On a class of $\overline{\partial}$-equations without solutions

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Abstract. In this note we construct $\overline{\partial}$-equations (inhomogeneous Cauchy-Riemann equations) without solutions. The construction involves Bochner-Martinelli type kernels and differentiation with respect to certain parameters in appropriate directions.

Keywords: $\overline{\partial}$-equation, Bochner-Martinelli type kernels

Classification: 32A25

1. Introduction

If D is an open set in $\mathbb{C}^n$ and f is a C^∞-function in D, one sets $\overline{\partial} f$ to be the $(0,1)$-form $\sum(\partial f/\partial \overline{z}_j)\, d\overline{z}_j$ where

$$\frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right) \quad (z_j = x_j + iy_j,\ x_j, y_j \in \mathbb{R},\ j = 1,\dots,n),$$

and, in general, if $u = \sum f_{j_1\dots j_q}\, d\overline{z}_{j_1} \wedge \dots \wedge d\overline{z}_{j_q}$ is a $(0,q)$-form with C^∞ coefficients in D then $\overline{\partial} u = \sum \overline{\partial} f_{j_1\dots j_q} \wedge d\overline{z}_{j_1} \wedge \dots \wedge d\overline{z}_{j_q}$.

Several constructions in complex analysis are reduced to the $\overline{\partial}$-equation, i.e., given a $(0,q)$-form v (in D) find a $(0,q-1)$-form u so that $\overline{\partial} u = v$; since $\overline{\partial}(\overline{\partial} u) = 0$, a necessary condition that the equation $\overline{\partial} u = v$ have a solution is that $\overline{\partial} v = 0$. It is usual to consider the quotient (the $(0,q)$-$\overline{\partial}$-cohomology)

$$H_{\overline{\partial}}^{(0,q)}(D) = \\ = \{(0,q)\text{-forms } v \text{ in } D \text{ with } \overline{\partial} v = 0\}/\{\overline{\partial} u : u \text{ is a } (0,q-1)\text{-form in } D\}$$

which measures the "insolvability" of the $\overline{\partial}$-equation in D (for $(0,q)$-forms).

If $K \subset \mathbb{C}^n$ is a convex compact set then $H_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - K)$ is infinite dimensional (see for example [4, p. 156]). The proof is based on Martineau's theorem of the representation of the $\overline{\partial}$-cohomology classes as holomorphic functions in an appropriate domain (depending on K). Here we will give a simple proof of this using the Bochner-Martinelli integral. In fact our proof works in more general settings, for example if we replace K by any compact set (not necessarily convex). We also cover the case of $(0,m-1)$-$\overline{\partial}$-cohomology if K is replaced by an appropriate closed neighborhood of some analytic varieties of codimension m.

After this brief introduction we come to the main point of this note which is to construct some classes of $\overline{\partial}$-equations without solution. The simplest such example is the one given in Rudin [6, p. 355]. As we pointed out, our construction involves Bochner-Martinelli type kernels and differentiation, in appropriate directions, with respect to certain parameters. We start by explaining this construction in a simple case and generalizing it gradually to more involved cases.

2. Examples of insolvable $\overline{\partial}$-equations

For $z \neq \zeta$, let us consider the Bochner-Martinelli kernel with singularity at ζ:

$$k(z,\zeta) =: \frac{1}{|z-\zeta|^{2n}} \sum_{j=1}^{n} (-1)^{j-1} (\overline{z}_j - \overline{\zeta}_j)\, d\overline{z}_1 \wedge \ldots (j) \ldots \wedge d\overline{z}_n.$$

We recall its basic properties: it is a $(0, n-1)$-form in $z \in \mathbb{C}^n - \{\zeta\}$, $\overline{\partial}_z k(z,\zeta) = 0$ and reproduces holomorphic functions, i.e., for a holomorphic function f in neighborhood of $\overline{D}$ (D is assumed to be a bounded domain in $\mathbb{C}^n$ with smooth boundary) and $\zeta \in D$,

$$\int_{z \in \partial D} f(z) k(z,\zeta) \wedge \omega(z) = c_n f(\zeta)$$

where $\omega(z) = dz_1 \wedge \ldots \wedge dz_n$ and $c_n = (2\pi i)^n/(n-1)!$ (see Kytmanov [5, Chapter 1]).

