\Omega_0}|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\ \textit{for all } \varphi\in V_2^{1,0}(\Omega_0\times(0,T)),\ \varphi = 0 \textit{ a.e. on } \partial\Omega_0\times(0,T) \\ (c_0 = \text{const} < +\infty). \end{cases}$$

Obviously, $W_2^{1,1}(Q) \subset V_2^{1,0}(Q)$.

Next, $\Omega' \subset\subset \Omega$ means: Ω' open, bounded and $\bar{\Omega}' \subset \Omega$. Given $0 < \nu < 1$ we define

$$\begin{aligned} C^{\nu,\nu/2}(Q) = \{v : Q \to \mathbb{R} \mid \forall\, \Omega' \subset\subset \Omega,\ \forall\, t' \in (0,T) \\ \exists\, K = \text{const} : |v(x,s) - v(y,t)| \le K(|x-y|^\nu + |s-t|^{\nu/2}) \\ \forall\, (x,s),(y,t) \in \Omega' \times (t',T)\} \end{aligned}$$

(notice that the constant K may depend on $\operatorname{dist}(\Omega', \partial\Omega)$ and t').

Let X be any normed vector space with norm $\|\cdot\|_X$. By $L^p(a,b;X)$ ($-\infty < a < b < +\infty$; $1 \le p \le +\infty$) we denote the vector space of all (classes of equivalent) Bochner measurable functions φ: $(a,b) \to X$ such that $\|\varphi(\cdot)\|_X \in L^p(a,b)$. Then $L^p(a,b;X)$ is a normed vector space with respect to the norm

$$\|\varphi\|_{L^p(a,b;X)} = \begin{cases} \Big(\int\limits_a^b \|\varphi(t)\|_X^p \,\mathrm{d}t \Big)^{1/p} & \text{if } 1 \le p < +\infty, \\ \operatorname*{ess\,sup}_{(a,b)} \|\varphi(t)\|_X & \text{if } p = +\infty. \end{cases}$$

The linear isometry $L^p(a,b;L^p(\Omega)) \cong L^p(\Omega \times (a,b))$ ($1 \le p < +\infty$) permits to identify these spaces.

Finally, set

$$L^p(Q,\mathbb{R}^N) = [L^p(Q)]^N, \quad W_2^{1,0}(Q;\mathbb{R}^N) = [W_2^{1,0}(Q)]^N \quad \text{etc.}$$

□

Definition. *A vector function $u \in V_2^{1,0}(Q;\mathbb{R}^N)$ is called a weak solution to* (1.1) *if*

$$(1.6) \qquad \begin{cases} -\int\limits_Q u^j \dfrac{\partial \varphi^j}{\partial t} \,\mathrm{d}x\,\mathrm{d}t + \int\limits_Q a_j^\alpha(x,t,u,\nabla u) D_\alpha \varphi^j \,\mathrm{d}x\,\mathrm{d}t = 0, \\ \forall\, \varphi \in W_2^{1,1}(Q),\ \operatorname{supp}(\varphi) \subset Q. \end{cases}$$

The interior Hölder continuity of weak solutions to (1.1) with coefficients $a_j^\alpha = a_j^\alpha(\xi)$ has been proved in [11] for dimensions $n = 2, 3$ and 4. For the case $n = 2$, an analogous result with coefficients $a_j^\alpha = a_j^\alpha(x,t,u,\xi)$ which are either Lipschitz continuous in x and measurable in t, or measurable in x and Lipschitz continuous in t (i.e. $\mu = 1$ in (1.3)) is presented in [6]. The Hölder continuity of weak solutions to nonlinear parabolic systems for arbitrary $n \ge 2$, but under additional restrictions on $\dfrac{\partial a_i^\alpha}{\partial \xi_\beta^j}$ has been established in [7] and [8]. In [3], the author proves for the case $n \le 2$ the interior Hölder continuity, and for dimensions $n \ge 3$ the interior *partial* Hölder continuity of weak solutions to nonlinear parabolic systems the coefficients of which fulfil an appropriate uniform continuity property with respect to x and t (notice that this paper also includes right-hand sides obeying strictly controlled growth conditions).

The aim of the present paper is to prove the interior Hölder continuity of any weak solution to (1.1) when $n = 2$ and the exponent μ in (1.3) is "sufficiently near to 1". Our main result is the following

Theorem. *Let* $n = 2$. *Let* (1.2)–(1.4) *be satisfied. Then there exists* $0 < \mu_0 < 1$ *such that: if* (1.3) *is fulfilled with* $\mu_0 < \mu < 1$, *then for any weak solution* $u \in V_2^{1,0}(Q;\mathbb{R}^N)$ *to* (1.1) *there holds*

$$u \in C^{\nu,\nu/2}(Q;\mathbb{R}^N).$$

We note that μ_0 is determined only by the exponent of integrability > 2 of the gradient of weak solutions to the nonlinear elliptic system associated with (1.1) (cf. [5]).

The paper is organized as follows. In Section 2 we prove some estimates on t-differences of weak solutions u to (1.1) which are based on an idea from [10]. The following section is concerned with the proof of the existence and regularity of $\dfrac{\partial u}{\partial t}$; here we make full use of the Fourier transform of vector valued functions. The results presented in these sections are of an independent interest. The proof of our main result is then given in Section 4. Following [11] we consider $\dfrac{\partial u}{\partial t}(\cdot,t)$ as right-hand side of the associated nonlinear elliptic system and apply then the theory of higher integrability of $\nabla u(\cdot,t)$ via reverse Hölder inequality.

2. Estimates on t-differences

Let $f \in L^p(Q)$ $(1 \le p < +\infty)$. We extend f by zero onto $\Omega \times (T,+\infty)$ and denote this extension again by f.

