Convex functions with non-Borel set of Gâteaux differentiability points

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Abstract. We show that on every nonseparable Banach space which has a fundamental system (e.g. on every nonseparable weakly compactly generated space, in particular on every nonseparable Hilbert space) there is a convex continuous function f such that the set of its Gâteaux differentiability points is not Borel. Thereby we answer a question of J. Rainwater (1990) and extend, in the same time, a former result of M. Talagrand (1979), who gave an example of such a function f on $\ell^1(\mathfrak{c})$.

Keywords: convex function, Gâteaux differentiability points, Borel set, fundamental system

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0. Introduction

In [18, Theorem 1], M. Talagrand proved the existence of a convex continuous function $f : \ell^1([0,\omega_{\mathfrak{c}}))$ $(= \ell^1(\mathfrak{c})) \to \mathbb{R}$ such that the set of points at which f is Gâteaux differentiable has any prescribed intersection with a fixed one-dimensional subspace of $\ell^1(\mathfrak{c})$.

Hence there is a continuous convex f on $X = \ell^1(\mathfrak{c})$ such that the set $G(f)$ of Gâteaux differentiability points of f is non-Borel. Clearly, f can be chosen even such that $G(f)$ does not have the Baire property in the restricted sense (i.e. $G(f) \cap F$ has not the Baire property in some closed $F \subset X$).

On the other hand, it is known that, for some Banach spaces X, $G(f)$ must be a "nice set" for each continuous convex function f on X:

(i) By the classical Mazur theorem the set $G(f)$ is a residual G_δ set whenever X is a separable Banach space (see [13, Theorem 1.20]).

(ii) If X is a weak Asplund space, then $G(f)$ is residual and thus it has the Baire property.

Rainwater ([15, p. 320]) asked whether the set $G(f)$ is necessarily Borel if $f : X \to \mathbb{R}$ is continuous convex and X is a GDS space. (A space X is GDS if $G(f)$ is dense for every continuous convex function $f : X \to \mathbb{R}$.) Notice that obviously every weak Asplund space is GDS and, of course, $\ell^1(\mathfrak{c})$ is not GDS.

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In our Theorem (see Section 3) we give a negative answer to the above question. In fact, for "almost all" standard nonseparable spaces X, we show that $G(f)$ can be non-Borel and, under continuum hypothesis, even that $G(f)$ may not have the Baire property in the restricted sense.

As Talagrand in [18], we construct a continuous convex function f with any prescribed intersection of $G(f)$ with a fixed one-dimensional subspace of X. However, the method of our construction of f is different.

We leave the following question open.

Question 1. *Is there some nonseparable Banach space X such that the set $G(f)$ has the Baire property in the restricted sense* (*or is even Borel*) *if $f : X \to \mathbb{R}$ is a continuous convex function?*

For example, we do not know whether $C(K)$, with K a Kunen compact, gives the positive answer to Question 1.

We do not know either the answer to the following question which is natural in connection with (ii) above.

Question 2. *Is there a Banach space X which is not a weak Asplund space and such that $G(f)$ has the Baire property for every continuous convex function f on X?*

We conclude the introduction by pointing out some further notation. We write $C(f)$ for the set of all points of continuity of f and $D(f)$ for the set of all points of discontinuity of f. By $f'_G(x)$ we denote the Gâteaux derivative of f at x, and $D^+_h f$ stands for the one-sided derivative of f at x in direction h, i.e. $D^+_h f(x) = \lim\limits_{t\to 0_+} \frac{f(x+th)-f(x)}{t}$.

We also use the notation $Ae = \{ae \mid a \in A\}$ for $e \in X$ and $A \subset \mathbb{R}$.

1. A construction of convex functions

We get our results by using only one method of construction of continuous convex functions which differs from Talagrand's one and originates in [20].

Lemma 1. *Let X be a Banach space, $Y \subset X$ a closed linear subspace of codimension one, $e \in X \setminus Y$ with $\|e\| = 1$, $g : \mathbb{R} \to (Y^*, w^*)$ and $M > 0$ be such that $\|g(r)\| \le M$ for all $r \in \mathbb{R}$. Then there is a continuous convex function $f : X \to \mathbb{R}$ such that $G(f) \cap \mathbb{R}\, e = C(g)e$.*

Proof: Every point $x \in X$ can be uniquely written in the form $x = y_x + r_x e$, where $y_x \in Y$ and $r_x \in \mathbb{R}$. We shall use this notation in what follows.