Let I be the set of n-tuples $a = (a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are non-negative integers. For every $a \in I$ let us define the differential form η_a by setting

$$\eta_a(z) = \left. \frac{\partial^{a_1+\ldots+a_n} k(z,\zeta)}{\partial \zeta_1^{a_1} \ldots \partial \zeta_n^{a_n}} \right|_{\zeta=0}.$$

Then η_a is a $(0, n-1)$-form with C^∞-coefficients in $\mathbb{C}^n - \{0\}$ where it is $\overline{\partial}$-closed, i.e., $\overline{\partial}\eta_a = 0$; this follows from the fact that $\overline{\partial}_z k(z,\zeta) = 0$.

We claim that for each finite subset $A \subset I$ and any $\lambda_a \in \mathbb{C}$, $a \in A$, the $\overline{\partial}$-equation $\overline{\partial} u = \sum_{a \in A} \lambda_a \eta_a$ has no solution in $\mathbb{C}^n - \{0\}$ unless $\lambda_a = 0$ for all $a \in A$ (of course now we assume that $n \geq 2$). To prove this let us assume that this equation has a solution u; then

$$\sum_{a \in A} \lambda_a \eta_a \wedge \omega = \overline{\partial} u \wedge \omega = d[u \wedge \omega]. \tag{1}$$

On the other hand

$$\int_{z \in \partial B(0,1)} f(z) \eta_a(z) \wedge \omega(z) = c_n \frac{\partial^{a_1+\ldots+a_n} f}{\partial z_1^{a_1} \ldots \partial z_n^{a_n}}(0) \tag{2}$$

for every holomorphic function f in a neighborhood of $\overline{B(0,1)}$, where $B(0,1) = \{z \in \mathbb{C}^n : |z| < 1\}$. Indeed for $\zeta \in B(0,1)$,

$$\int_{z\in\partial B(0,1)} f(z)k(z,\zeta)\wedge\omega(z) = c_n f(\zeta),$$

and applying the differential operator $\partial^{a_1+\ldots+a_n}/\partial\zeta_1^{a_1}\ldots\partial_n^{a_n}$ and evaluating at $\zeta = 0$ we obtain (2).

Next applying (2) with $f = f_\beta(z) = z_1^{\beta_1}\ldots z_n^{\beta_n}$, for each $\beta = (\beta_1,\ldots,\beta_n) \in A$, we obtain

$$\int_{z\in\partial B(0,1)} f_\beta(z)\eta_a(z)\wedge\omega(z) = \begin{cases} 0 & \text{if } a \neq \beta \\ c_n\beta_1!\ldots\beta_n! & \text{if } a = \beta. \end{cases} \tag{3}$$

But (1) and Stokes's theorem give

$$\int_{\partial B(0,1)} \sum_{a\in A} \lambda_a f_\beta \eta_a \wedge \omega = \int_{\partial B(0,1)} d[f_\beta u \wedge \omega] = 0.$$

The above equation, taking into consideration (3), gives that $\lambda_\beta = 0$, and the claim follows.

It follows from what we have just proved that the set $\{[\eta_a] : a \in I\}$ is linearly independent in $H_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\})$; it follows that these vector spaces are infinite dimensional: $\dim H_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\}) = \infty$. More generally if $K \subset \mathbb{C}^n$ is a compact set with $0 \in K$ then the cohomology classes $\{[\eta_a] : a \in I\}$ are linearly independent in $H_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - K)$. This can be proved in the same exactly way replacing the sphere $S(0,1)$ by a sufficiently large sphere surrounding the compact set K. In particular $\dim H_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - K) = \infty$; as we said in the introduction, in the case K is a convex compact set in $\mathbb{C}^n$, this is proved (by a different method) in Henkin-Leiterer [4, p. 156].