The Steklov average of f with respect to t is defined by

$$f_\lambda(x,t) = \frac{1}{\lambda} \int\limits_t^{t+\lambda} f(x,s)\,\mathrm{d}s \quad \text{for a.a.} \quad (x,t) \in Q,\ \lambda > 0.$$

It is readily seen that, for any $0 \le t_0 < t_1 < T$,

$$\int\limits_{t_0}^{t_1}\int\limits_\Omega |f_\lambda|^p \,\mathrm{d}x\,\mathrm{d}t \le \int\limits_{t_0}^{T}\int\limits_\Omega |f|^p \,\mathrm{d}x\,\mathrm{d}t \quad \forall\, 0 < \lambda < T - t_1, \tag{2.1}$$

and that $f_\lambda \to f$ in $L^p(Q)$ as $\lambda \to 0$. The function f_λ possesses the weak t-derivative

$$\frac{\partial f_\lambda}{\partial t}(x,t) = \frac{1}{\lambda}(f(x,t+\lambda) - f(x,t)) \text{ for a.a. } (x,t) \in Q,\ \forall\, \lambda > 0. \tag{2.2}$$

In addition, if there exists the weak spatial derivative $D_\alpha f \in L^p(Q)$ ($\alpha \in \{1,\dots,n\}$) then

$$(D_\alpha f_\lambda)(x,t) = (D_\alpha f)_\lambda(x,t) \text{ for a.a. } (x,t) \in Q,\ \forall\, \lambda > 0. \tag{2.3}$$

□

Assume (1.2), (1.3). Let $u \in V_2^{1,0}(Q;\mathbb{R}^N)$ be a weak solution to (1.1). Let $\Omega' \subset\subset \Omega$ and $0 < t_1 < T$. Observing (2.2) and (2.3) we may localize (1.6) with respect to t:

$$\begin{cases} \displaystyle\int_\Omega \frac{\partial u^j_\lambda}{\partial t}(x,t)\psi^j(x)\,\mathrm{d}x + \int_\Omega (a^\alpha_j)_\lambda(x,t) D_\alpha \psi^j(x)\,\mathrm{d}x = 0 \\ \text{for a.a. } t \in (0,t_1),\ \forall\, 0 < \lambda < T - t_1,\ \forall\, \psi \in W^1_2(\Omega;\mathbb{R}^N) \\ \text{with } \psi = 0 \text{ a.e. in } \Omega \setminus \Omega' \end{cases} \tag{2.4}$$

(cf. [10]; notice that the set of measure zero of those t for which (2.4) fails, does not depend on λ).

Define

$$(\Delta_h f)(x,t) = f(x,t+h) - f(x,t).$$

The localized version (2.4) is the point of departure for proving the following result whose idea of proof is developed in [10].

Lemma 1. *Let $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, $0 \le t_0 < t_1 < T$. Then*

$$\begin{aligned} \frac{1}{h}\int_{t_0}^{t_1}\int_{\Omega''} |\Delta_h u|^2\,\mathrm{d}x\,\mathrm{d}t &\le c\Big(1 + \int_Q (|u|^{2(n+2)/n} + |\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \times \\ &\quad\times \Big(\int_{t_0}^{t_1}\int_{\Omega'} (|\Delta_h u|^2 + |\Delta_h \nabla u|^2)\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \end{aligned} \tag{2.5}$$

for all $0 < h < T - t_1$, *where* $c = \text{const}$ *depends on* $\operatorname{dist}(\Omega'', \partial\Omega')$ [2].

Proof: Let $\zeta \in C_c^\infty(\Omega')$ (= set of all infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω') be a cut-off function such that $0 \le \zeta \le 1$ in Ω', $\zeta \equiv 1$ in Ω''.

[2] In what follows, by c we denote positive constants which may change their numerical value from line to line, but are independent of h.

Let $0 < h < T - t_1$. Setting $\lambda = h$ in (2.2) gives

$$\frac{\partial u_h}{\partial t}(x,t) = \frac{1}{h}(\Delta_h u)(x,t) \text{ for a.a. } (x,t) \in \Omega \times (0, t_1).$$

We may insert $\psi(x) = (\Delta_h u)(x,t)\zeta(x)$ $((x,t) \in \Omega \times (t_0, t_1))$ into (2.4). Integrating over the interval (t_0, t_1) and observing (2.1) yields

$$\begin{aligned}
&\frac{1}{h}\int_{t_0}^{t_1}\int_{\Omega'} |\Delta_h u|^2 \zeta \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{t_0}^{t_1}\int_{\Omega'} (a_j^\alpha)_h((\Delta_h D_\alpha u^j)\zeta + (\Delta_h u^j)D_\alpha\zeta)\,\mathrm{d}x\,\mathrm{d}t \\
&\le \left(\int_Q (a_j^\alpha)^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \left(\int_{t_0}^{t_1}\int_{\Omega'} [(\Delta_h D_\alpha u^j)\zeta + (\Delta_h u^j)D_a\zeta]^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.
\end{aligned}$$

Hence (2.5) holds. □

From (2.5) it follows that

$$\int_{t_0}^{t_1}\int_{\Omega''} |\Delta_h u|^2\,\mathrm{d}x\,\mathrm{d}t \le c\left(1 + \int_Q (|u|^{2(n+2)/n} + |\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}t\right)h \tag{2.6}$$

for all $0 < h < T - t_1$. Based on this estimate we have

Proposition 1. *Assume* (1.2)–(1.4). *Let* $u \in V_2^{1,0}(Q;\mathbb{R}^N)$ *be a weak solution to* (1.1). *Then, for any* $\Omega' \subset\subset \Omega$ *and* $0 < t_0 < t_1 < T$,

$$\int_{t_0}^{t_1}\int_{\Omega'} |\Delta_h u|^2\,\mathrm{d}x\,\mathrm{d}t \le c\,h^{1+\mu}, \tag{2.7}$$

$$\operatorname*{ess\,sup}_{(t_0,t_1)} \int_{\Omega'} |\Delta_h u|^2\,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\Omega'} |\Delta_h \nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le c\,h^{2\mu} \tag{2.8}$$

for all $0 < h < T - t_1$ $(c = \text{const})$.

Proof: Let $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, $0 < t_0' < t_0 < t_1 < T$. Let $\zeta \in C_c^\infty(\Omega')$ be a cut-off function such that $0 \le \zeta \le 1$ in Ω', $\zeta \equiv 1$ on Ω'', and let $\rho \in C^\infty(\mathbb{R})$ satisfy $\rho \equiv 0$ in $(-\infty, t_0']$, $\rho \equiv 1$ in $(t_0, +\infty)$ and $0 \le \rho \le 1$ in $\mathbb{R}$. Let $0 < h < T - t_1$.