Let us define $a_r : X \to \mathbb{R}$, a continuous affine function for every $r \in \mathbb{R}$, by

$$a_r(x) = g(r)(y_x) + r^2 + 2r(r_x - r) \quad (= g(r)(y_x) + r_x^2 - (r_x - r)^2). \tag{1}$$

So

$$a_r(x) \le \|g(r)\|\|y_x\| + r_x^2 \le M\|y_x\| + r_x^2 \quad \text{and} \quad a_r(re) = r^2 \tag{2}$$

for every $r \in \mathbb{R}$ and $x \in X$.

Let us define the function $f : X \to \mathbb{R}$ by

$$f(x) = \sup_{r \in \mathbb{R}} a_r(x). \tag{3}$$

Since all a_r are affine, f is convex. We see from (2) that

$$f(x) \leq M\|y_x\| + r_x^2 \tag{4}$$

and so f is locally bounded from above. As such it is a locally Lipschitz function. (It follows e.g. from the fact that due to (3), or the convexity itself, it is also locally bounded from below and we can use [13, Proposition 1.6] and the remark following its proof.) Due to (4), (3), and (2), we have the identity

$$f(re) = r^2 \text{ for every } r \in \mathbb{R}. \tag{5}$$

1. Let $r \in C(g)$ and $h \in X$ be fixed. We shall show that

$$D_h^+ f(re) = g(r)(y_h) + 2rr_h,$$

i.e. that the continuous linear function $\varphi(h) = g(r)(y_h) + 2rr_h$ is the Gâteaux derivative of f at re.

Since $D_h^+ f(re) = 2rr_h$ for $h \in \mathbb{R}e$ by (5) and f is convex and continuous, it suffices to prove that $D_h^+ f(re) = g(r)(h)$ for $h \in Y$ (the sufficiency of this condition is quite simple, it follows easily from the fact that f is Gâteaux differentiable at re if and only if the subdifferential $\partial f(re) \subset X^*$ of f at re contains (at most) one element).

Let $\varepsilon > 0$ be arbitrary. The mapping $g : \mathbb{R} \to (Y^*, w^*)$ is continuous at r and so there is a $\delta > 0$ such that

$$|g(s)(h) - g(r)(h)| < \varepsilon \quad \text{for every} \quad s \in (r - \delta, r + \delta). \tag{6}$$

Hence, for $t > 0$ and $s \in (r - \delta, r + \delta)$, we have, using also (1),

$$a_s(th + re) \leq g(s)(th) + r^2 \leq g(r)(th) + r^2 + t\varepsilon. \tag{7}$$

Otherwise, if $|s - r| \geq \delta$, then the following relations hold.

$$\begin{aligned} a_s(th + re) &= g(s)(th) + r^2 - (s - r)^2 \leq g(s)(th) + r^2 - \delta^2 = \\ &= g(r)(th) + r^2 + t\varepsilon + [g(s)(th) - g(r)(th) - t\varepsilon - \delta^2]. \end{aligned} \tag{8}$$

There is a $\theta = \theta(r, \varepsilon, h) > 0$ such that, for $|t| < \theta$,

$$g(s)(th) - g(r)(th) - t\varepsilon - \delta^2 \leq 2M\theta\|h\| + \theta\varepsilon - \delta^2 \leq 0, \tag{9}$$

and we conclude that, by (7), (8), and (9),

$$a_s(th+re) \le g(r)(th) + r^2 + t\varepsilon \tag{10}$$

for all $s \in \mathbb{R}$ and $|t| < \theta$. By (3) and (10) we get the inequality

$$f(th+re) \le g(r)(th) + r^2 + t\varepsilon \tag{11}$$

for $|t| < \theta$.