Now we are going to generalize the above construction by replacing the set $\mathbb{C}^n - \{0\} = \mathbb{C}^n - \{z_1 = z_2 = \ldots = z_n = 0\}$ by a more general set of the form $\mathbb{C}^n - \{z \in \mathbb{C}^n : h_1(z) = \ldots = h_m(z) = 0\}$ (with $h_j \in O(\mathbb{C}^n)$ and $m \leq n$) and the set $\mathbb{C}^n - K$ by a set of the form $D - A$, where $D \subset \mathbb{C}^n$ is open and A is an appropriate closed set of D and satisfying a simple geometric condition; moreover we will construct explicitly an infinite set of linearly independent cohomology classes in $H_{\overline{\partial}}^{(0,m-1)}(D - A)$ (thus giving a large class of $\overline{\partial}$-equations without solutions). More precisely we will prove the following

Theorem 1. *Let D be a domain in $\mathbb{C}^n$, $h_1,\ldots,h_m \in O(D)$ holomorphic functions on D and $V = \{z \in D : h_1(z) = \ldots = h_m(z) = 0\}$. Suppose $A \subset D$ is a closed* (*in D*) *subset, containing V, such that there exist a point $p \in V$ which* *is a regular point of* V *(in the sense that* $dh_1 \wedge \ldots \wedge dh_m \neq 0$ *at the point* p*) and a complex submanifold* X *of* D *of dimension* m*, meeting* V *only at* p *and transversally, so that* $A \cap X$ *is a compact set. Then* $\dim H_{\overline{\partial}}^{(0,m-1)}(D-A) = \infty$.

Proof: Let us consider the $(0, m-1)$-form:

$$\theta(z,\zeta) = \frac{\sum_{j=1}^{m} (-1)^{j-1} (\overline{h}_j(z) - \overline{h}_j(\zeta)) \overline{\partial h_1}(z) \wedge \ldots (j) \ldots \wedge \overline{\partial h_m}(z)}{\left[\sum_{j=1}^{m} |h_j(z) - h_j(\zeta|^2 \right]^m};$$

this is a $(0, m-1)$-form in z defined for $z \in D - \{z \in D : h_j(z) = h_j(\zeta),\ 1 \leq j \leq m\}$; its coefficients depend on the parameter $\zeta \in D$.

Also let us consider a holomorphic vector field $w(\zeta)$ tangent to X in a neighborhood of p with $w(p) \neq 0$. Then express $w(\zeta)$ in terms of the basic fields $\partial/\partial\zeta_1, \ldots, \partial/\partial\zeta_n$ to obtain

$$w(\zeta) = \sum_{j=1}^{n} c_j(\zeta) \left(\frac{\partial}{\partial \zeta_j} \right)_\zeta$$

for some holomorphic functions $c_j(\zeta)$ in a neighborhood of p. Since $w(p) \neq 0$ we may assume that $c_1(p) \neq 0$ and hence $c_1(\zeta) \neq 0$ in a neighborhood U of p; in U define

$$\xi(\zeta) = \left(\frac{\partial}{\partial \zeta_1} \right)_\zeta + \sum_{j=2}^{n} \frac{c_j(\zeta)}{c_1(\zeta)} \left(\frac{\partial}{\partial \zeta_j} \right)_\zeta .$$

Now for $k = 0, 1, 2, \ldots$, define η_k by the formula

$$\eta_k(z) = \xi_\zeta^k \theta(z,\zeta)|_{\zeta=p};$$

here ξ_ζ^k acts in the variable ζ on each coefficient of the form $\theta(z,\zeta)$. It is clear that η_k is a $(0, m-1)$-form with C^∞ coefficients on $D - V \supset D - A$. Moreover $\overline{\partial}_z \theta(z,\zeta) = 0$ for each fixed ζ; this is a straightforward computation. Therefore $\overline{\partial} \eta_k = 0$ in $D - A$ for every k and consequently the $(0, m-1)$-forms η_k define cohomology classes in $H_{\overline{\partial}}^{(0,m-1)}(D-A)$, denoted by $[\eta_k]$.
We will prove that the set $\{[\eta_k] : k = 0, 1, 2, \ldots\}$ is $\mathbb{C}$-linearly independent in $H_{\overline{\partial}}^{(0,m-1)}(D-A)$. For this let $\lambda_0, \ldots, \lambda_N \in \mathbb{C}$ so that