We form the difference Δ_h in (2.4) for a.a. $t \in (t_0', t_1)$ [3] $(0 < \lambda < T - t_1 - h)$, insert $\psi(x) = (\Delta_h u)(x,t)\zeta^2(x)\rho^2(t)$ into (2.4), integrate over the interval (t_0', t) $(t \in (t_0', t_1))$ and let tend $\lambda \to 0$. It follows that

$$
\tag{2.9}
\begin{aligned}
&\frac{1}{2}\int\limits_\Omega |\Delta_h u(x,t)|^2 \zeta^2(x)\,\mathrm{d}x \rho^2(t) + \int\limits_{t_0'}^{t}\int\limits_\Omega (\Delta_h a_j^\alpha)(\Delta_h D_\alpha u^j)\zeta^2\rho^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = -2\int\limits_{t_0'}^{t}\int\limits_\Omega (\Delta_h a_j^\alpha)(\Delta_h u^j)\zeta(D_\alpha\zeta)\rho^2\,\mathrm{d}x\,\mathrm{d}s + \int\limits_{t_0'}^{t}\int\limits_\Omega |\Delta_h u|^2\zeta^2\rho\rho'\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

By (1.3) and (1.4),

$$
\begin{aligned}
&(\Delta_h a_j^\alpha)(\Delta_h D_\alpha u^j) \geq \\
&\geq \frac{\nu_0}{2}|\Delta_h \nabla u|^2 - c\Big\{h^{2\mu}(1 + |u(x,t)|^{2(n+2)/n} + |u(x,t+h)|^{2(n+2)/n} \\
&\qquad\qquad + |\nabla u(x,t)|^2 + |\nabla u(x,t+h)|^2) + |\Delta_h u|^2\Big\}
\end{aligned}
$$

for a.a. $(x,t) \in \Omega \times (t_0', t_1)$. The first integral on the right of (2.9) can be estimated by the aid of (1.3). Thus,

$$
\tag{2.10}
\begin{aligned}
&\frac{1}{2}\int\limits_\Omega |\Delta_h u(x,t)|^2\zeta^2(x)\,\mathrm{d}x\rho^2(t) + \frac{\nu_0}{2}\int\limits_{t_0'}^{t}\int\limits_\Omega |\Delta_h \nabla u|^2\zeta^2\rho^2\,\mathrm{d}x\,\mathrm{d}s \\
&\leq c\,h^{2\mu}\int\limits_Q (1 + |u|^{2(n+2)/n} + |\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + c(1 + \max|\nabla\zeta|^2 + \max(\rho')^2)\int\limits_{t_0'}^{t_1}\int\limits_{\Omega'} |\Delta_h u|^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

for a.a. $t \in (t_0', t_1)$. Now we insert (2.6) (with t_0' in place of t_0, Ω' in place of Ω'') to the right-hand side of the latter inequality to obtain

$$
\tag{2.11}
\operatorname*{ess\,sup}_{(t_0,t_1)}\int\limits_{\Omega''} |\Delta_h u|^2\,\mathrm{d}x + \int\limits_{t_0}^{t_1}\int\limits_{\Omega''} |\Delta_h \nabla u|^2\,\mathrm{d}x\,\mathrm{d}s \leq c(h^{2\mu} + h).
$$

Next, given any $\Omega''' \subset\subset \Omega''$ we combine the inequality just obtained and (2.5) (with Ω''' in place of Ω'', Ω'' in place of Ω'). Hence

$$
\tag{2.12}
\int\limits_{t_0}^{t_1}\int\limits_{\Omega'''} |\Delta_h u|^2\,\mathrm{d}x\,\mathrm{d}t \leq c(h^\mu + h^{1/2})h.
$$

[3] Notice that $\Delta_h f_\lambda = (\Delta_h f)_\lambda$.

If $0 < \mu \le \frac{1}{2}$ we have finished (i.e. (2.7) and (2.8) hold with Ω'''). However, if $\frac{1}{2} < \mu < 1$ we consider (2.10) with Ω''' in place of Ω' and insert (2.12) therein. We obtain estimates of the type (2.11) and (2.12) with appropriately chosen subdomains of Ω''' and right-hand sides $c(h^{2\mu} + h^{1+1/2})$ and $c(h^{\mu} + h^{(1+1/2)/2})h$, respectively. Clearly, after a finite number of steps,

$$\sum_{k=0}^{m} \left(\frac{1}{2}\right)^k \ge 1 + \mu.$$

□

3. Existence and regularity of $\frac{\partial u}{\partial t}$

Let $0 < t_0 < t_1 < T$ and $\rho \in C_c^\infty((t_0, t_1))$, $0 \le \rho \le 1$ on (t_0, t_1) be fixed. Given $\varphi \in W_2^{1,1}(Q;\mathbb{R}^N)$, $\operatorname{supp}(\varphi) \subset Q$ we replace φ in (1.6) by $\varphi\rho$ to obtain

$$-\int\limits_Q u^j \rho \frac{\partial \varphi^j}{\partial t}\, \mathrm{d}x\, \mathrm{d}t = -\int\limits_Q a_j^\alpha \rho D_\alpha \varphi^j\, \mathrm{d}x\, \mathrm{d}t + \int\limits_Q u^j \rho' \varphi^j\, \mathrm{d}x\, \mathrm{d}t \tag{3.1}$$

$\Big($where $\rho' = \dfrac{\mathrm{d}\rho}{\mathrm{d}t}\Big)$.

Define

$$v(x,t) = \begin{cases} u(x,t)\rho(t) & \text{for a.a. } (x,t) \in \Omega \times (t_0,t_1), \\ 0 & \text{for a.a. } (x,t) \in \Omega \times (\mathbb{R} \setminus (t_0,t_1)), \end{cases}$$

$$w(x,t) = \begin{cases} u(x,t)\rho'(t) & \text{for a.a. } (x,t) \in \Omega \times (t_0,t_1), \\ 0 & \text{for a.a. } (x,t) \in \Omega \times (\mathbb{R} \setminus (t_0,t_1)), \end{cases}$$

$$\tilde{a}_j^\alpha(x,t,u,\xi) = \begin{cases} a_j^\alpha(x,t,u,\xi)\rho(t) & \text{for a.a. } (x,t) \in \Omega \times (t_0,t_1), \\ 0 & \text{for a.a. } (x,t) \in \Omega \times (\mathbb{R} \setminus (t_0,t_1)) \\ & \forall\, u \in \mathbb{R}^N,\ \forall\, \xi \in \mathbb{R}^{nN}. \end{cases}$$

Then (3.1) takes the form

$$-\int\limits_Q v^j \frac{\partial \varphi^j}{\partial t}\, \mathrm{d}x\, \mathrm{d}t = -\int\limits_Q \tilde{a}_j^\alpha D_\alpha \varphi^j\, \mathrm{d}x\, \mathrm{d}t + \int\limits_Q w^j \varphi^j\, \mathrm{d}x\, \mathrm{d}t. \tag{3.2}$$