The one-sided derivative of the convex function f at re in direction $h \in Y$ can be now, using (5) and (11), estimated by

$$D_h^+ f(re) = \lim_{t\to 0_+} \frac{1}{t}(f(th+re) - f(re)) \le$$
$$\le \lim_{t\to 0_+} \frac{1}{t}(g(r)(th) + r^2 + t\varepsilon - r^2) = g(r)(h) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get

$$D_h^+ f(re) \le g(r)(h). \tag{12}$$

We also have

$$g(r)(h) \ge D_h^+ f(re) \ge -D_{-h}^+ f(re) \ge g(r)(h). \tag{13}$$

The last equality follows from (12) used for $-h$ instead of h. The second inequality uses the convexity of f. Now, (13) gives

$$D_h^+ f(re) = g(r)(h)$$

and we have proved that $re \in G(f)$.

2. Now, let $r \notin C(g)$. Since $a_s + s^2$ is contained in the subdifferential of f at se for $s \in \mathbb{R}$ due to (3) and (5), and g is not continuous at r, we see that the selection of the subdifferential which maps every se to the element $a_s + s^2$ of the subdifferential of f at se is not norm to weak star continuous at re, because $(a_s + s^2) \restriction Y = g(s)$ by (1), and thus the function f is not Gâteaux differentiable at re (see [13, the proof of Proposition 2.8]). $\square$

Now we shall describe a way how to obtain a function $g : \mathbb{R} \to (Y^*, w^*)$ with a prescribed set $D(f)$ of discontinuity points. To this end we need the existence of a special family of functionals in Y^*.

Definition. We say that a set $\mathcal{F} \subset Y^*$ of functionals on the Banach space Y is a *system with property* (Z_0) if every injective sequence of elements of $\mathcal{F}$ converges to zero in (Y^*, w^*). An uncountable system of nonzero functionals of norm at most one, $\mathcal{F} \subset Y^*$, with property (Z_0) is called *a Z-system* or, equivalently, *a system with property (Z).*

Remark 1. If $\mathcal{F}^*$ is an uncountable family of elements of Y^* with property (Z_0), then the family $\mathcal{F} = \{f \in Y^* \mid f = \frac{g}{\max(1, \|g\|)}, g \in \mathcal{F}^* \setminus \{0\}\}$ is a system with property (Z). The cardinality of $\mathcal{F}$ is the same as that of $\mathcal{F}^*$ because the sets of functionals $g \in \mathcal{F}^*$ with norm greater than one and the same $\frac{g}{\|g\|}$ are clearly finite.

Lemma 2. *Let Y be a Banach space and $\mathcal{F} \subset Y^*$ have property (Z). Let $D \subset \mathbb{R}$ be a set of cardinality at most* $\operatorname{card} \mathcal{F}$. *Then there is a mapping $g : \mathbb{R} \to (Y^*, w^*)$ such that $D = D(g)$.*

PROOF: We put $g(r) = 0 \in Y^*$ if $r \notin D$. Let $g \restriction (D \setminus D^o)$ be any one-to-one map of $D \setminus D^o$ into $\mathcal{F}$, and $g(r) = [\chi_{\mathbb{Q}}(r) \min(1, \operatorname{dist}(r, \mathbb{R} \setminus D^o))] \cdot f$ for $r \in D^o$, where f is any element of $Y^* \setminus \{0\}$ with $\|f\| = 1$ and $\chi_{\mathbb{Q}}$ denotes the characteristic function of the set of all rational numbers. (We denote the interior of D by D^o.)

The mapping $g : \mathbb{R} \to (Y^*, w^*)$ is obviously discontinuous at each point $t \in D^o$ and $\|g(r)\| \leq 1$ for $r \in \mathbb{R}$.

Let $t \in D \setminus D^o$. There is a $y \in Y$ such that $g(t)(y) \neq 0$. But $t \in \overline{\mathbb{R} \setminus D}$ and $g(s)(y) = 0$ for $s \notin D$. Hence $g : \mathbb{R} \to (Y^*, w^*)$ is not continuous at t.

Now, let $t \in \mathbb{R} \setminus D$ and so $g(t) = 0$. We have $g(r) = 0$ for $r \notin D$, $\|g(r)\| \leq \operatorname{dist}(r, \mathbb{R} \setminus D^o) \leq |r - t|$ for $r \in D^o$ and $\lim\limits_{r \to t, r \in D \setminus D^o} g(r) = 0$ by property (Z) of $\mathcal{F}$. Therefore g is continuous at t. □

Due to Lemma 1 and Lemma 2 we come immediately to the following conclusion.