$$\sum_{k=0}^{N} \lambda_k [\eta_k] = 0 \text{ in } H_{\overline{\partial}}^{(0,m-1)}(D-A);$$

this means that there is a $(0, m-2)$-form u with C^∞ coefficients in $D - A$ such that

$$\sum_{k=0}^{N} \lambda_k \eta_k = \overline{\partial} u \text{ in } D - A \quad \text{(here we assume } m \geq 2).$$

Since $X \cap A$ is compact there is a domain $G \subset\subset X$ with smooth boundary which contains $X \cap A$, i.e., $\partial G \subset X - A$.
Then, for every holomorphic function f on D, we have

$$\int_{\partial G} f \overline{\partial} u \wedge \omega(h) = \int_{\partial G} d\,[f u \wedge \omega(h)] = 0,$$

by Stokes's theorem, where $\omega(h) = \partial h_1 \wedge \ldots \wedge \partial h_m$. Therefore

$$\sum_{k=0}^{N} \lambda_k \int_{\partial G} f \eta_k \wedge \omega(h) = 0. \tag{4}$$

Since $d[f\eta_k \wedge \omega(h)] = 0$ (with differential forms restricted to X) we have

$$\int_{\partial G} f \eta_k \wedge \omega(h) = \int_{\partial B} f \eta_k \wedge \omega(h) \tag{5}$$

where $B \subset X$ is a small domain with smooth boundary containing the point p; here we used also our assumption that $X \cap V = \{p\}$.
On the other hand

$$\int_{\partial B} f \eta_k \wedge \omega(h) = c_m \xi^k f(p) \tag{6}$$

where $c_m = (2\pi i)^m/(m-1)!$, provided that B is sufficiently small.
Indeed since p is a regular point of V and X meets V at p and transversally it follows that B can be chosen sufficiently small so that for each $\zeta \in B$ the map:

$$B \ni z \to (h_1(z) - h_1(\zeta), \ldots, h_m(z) - h_m(\zeta))$$

is one-to-one; hence the multiplicity of the zero ζ of this map is 1. Thus by [1, p. 25]

$$\int_{z \in \partial B} f(z) \theta(\zeta, z) \wedge \omega(h)(z) = c_m f(\zeta) \quad \text{for} \quad \zeta \in B.$$

Applying ξ_ζ^k to both sides of this equation (this can be done since ξ_ζ are tangent to X near the point p) and evaluating at $\zeta = p$ we obtain (3).
Now apply (4) and (5) with $f = (z_1 - p_1)^s$ for $s = 0, 1, 2, \ldots$ to obtain

$$\sum_{k=0}^{N} \lambda_k \int_{\partial B} (z_1 - p_1)^s \eta_k \wedge \omega(h) = 0, s = 0, 1, 2, \ldots\ . \tag{7}$$

But by (6)

$$\int_{\partial B}(z_1-p_1)^s\eta_k\wedge\omega(h)=c_m\left(\xi^k(\zeta_1-p_1)^s\right)\Big|_{\zeta=p}. \tag{8}$$

Also, as a simple computation shows,

$$\xi^k=\frac{\partial^k}{\partial\zeta_1^k}+L_k$$

where L_k is a differential operator with the property $L_k[(\zeta_1-p_1)^s]|_{\zeta=p}=0$ for all s. Hence

$$\left(\xi^k(\zeta_1-p_1)^s\right)\Big|_{\zeta=p}=\frac{\partial^k}{\partial\zeta_1^k}\left((\zeta_1-p_1)^s\right)|_{\zeta=p}=\begin{cases}k! & \text{if } k=s\\ 0 & \text{if } k\neq s.\end{cases}$$

This, combined with (7) and (8), gives that $\lambda_s=0$, $s=0,1,\dots,N$.
Thus the linear independence of the classes $[\eta_k]$, $k=0,1,2,\dots$ has been established and the proof is complete.