□

Let $\Omega' \subset\subset \Omega$ (without loss of generality we may assume that $\partial\Omega'$ is smooth). By introducing the Steklov average as above, from (3.2) it follows that

$$\begin{cases} \displaystyle\int\limits_{\Omega'} \frac{\partial v_\lambda^j}{\partial t} \psi^j\, \mathrm{d}x = -\int\limits_{\Omega'} (\tilde{a}_j^\alpha)_\lambda D_\alpha \psi^j\, \mathrm{d}x + \int\limits_{\Omega'} w_\lambda^j \psi^j\, \mathrm{d}x \\ \text{for a.a. } t \in (0,t_1),\ \forall\, \psi \in \overset{\circ}{W}{}_2^1(\Omega';\mathbb{R}^N),\ \forall\, 0 < \lambda < T - t_1. \end{cases} \tag{3.3}$$

Clearly,

$$v_\lambda^j = w_\lambda^j = (\tilde{a}_j^\alpha)_\lambda = 0 \quad \text{for a.a.} \quad (x,t) \in \Omega' \times ((-\infty,0) \cup (t_1,+\infty))$$

for all $0 < \lambda < \min\{t_0, T-t_1\}$ ($j = 1,\dots,N$; $\alpha = 1,\dots,n$). Thus, (3.3) is equivalent to

$$(3.3') \qquad \begin{cases} \int\limits_{\Omega'} \frac{\partial v_\lambda^j}{\partial t} \psi^j \,\mathrm{d}x = -\int\limits_{\Omega'} (\tilde{a}_j^\alpha)_\lambda D_\alpha \psi^j \,\mathrm{d}x + \int\limits_{\Omega'} w_\lambda^j \psi^j \,\mathrm{d}x \\ \text{for a.a.} \quad t \in \mathbb{R},\ \forall\, \psi \in \overset{\circ}{W}{}_2^1(\Omega';\mathbb{R}^N),\ \forall\, 0 < \lambda < \min\{t_0, T-t_1\}. \end{cases}$$

Let $0 < h < T - t_1$ and $0 < \lambda < \min\{t_0, T-t_1-h\}$. Then from (3.3') we obtain

$$(3.4) \qquad \begin{aligned} &\int\limits_{\mathbb{R}} \int\limits_{\Omega'} \Big(\frac{\partial}{\partial t} \Delta_h v_\lambda^j\Big) \varphi^j \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int\limits_{\mathbb{R}} \int\limits_{\Omega'} (\Delta_h (\tilde{a}_j^\alpha)_\lambda) D_\alpha \varphi^j \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}} \int\limits_{\Omega'} (\Delta_h w_\lambda^j) \varphi^j \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for all $\varphi \in L^2(\mathbb{R}; \overset{\circ}{W}{}_2^1(\Omega';\mathbb{R}^N))$. □

Let $\varphi \in L^2(\mathbb{R}; \overset{\circ}{W}{}_2^1(\Omega';\mathbb{C}^N))$. Then (3.4) is separately true for the real and the imaginary part of φ, and thus for $\bar{\varphi}$ (= the conjugate complex of φ).

In what follows, we identify real valued functions in the canonical way with complex valued functions. □

Let H be a complex Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$. The Fourier transform of $\varphi \in L^1(\mathbb{R};H) \cap L^2(\mathbb{R};H)$ is defined by

$$(\mathcal{F}\varphi)(t) = \hat{\varphi}(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-it\tau} \varphi(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R}.$$

We note that $\mathcal{F}$ is a unitary mapping on the dense subset of all step functions in $L^2(\mathbb{R};H)$; then $\mathcal{F}$ may be defined on the whole space $L^2(\mathbb{R};H)$ by continuous extension.

The inverse Fourier transform of $\varphi \in L^2(\mathbb{R};H) \cap L^1(\mathbb{R};H)$ is given by

$$(\mathcal{F}^{-1}\varphi)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{it\tau} \varphi(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R};$$

there holds

$$\mathcal{F} \circ \mathcal{F}^{-1} = \mathcal{F}^{-1} \circ \mathcal{F} = \text{ id}.$$

Let $0 < \theta < 1$. We define:

$$H^\theta(\mathbb{R}; H) = \left\{ \varphi \in L^2(\mathbb{R}; H) \,\Big|\, \int_{\mathbb{R}} (1 + |t|^{2\theta}) \|\hat{\varphi}(t)\|^2 \, \mathrm{d}t < +\infty \right\}.$$

Clearly, $H^\theta(\mathbb{R}; H)$ is a Hilbert space with respect to the scalar product

$$(\varphi, \psi)_{H^\theta(\mathbb{R};H)} = \int_{\mathbb{R}} (1 + |t|^{2\theta}) (\hat{\varphi}(t), \hat{\psi}(t)) \, \mathrm{d}t.$$

It is well known that

$$\int_{\mathbb{R}} (1 + |t|^{2\theta}) \|\hat{\varphi}(t)\|^2 \, \mathrm{d}t = \int_{\mathbb{R}} \|\varphi(t)\|^2 \, \mathrm{d}t + c_\theta \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|\varphi(s) - \varphi(t)\|^2}{|s - t|^{1+2\theta}} \, \mathrm{d}s \, \mathrm{d}t,$$

where

$$\frac{1}{c_\theta} = 2 \int_{\mathbb{R}} \frac{1 - \cos t}{|t|^{1+2\theta}} \, \mathrm{d}t \qquad (0 < \theta < 1),$$

and

$$H^\theta(\mathbb{R}; H) \subset L^{2/(1-2\theta)}(\mathbb{R}; H) \ \text{ continuously } \ \left(0 < \theta < \frac{1}{2}\right).$$

Finally, if $\varphi \in L^2(a, b; H)$ $(-\infty < a < b < +\infty)$ and $q > 0$ then

$$\int_a^b \int_a^b \frac{\|\varphi(s) - \varphi(t)\|^2}{|s - t|^q} \mathrm{d}s \, \mathrm{d}t = 2 \int_0^{b-a} \frac{1}{h^q} \left(\int_a^{b-a} \|\varphi(t + h) - \varphi(t)\|^2 \, \mathrm{d}t \right) \mathrm{d}h.$$

□

To proceed we make use of (2.7) to obtain

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\Omega'} |\Delta_h v|^2 \, \mathrm{d}x \, \mathrm{d}t &= \int_0^{t_1} \int_{\Omega'} |\Delta_h v|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\le 2 \int_{t_0-h}^{t_1-h} \int_{\Omega'} |\Delta_h u|^2 \, \mathrm{d}x \, \mathrm{d}t + 2h^2 \max(\rho')^2 \int_{t_0-h}^{t_1-h} \int_{\Omega'} |u|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\le c\, h^{1+\mu}
\end{aligned}$$

for all $0 < h < \frac{1}{2} \min\{t_0, T - t_1\}$. Thus, $v \in H^{1/2}(\mathbb{R}; L^2(\Omega'; \mathbb{C}^N))$.