Lemma 3. *Let X be a Banach space, Y be a closed linear subspace of X and R be a one-dimensional subspace of X, and let $X = Y \oplus R$. Let $\mathcal{F} \subset Y^*$ be a set with property (Z). Then, for every $D \subset R$ of cardinality at most that of $\mathcal{F}$, there is a convex continuous function $f : X \to \mathbb{R}$ with $R \setminus G(f) = D$.*

PROOF: We use Lemma 2 to get $g : \mathbb{R} \to Y^*$ and we use it as in Lemma 1 to get f with the desired properties. □

Remark 2. Let us point out that our above construction works with small changes as well if we suppose that $X = Y \oplus H$, where H is a nontrivial Hilbert subspace of X, Y is a closed linear subspace of X and $\mathcal{F} \subset Y^*$ is a Z-system. Having a set $D \subset H$ of cardinality at most that of $\mathcal{F}$, we can construct $g : H \to Y^*$ like in Lemma 2 and we get, as in Lemma 1, a convex continuous function $f : X \to \mathbb{R}$ such that $G(f) \cap H = H \setminus D$.

We can see immediately that the orthonormal basis in any nonseparable Hilbert space H gives an example of a set of functionals having property (Z). Thus, by

Lemma 3, for any one-dimensional subspace R and any $D \subset R$ of cardinality at most the weight of H, there is a convex continuous function f with $R \setminus G(f) = D$. If the weight of H is continuum, we get a result analogous to that of M. Talagrand for H instead of $\ell^1([0, \omega_{\mathfrak{c}}))$.

2. Systems with property (Z)

Now, we are going to give several sufficient and several necessary conditions for a Banach space to admit a family of functionals having property (Z) of given cardinality.

We recall that a system $(x_a, f_a)_{a \in A} \subset X \times X^*$ which is *biorthogonal* (i.e. $f_a(x_b) = \delta_{a,b}$ for $a, b \in A$) is called a *fundamental system* of X if the linear span of the set $\{x_a \mid a \in A\}$ is dense in X, i.e. if $\{x_a \mid a \in A\}$ is *complete*.

It is easy to check and well-known (remarked also in [6]) that a system $(x_a)_{a \in A}$ is minimal if and only if there are $f_a, a \in A$, such that $(x_a, f_a)_{a \in A}$ is biorthogonal. Hence there is a fundamental system $(x_a, f_a)_{a \in A}$ on X if and only if there is a minimal system $(x_a)_{a \in A}$ which is complete in X. We recall that $(x_a)_{a \in A}$ is a *minimal system* in a Banach space X if no proper subsystem has the same closed linear span as $\{x_a \mid a \in A\}$ itself (see [6]).

Proposition 1. *Let X be a nonseparable Banach space with a (uncountable) fundamental system $(x_a, f_a)_{a \in A}$. Then there is a system $\mathcal{F}$ with property (Z) and the same cardinality as A.*

Proof: We put $\mathcal{F} = \{\frac{f_a}{\|f_a\|} \mid a \in A\}$ and notice that $\mathcal{F}$ is of the same cardinality as A by the properties of the fundamental system. Now any injective sequence (f_n) of elements of $\mathcal{F}$ converges pointwise to zero on $\{x_a \mid a \in A\}$. Since the functionals in $\mathcal{F}$ are of norm one and the linear span of $\{x_a \mid a \in A\}$ is dense in X, the sequence (f_n) converges pointwise to zero on X and $\mathcal{F}$ has property (Z). □

Proposition 2. *Let X, Y be Banach spaces and $L : X \to Y$ be a continuous linear surjection. If Y^* contains a system $\mathcal{F}_Y$ with property (Z), then X^* contains a system $\mathcal{F}_X$ with property (Z) such that $\operatorname{card} \mathcal{F}_X = \operatorname{card} \mathcal{F}_Y$.*

In particular, if Y is a complemented subspace of X and $\mathcal{F}_Y \subset Y^*$ is a system with property (Z), then there is a system $\mathcal{F}_X \subset X^*$ with property (Z) and the same cardinality as $\mathcal{F}_Y$.