□

Remarks. (i) With the notation of Theorem 1, suppose that the Jacobian $\partial(h_1,\dots,h_m)/\partial(z_1,\dots,z_m)$ is different from zero at the point p of V. Then $X=\{z\in D: z_{m+1}=p_{m+1},\dots,z_n=p_n\}$ meets V at p transversally and if the closed subset $A\subset D$, which contains V, is such that there is a neighborhood U of p and a compact subset $K\subset X\cap U$ so that $p\in K$ and $X\cap U-K\subset D-A$, then dim $H_{\overline{\partial}}^{(0,m-1)}(D-A)=\infty$. In fact a similar proof shows that the set $\{[\eta_a]$: where $a=(a_1,\dots,a_m)$ with $a_1,\dots,a_m$ being non-negative integers$\}$, is linearly independent in $H_{\overline{\partial}}^{(0,m-1)}(D-A)$ where the $(0,m-1)$-forms η_a are defined as follows:

$$\eta_a(z)=\frac{\partial^{a_1+\ldots+a_m}\theta(z,\zeta)}{\partial\zeta_1^{a_1}\ldots\partial\zeta_m^{a_m}}\Big|_{\zeta=p}.$$

A special case of this is when $A=V$ and we obtain, in particular, that dim $H_{\overline{\partial}}^{(0,m-1)}(D-V)=\infty$; this is proved (by a different method) in Gunning [2, p. 163].

(ii) Notice that Theorem 1 holds even for $m=1$; then it simply says that dim $O(D-A)=\infty$. Thus Theorem 1 may be considered as a generalization of this fact and the space $H_{\overline{\partial}}^{(0,m-1)}(D-A)$ plays, in a sense, the role of $O(D-A)$.

(iii) A similar construction can also be carried out with the Cauchy-Fantappie type kernels of [3].

3. Replacing D by a complex manifold

Let M be an n-dimensional complex manifold on which global holomorphic functions give coordinates at any point of M, i.e., for any point p of M there exist holomorphic functions $\zeta_1, \dots, \zeta_n$ on M so that $(\zeta_1, \dots, \zeta_n)$ restricted to a sufficiently small neighborhood of p define holomorphic coordinates for M at p. For example a Stein manifold (i.e., a closed submanifold of some $\mathbb{C}^n$) or any open subset of a Stein manifold satisfies this condition. For such manifolds we can prove the following

Theorem 2. *Let M be an n-dimensional complex manifold on which global holomorphic functions give coordinates at any point of M. Let $h_1, \dots, h_m \in O(M)$ be holomorphic functions on M and $V = \{z \in M : h_1(z) = \dots = h_m(z) = 0\}$. Suppose $A \subset M$ is a closed subset, containing V, such that there exist a point $p \in V$ which is a regular point of V and a complex submanifold X of M of dimension m, meeting V only at p and transversally, so that $A \cap X$ is a compact set. Then* $\dim H^{(0,m-1)}_{\overline{\partial}}(M - A) = \infty$.

Proof: Choose functions $\zeta_1, \dots, \zeta_n$, holomorphic on M, which give coordinates at p and $\zeta_1(p) = \dots = \zeta_n(p) = 0$. Then, as in the proof of Theorem 1, we may choose a holomorphic vector field of the form

$$\xi(\zeta) = \left(\frac{\partial}{\partial \zeta_1}\right)_\zeta + \sum_{j=2}^{n} \frac{c_j(\zeta)}{c_1(\zeta)} \left(\frac{\partial}{\partial \zeta_j}\right)_\zeta$$

in a neighborhood U (in M) of p which is tangential to X at any point of $X \cap U$. Then the proof can be continued as in the case of Theorem 1, using at the end the functions z_1^s, $s = 0, 1, 2 \dots$, which are holomorphic functions on all of M. $\square$

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(*Received* May 13, 1997)