Next, observing the Plancherel formula and that

$$\widehat{\frac{\mathrm{d}}{\mathrm{d}t}\Delta_h v_\lambda}(t) = it\,\widehat{\Delta_h v_\lambda}(t) \quad \text{for a.a.} \quad t \in \mathbb{R},$$

we find

$$\int_{\mathbb{R}}\int_{\Omega'} \Big(\frac{\partial}{\partial t}\Delta_h v_\lambda^j\Big)\bar{\varphi}^j \,\mathrm{d}x\,\mathrm{d}t = i\int_{\mathbb{R}} t(\widehat{\Delta_h v_\lambda}, \hat{\varphi})_{L^2}\,\mathrm{d}t \;^{4}$$

for all $\varphi \in L^2(\mathbb{R}; \overset{\circ}{W}{}^1_2(\Omega'; \mathbb{C}^N))$ and all $0 < \lambda < \min\{t_0, T - t_1 - h\}$. Obviously,

$$\widehat{\Delta_h v_\lambda}(t) = \frac{e^{i\lambda t} - 1}{i\lambda t}\widehat{\Delta_h v}(t) \quad \text{for a.a.} \quad t \in \mathbb{R},$$

and thus

$$\widehat{\Delta_h v_\lambda} \longrightarrow \widehat{\Delta_h v} \;\text{ in }\; L^2(\mathbb{R}; L^2(\Omega'; \mathbb{C}^N)) \;\text{ as }\; \lambda \to 0.$$

The passage to the limit $\lambda \to 0$ in (3.4) gives

$$\begin{aligned} &i\int_{\mathbb{R}} t(\widehat{\Delta_h v}, \hat{\varphi})_{L^2}\,\mathrm{d}t \\ &= -\int_{\mathbb{R}}\int_{\Omega'} (\Delta_h \tilde{a}^\alpha_j) D_\alpha \bar{\varphi}^j \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}\int_{\Omega'} (\Delta_h w^j)\bar{\varphi}^j \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.5}$$

for all $\varphi \in L^2(\mathbb{R}; \overset{\circ}{W}{}^1_2(\Omega'; \mathbb{C}^N))$ with $\dfrac{\mathrm{d}\varphi}{\mathrm{d}t} \in L^2(\mathbb{R}; L^2(\Omega'; \mathbb{C}^N))$ $\Big(0 < h < \frac{1}{2}\min\{t_0, T - t_1\}\Big)$. By an approximation argument, (3.5) holds for all $\varphi \in L^2(\mathbb{R}; \overset{\circ}{W}{}^1_2(\Omega'; \mathbb{C}^N)) \cap H^{1/2}(\mathbb{R}; L^2(\Omega'; \mathbb{C}^N))$. □

We are now able to prove

Proposition 2. *Let $2 \le \sigma < 3$. Let (1.2)–(1.4) be satisfied with $\frac{\sigma - 1}{2} < \mu < 1$. Then*

$$\frac{\mathrm{d}v}{\mathrm{d}t} \in L^{2/(3-\sigma)}(\mathbb{R}; L^2(\Omega'; \mathbb{R}^N)) \;^{5}.$$

Proof: We estimate the integrals on the right of (3.5) for any $\varphi \in L^2(\mathbb{R}; \overset{\circ}{W}{}^1_2(\Omega'; \mathbb{C}^N))$. To this end, let $0 < h < \frac{1}{2}\min\{t_0, T - t_1\}$. Firstly, we

[4] By $(\zeta, \eta)_{L^2} = \int_{\Omega'} \zeta(x)\overline{\eta(x)}\,\mathrm{d}x$ we denote the scalar product in $L^2(\Omega'; \mathbb{C}^N)$ ($\Omega' \subset\subset \Omega$ fixed); $\|\cdot\|_{L^2} = (\cdot,\cdot)^{1/2}_{L^2}$.

[5] Here $\frac{\mathrm{d}v}{\mathrm{d}t}$ has to be understood in the sense of vector-valued distributions (cf. e.g. [1, Appendices]).

have

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\Omega'} (\Delta_h \tilde{a}_j^\alpha) D_\alpha \bar{\varphi}^j \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{t_0-h}^{t_1-h}\int_{\Omega'} a_j^\alpha(x, t+h), u(x,t+h), \nabla u(x,t+h))\rho(t+h) D_\alpha \bar{\varphi}^j(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{t_0}^{t_1}\int_{\Omega'} a_j^\alpha(x,t,u(x,t),\nabla u(x,t))\rho(t) D_\alpha \bar{\varphi}^j(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{t_0-h}^{t_1-h}\int_{\Omega'} [a_j^\alpha(x,t+h,u(x,t+h),\nabla u(x,t+h)) \\
&\qquad\qquad - a_j^\alpha(x,t,u(x,t),\nabla u(x,t))]\rho(t+h) D_\alpha \bar{\varphi}^j(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{t_0-h}^{t_1}\int_{\Omega'} a_j^\alpha(x,t,u(x,t),\nabla u(x,t))[\rho(t+h)-\rho(t)] D_\alpha \bar{\varphi}^j(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= I_1 + I_2.
\end{aligned}
$$

To estimate I_1, we make use of (1.3) and (2.8) $\Big($ with $\frac{t_0}{2}$ in place of $t_0\Big)$. It follows that

$$
\begin{aligned}
|I_1| &\le c \int_{t_0-h}^{t_1-h}\int_{\Omega'} \Big\{ h^\mu (1 + |u(x,t)|^{(n+2)/n} + |u(x,t+h)|^{(n+2)/n} \\
&\qquad + |\nabla u(x,t)| + |\nabla u(x,t+h)|) + |\Delta_h u| + |\Delta_h \nabla u| \Big\} |\nabla \varphi| \,\mathrm{d}x\,\mathrm{d}t \\
&\le c \Bigg\{ h^{2\mu} \int_{t_0/2}^{t_1}\int_{\Omega'} (1 + |u|^{2(n+2)/n} + |\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{t_0/2}^{t_1}\int_{\Omega'} (|\Delta_h u|^2 + |\Delta_h \nabla u|^2)\,\mathrm{d}x\,\mathrm{d}t \Bigg\}^{1/2} \Bigg(\int_{\mathbb{R}}\int_{\Omega'} |\nabla \varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{1/2} \\
&\le c\, h^\mu \Bigg(\int_{\mathbb{R}}\int_{\Omega'} |\nabla \varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{1/2}.
\end{aligned}
$$

Clearly,

$$|I_2| \le c \max|\rho'| h \left(1 + \int\limits_{t_0/2}^{t_1} \int\limits_{\Omega'} (|u|^{2(n+2)/2} + |\nabla u|^2)\, \mathrm{d}x\, \mathrm{d}t \right)^{1/2} \times \\ \times \left(\int\limits_{\mathbb{R}} \int\limits_{\Omega'} |\nabla \varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2}.$$

Secondly, using (2.7) (or (2.8)) we find

$$\left| \int\limits_{\mathbb{R}} \int\limits_{\Omega'} (\Delta_h w^j) \bar{\varphi}^j \,\mathrm{d}x\,\mathrm{d}t \right| \le c\, h^{\mu} \left(\int\limits_{\mathbb{R}} \int\limits_{\Omega'} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2}$$

($c =$ const depending on $\max|\rho'|$ and $\max|\rho''|$).

Thus, (3.5) implies

$$\left| \int\limits_{\mathbb{R}} t (\widehat{\Delta_h v}, \hat{\varphi})_{L^2} \,\mathrm{d}t \right| \le c\, h^{\mu} \left(\int\limits_{\mathbb{R}} \int\limits_{\Omega'} (|\varphi|^2 + |\nabla \varphi|^2) \,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \tag{3.6}$$

for all $\varphi \in L^2(\mathbb{R}; \overset{\circ}{W}{}^1_2(\Omega', \mathbb{C}^N)) \cap H^{1/2}(\mathbb{R}; L^2(\Omega'; \mathbb{C}^N))$ and all $0 < h < \frac{1}{2}\min\{t_0, T - t_1\}$. The function $\varphi = \mathcal{F}^{-1}(\mathrm{sign}(\cdot)\widehat{\Delta_h v})$ is admissible in (3.6). We have

$$\begin{aligned} \int\limits_{\mathbb{R}} \int\limits_{\Omega'} (|\varphi|^2 + |\nabla \varphi|^2) \,\mathrm{d}x\,\mathrm{d}t &= \int\limits_{\mathbb{R}} \int\limits_{\Omega'} (|\widehat{\Delta_h v}|^2 + |\nabla (\widehat{\Delta_h v})|^2) \,\mathrm{d}x\,\mathrm{d}t \\ &= \int\limits_{\mathbb{R}} \int\limits_{\Omega'} (|\Delta_h v|^2 + |\Delta_h \nabla v|^2) \,\mathrm{d}x\,\mathrm{d}t \;{}^{6} \\ &\le c\, h^{2\mu} \end{aligned}$$

with $c =$ const depending on $\max|\rho'|$. Here we have used once more (2.8) $\Big($with $\frac{t_0}{2}$ in place of $t_0\Big)$. Observing that $\widehat{\Delta_h v}(t) = (e^{iht} - 1)\hat{v}(t)$ for a.a. $t \in \mathbb{R}$, from (3.6) we deduce

$$\int\limits_{\mathbb{R}} |t|\, |e^{iht} - 1|^2 \, \|\hat{v}(t)\|^2_{L^2} \,\mathrm{d}t \le c\, h^{2\mu} \quad \forall\, 0 < h < \frac{1}{2}\min\{t_0, T - t_1\}. \tag{3.7}$$

[6] We have $\nabla(\widehat{\Delta_h v}) = \widehat{\nabla(\Delta_h v)}$.

This estimate implies the claim of Proposition 2. To see this, set $h_0 = \frac{1}{2}\min\{t_0, T - t_1\}$. Let $2 \le \sigma < 3$. We have

$$\int_0^{h_0} \frac{|e^{iht} - 1|^2}{h^\sigma}\,\mathrm{d}h \ge |t|^{\sigma-1} \int_0^{h_0} \frac{|e^{i\tau} - 1|^2}{\tau^\sigma}\,\mathrm{d}\tau \quad \forall\,|t| \ge 1. \tag{3.8}$$

Now we proceed in two steps. Firstly, (3.7) and (3.8) (with $\sigma = 2$) imply

$$\begin{aligned} c\int_0^{h_0} h^{2(\mu-1)}\,\mathrm{d}h &\ge \int_{\mathbb{R}} |t|\Big(\int_0^{h_0} \frac{|e^{iht} - 1|^2}{h^2}\,\mathrm{d}h\Big)\|\hat{v}(t)\|_{L^2}^2\,\mathrm{d}t \\ &\ge \int_0^{h_0} \frac{|e^{i\tau} - 1|^2}{\tau^2}\,\mathrm{d}\tau \int_{\{t\,|\,|t|\ge 1\}} t^2\|\hat{v}(t)\|_{L^2}^2\,\mathrm{d}t. \end{aligned}$$

Obviously,

$$\int_{\{t\,|\,|t|<1\}} t^2\|\hat{v}(t)\|_{L^2}^2\,\mathrm{d}t \le \int_{\mathbb{R}} \|\hat{v}(t)\|_{L^2}^2\,\mathrm{d}t \le \int_Q |u|^2\,\mathrm{d}x\,\mathrm{d}t,$$

and therefore

$$\int_{\mathbb{R}} t^2\|\hat{v}(t)\|_{L^2}^2\,\mathrm{d}t \le c\left(1 + \int_0^{h_0} h^{2(\mu-1)}\,\mathrm{d}h\right) < +\infty \;^{7}.$$

It is well known that this estimate (together with $v \in L^2(\mathbb{R}; L^2(\Omega'; \mathbb{R}^N)))$ is equivalent to

$$\frac{\mathrm{d}v}{\mathrm{d}t} \in L^2(\mathbb{R}; L^2(\Omega', \mathbb{R}^N)).$$

Secondly, observing that $\widehat{\frac{\mathrm{d}v}{\mathrm{d}t}} = it\hat{v}$ for a.a. $t \in \mathbb{R}$, and combining (3.7) and (3.8) (with $2 < \sigma < 3$) we find

$$\begin{aligned} c\int_0^{h_0} h^{2\mu-\sigma}\,\mathrm{d}h &\ge \int_0^{h_0} \frac{|e^{i\tau} - 1|^2}{\tau^\sigma}\,\mathrm{d}\tau \int_{\{t\,|\,|t|\ge 1\}} |t|^{\sigma-2}\|it\hat{v}(t)\|_{L^2}^2\,\mathrm{d}t \\ &= \int_0^{h_0} \frac{|e^{i\tau} - 1|^2}{\tau^\sigma}\,\mathrm{d}\tau \int_{\{t\,|\,|t|\ge 1\}} |t|^{\sigma-2}\Big\|\widehat{\frac{\mathrm{d}v}{\mathrm{d}t}}(t)\Big\|_{L^2}^2\,\mathrm{d}t. \end{aligned}$$

[7] Recall that $\mu > \frac{\sigma-1}{2} \ge \frac{1}{2}$.

Hence

$$\int_{\mathbb{R}} |t|^{2(\sigma/2-1)} \Big\| \widehat{\frac{\mathrm{d}v}{\mathrm{d}t}}(t) \Big\|_{L^2}^2 \,\mathrm{d}t \le c\bigg(1 + \int_0^{h_0} h^{2\mu-\sigma}\,\mathrm{d}h \bigg) < +\infty$$

(for $2\mu - \sigma > -1$). Thus

$$\frac{\mathrm{d}v}{\mathrm{d}t} \in L^q(\mathbb{R}; L^2(\Omega'; \mathbb{R}^N)), \quad q = \frac{2}{1 - 2(\frac{\sigma}{2} - 1)} = \frac{2}{3-\sigma}$$

(cf. above). □

Let $\Omega' \subset\subset \Omega$, $0 < t_0 < t_1 < T$, and let $\rho \in C_c^\infty((0,T))$ satisfy $0 \le \rho \le 1$ on $(0,T)$, $\rho \equiv 1$ on (t_0, t_1). Let the assumptions of Proposition 2 be fulfilled. Then, for any weak solution $u \in V_2^{1,0}(Q; \mathbb{R}^N)$ to (1.1) we have

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in L^{2/(3-\sigma)}(t_0, t_1; L^2(\Omega'; \mathbb{R}^N)) \quad \Big(2 \le \sigma < 3,\ \frac{\sigma-1}{2} < \mu < 1 \Big). \tag{3.9}$$

Hence u possesses the weak derivative $\frac{\partial u}{\partial t}$ such that

$$\int_{t_0}^{t_1} \Big(\int_{\Omega'} \Big| \frac{\partial u}{\partial t} \Big|^2 \,\mathrm{d}x \Big)^{1/(3-\sigma)} \,\mathrm{d}t < +\infty.$$

□

4. Proof of the Theorem

First of all, from (1.6) we infer

$$\begin{cases} \displaystyle\int_\Omega \frac{\partial u^j}{\partial t}(x,t)\psi^j(x)\,\mathrm{d}x + \int_\Omega a_j^\alpha(x,t,u,\nabla u) D_\alpha \psi^j(x)\,\mathrm{d}x = 0 \\ \text{for a.a. } \ t \in (0,T),\ \forall \psi \in W_2^1(\Omega;\mathbb{R}^N),\ \operatorname{supp}(\psi) \subset \Omega \end{cases} \tag{4.1}$$

(cf. (3.9)).

Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega''' \subset\subset \Omega$ (without loss of generality, we may assume that $\partial\Omega'$ is smooth). Let $\zeta \in C_c^\infty(\Omega''')$ be a cut-off function such that $0 \le \zeta \le 1$ in Ω''', $\zeta \equiv 1$ on Ω''. Inserting $\psi(x) = u(x,t)\zeta^2(x)$ into (4.1) gives

$$\begin{aligned} &\int_{\Omega'''} a_j^\alpha(x,t,u,\nabla u)(D_\alpha u^j)\zeta^2\,\mathrm{d}x \\ &\quad = -\int_{\Omega'''} \frac{\partial u^j}{\partial t} u^j \zeta^2\,\mathrm{d}x - 2\int_{\Omega'''} a_j^\alpha(x,t,u,\nabla u) u^j \zeta D_\alpha\zeta\,\mathrm{d}x, \end{aligned}$$

and therefore

$$\int_{\Omega''} |\nabla u(x,t)|^2 \,\mathrm{d}x \le c\left(1 + \operatorname*{ess\,sup}_{(0,T)} \int_{\Omega} |u|^2 \,\mathrm{d}x + \int_{\Omega'''} \left|\frac{\partial u}{\partial t}(x,t)\right|^2 \mathrm{d}x\right) \tag{4.2}$$

for a.a. $t \in (0,T)$.

On the other hand, for a.a. $t \in (0,T)$, $u(\cdot,t)$ may be considered as weak solution to a nonlinear elliptic system with right-hand side $\frac{\partial u}{\partial t}(\cdot,t) \in L^2(\Omega''', \mathbb{R}^N)$ (recall that $a_j^\alpha(\cdot,0,0,0) \in L^\sigma(\Omega)$ $(\sigma > 2)$; cf. (1.2)).

Thus, by reverse Hölder inequality, there exists a $p > 2$ such that

$$\left(\int_{\Omega'} |\nabla u|^p \,\mathrm{d}x\right)^{1/p} \le c\left\{\int_{\Omega''} \left(|\nabla u|^2 + \left|\frac{\partial u}{\partial t}\right|^2\right) \mathrm{d}x\right\}^{1/2} \quad \text{for a.a. } t \in (0,T), \tag{4.3}$$

where neither p nor $c = \text{const}$ depend on t (cf. [5, pp. 137-139]). Without loss of generality we may assume that $2 < p \le 4$. Now we add

$$\left(\int_{\Omega'} |u|^p \,\mathrm{d}x\right)^{1/p}$$

to both sides of (4.3) and make use of the well-known multiplicative inequalities $(n = 2)$ (cf. e.g. [9]). Thus, combining (4.2) and (4.3) gives

$$\|u(\cdot,t)\|_{W_p^1(\Omega';\mathbb{R}^N)} \le c\left\{1 + \left(\int_{\Omega'''} \left|\frac{\partial u}{\partial t}(x,t)\right|^2 \mathrm{d}x\right)^{1/2}\right\}$$

for a.a. $t \in (0,T)$.

From the Sobolev imbedding theorem $(n = 2)$ we obtain: for a.a. $t \in (0,T)$ there exists a representative $\tilde{u}(\cdot,t) \in u(\cdot,t)$ such that

$$|\tilde{u}(x,t) - \tilde{u}(y,t)| \le c|x-y|^{1-2/p}\left\{1 + \left(\int_{\Omega'''} \left|\frac{\partial u}{\partial t}(z,t)\right|^2 \mathrm{d}z\right)^{1/2}\right\} \quad \forall\, x,y \in \Omega'. \tag{4.4}$$

Define $\mu_0 = 1 - \frac{p-2}{2p}$. Let $\mu_0 < \mu < 1$ in (1.3). We fix $1 + 2\mu_0 < \sigma < 1 + 2\mu$. Then

$$2\left(1 + \frac{1}{p}\right) < \sigma < 3, \quad \frac{\sigma - 1}{2} < \mu, \quad \frac{2}{3-\sigma} > \frac{2}{1 - \frac{2}{p}}$$

and

$$\left(\int\limits_{\Omega'''}\left|\frac{\partial u}{\partial t}(z,\cdot)\right|^2 \mathrm{d}z\right)^{1/2} \in L^{2/(3-\sigma)}(T_0,T_1)$$

for any $0 < T_0 < T_1 < T$ (cf. (3.9)), i.e. (A1) of the appendix below is satisfied with $\alpha = 1 - \frac{2}{p}$.

Finally, given $(x_0,t_0) \in \Omega' \times (T',T_1)$ $(T_0 < T' < T_1)$ and $0 < r < \frac{1}{2}\min\{\mathrm{dist}(\Omega',\partial\Omega'''), \sqrt{T'-T_0}\}$ we have

$$(u^j(x,s) - u^j(x,t))^2 \le r^2 \int\limits_{t_0-r^2}^{t_0} \left(\frac{\partial u^j}{\partial \tau}(x,\tau)\right)^2 \mathrm{d}\tau \qquad (j = 1,\dots,N)$$

for a.a. $x \in B_r(x_0)$ [8] and a.a. $s,t \in (t_0 - r^2, t_0)$. Thus,

$$\begin{aligned}\int\limits_{Q_r} |u(x,s)-u(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t &\le r^4 \int\limits_{t_0-r^2}^{t_0}\int\limits_{B_r}\left|\frac{\partial u}{\partial \tau}\right|^2 \mathrm{d}x\,\mathrm{d}\tau \\ &\le r^{4+2(\sigma-2)}\left\{\int\limits_{T_0}^{T_1}\left(\int\limits_{\Omega'''}\left|\frac{\partial u}{\partial \tau}\right|^2 \mathrm{d}x\right)^{\frac{1}{3-\sigma}} \mathrm{d}\tau\right\}^{3-\sigma},\end{aligned}$$

i.e. (A2) is satisfied with $\beta = \sigma - 2$.

By Lemma 2 (Appendix),

$$u \in C^{\gamma,\gamma/2}(Q;\mathbb{R}^N), \quad \gamma = \sigma - 2\Big(1 + \frac{1}{p}\Big).$$

□

Appendix

Define

$$\begin{aligned}B_r &= B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\},\\ Q_r &= Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0).\end{aligned}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, $-\infty < T_0 < T_1 < +\infty$. Set $Q = \Omega \times (T_0,T_1)$. We have the following

[8] cf. the appendix for the notations.

Lemma 2. *Let $w \in L^2(Q)$. Suppose that for any $\Omega' \subset\subset \Omega$ and $T_0 < T' < T_1$ there holds*

$$\text{(A1)} \quad \begin{cases} \text{for a.a. } t \in (T_0, T_1)\ \exists\, \tilde{w}(\cdot, t) \in w(\cdot, t) : \\ |\tilde{w}(x,t) - \tilde{w}(y,t)| \le |x-y|^\alpha g(t) \quad \forall\, x, y \in \Omega' \\ (0 < \alpha \le 1,\ g \in L^q(T_0, T_1) \Big(q > \frac{2}{\alpha}\Big),\ g(t) \ge 0 \\ \text{for a.a. } t \in (T_0, T_1)), \end{cases}$$

$$\text{(A2)} \quad \begin{cases} \displaystyle\int_{Q_r} (w(x,s) - w(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t \le C_0 r^{n+2+2\beta} \\ \forall\, 0 < r < \frac{1}{2}\min\{\operatorname{dist}(\Omega', \partial\Omega), \sqrt{T' - T_0}\},\ \forall\, (x_0, t_0) \in \\ \Omega' \times (T', T_1) \ \text{and for a.a. } s \in (t_0 - r^2, t_0)\ (0 < \beta < 1)\ ^{9}. \end{cases}$$

Then

$$\text{(A3)} \qquad w \in C^{\gamma, \gamma/2}(Q), \qquad \gamma = \min\Big\{\alpha - \frac{2}{q}, \beta\Big\}.$$

Proof: Let $|E|$ denote the n-dimensional (resp. $(n+1)$-dimensional) Lebesgue measure of a set $E \subset \mathbb{R}^n$ (resp. $E \subset \mathbb{R}^{n+1}$).

Let $(x_0, t_0) \in \Omega' \times (T', T_1)$, $0 < r < \frac{1}{2}\min\{\operatorname{dist}(\Omega', \partial\Omega), \sqrt{T' - T_0}\}$. For any $(x,t) \in Q_r = Q_r(x_0, t_0)$,

$$\begin{aligned} &\tilde{w}(x,t) - \frac{1}{|Q_r|}\int_{Q_r} w(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ \text{(A4)} \quad &= \frac{1}{|B_r|}\int_{B_r} (\tilde{w}(x,t) - \tilde{w}(y,t))\,\mathrm{d}y + \frac{1}{|Q_r|}\int_{Q_r} (w(y,t) - w(y,s))\,\mathrm{d}y\,\mathrm{d}s \\ &= I_1 + I_2. \end{aligned}$$

By (A1),

$$I_1^2 \le \frac{1}{|B_r|}\int_{B_r} (\tilde{w}(x,t) - \tilde{w}(y,t))^2\,\mathrm{d}y \le 2^{2\alpha} r^{2\alpha} (g(t))^2,$$

$$\begin{aligned} \int_{Q_r} I_1^2 \,\mathrm{d}x\,\mathrm{d}t &\le 2^{2\alpha}|B_1| r^{n+2\alpha} \int_{t_0 - r^2}^{t_0} (g(t))^2\,\mathrm{d}t \\ &\le 2^{2\alpha}|B_1| r^{n+2+2(\alpha - 2/q)} \left(\int_{T_0}^{T_1} (g(t))^q\,\mathrm{d}t\right)^{2/q}, \end{aligned}$$

[9] In (A2) the constant C_0 may depend on $\operatorname{dist}(\Omega', \partial\Omega)$ and $T' - T_0$.

and by (A2),

$$\int_{Q_r} I_2^2 \,\mathrm{d}x\,\mathrm{d}t \le C_0 r^{n+2+2\beta}.$$

Thus,

$$\int_{Q_r} \left(w(x,t) - \frac{1}{|Q_r|} \int_{Q_r} w(y,s)\ \mathrm{d}y\ \mathrm{d}s \right)^2 \mathrm{d}x\,\mathrm{d}t$$
$$= \int_{t_0-r^2}^{t_0} \int_{B_r} \left(\tilde{w}(x,t) - \frac{1}{|Q_r|} \int_{Q_r} w(y,s)\,\mathrm{d}y\,\mathrm{d}s \right)^2 \mathrm{d}x\,\mathrm{d}t$$
$$\le c\, r^{n+2+2\gamma}.$$

Then (A3) follows from the well-known integral characterization of Hölder continuous functions (cf. [2], [4]).

□

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Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany

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