Proof: It suffices to put $\mathcal{F}_X^* = \{f \circ L \mid f \in \mathcal{F}_Y\}$ and use Remark 1. If Y is a complemented subspace of X we take some continuous linear projection of X onto Y for L first. □

Example 1. The Banach space $\ell^\infty(\mathbb{N})$ admits a system of functionals with property (Z) and cardinality $\mathfrak{c}$ because it is proved in [6, Theorem] that $\ell^\infty(\mathbb{N})$ contains a minimal complete system of cardinality $\mathfrak{c}$.

Example 2. Every Banach space that is weakly Lindelöf determined (i.e. the closed dual unit ball is a Corson compact endowed with the weak-star topology, see [1, Proposition 1.2]) admits a fundamental system, even a Markushevich basis ([19, Theorem 2]). Thus also all weakly countably determined or weakly compactly generated spaces as well as all reflexive spaces have a fundamental system.

Example 3. Every space $C(K)$ of continuous functions on a Valdivia compact K has a Markushevich basis and thus also a fundamental system. The existence of a Markushevich basis of $C(K)$ follows by the standard induction procedure from [3, Remark 7.7, p. 256].

Example 4. Unfortunately, there are nonseparable spaces which do not have a fundamental system. The space $\ell^\infty_c(\Gamma)$ of bounded functions having countable support in Γ for any set Γ of cardinality greater than $\mathfrak{c}$ has no fundamental system ([14, Theorem 3]). For more examples of subspaces of $\ell^\infty(\Gamma)$ without a fundamental system see [8, Theorem 1]. Nevertheless, every space $\ell^\infty_c(\Gamma)$ with Γ infinite contains clearly a complemented subspace isometric to $\ell^\infty(\mathbb{N})$ (any restriction of elements of $\ell^\infty_c(\Gamma)$ to a countable subset of Γ is a projection to such a subspace). In Example 1 above we remarked that $\ell^\infty(\mathbb{N})$ admits a system of functionals with property (Z) and cardinality $\mathfrak{c}$. So by Proposition 2 there is a system of functionals on $\ell^\infty_c(\Gamma)$ with property (Z) and cardinality $\mathfrak{c}$.

Example 5. The space $C(K)$ of continuous functions on the Kunen compact, the existence of which was proved under the continuum hypothesis (see e.g. [12, Chapter 7]), is another well-known example of a space having no fundamental system. We shall show that $(C(K)^*, w^*)$ is hereditarily separable and that (therefore) there is no system with property (Z) in $C(K)^*$.

Before it we notice several obvious properties of a system with property (Z_0), and thus also of a system with property (Z), in the following proposition.

Proposition 3. *If $\mathcal{F} \subset X^*$ has property (Z_0), then $(\mathcal{F}, w^*)$ is a discrete space and $K = \mathcal{F} \cup \{0\}$ is a compact and sequentially compact subset of (X^*, w^*).*

Proof: Let us suppose that $0 \neq f \in X^*$ is a w^*-accumulation point of $\mathcal{F}$. We find w^*-open subsets U, V of X^* such that $0 \in U$, $f \in V$ and $U \cap V = \emptyset$. Now $\mathcal{F} \cap V$ is infinite and so we can choose an injective sequence (f_n) in $V \cap \mathcal{F} \subset X^* \setminus U$. The property ($\mathrm{Z}_0$) of $\mathcal{F}$ ensures that (f_n) tends to zero in (X^*, w^*), but this is a contradiction.

It follows that $(\mathcal{F}, w^*)$ is discrete and K is w^*-closed and thus w^*-compact because $K \subset B_{X^*}$. Obviously, K is w^*-sequentially compact. □

Corollary. *If X^* contains a system with property (Z), then (X^*, w^*) is not hereditarily separable.*

Remark 3. Example 6, and the remark before it, show that we cannot write separable instead of hereditarily separable in the above corollary.

Proposition 4. *If K is a nonmetrizable scattered compact space, or equivalently, an uncountable scattered compact space, such that K^n is hereditarily separable for every $n \in \mathbb{N}$, e.g. the Kunen compact considered in Example 5, then $(C(K)^*, w^*)$ is hereditarily separable and therefore there is no system with property (Z) in $C(K)^*$.*

PROOF: Let us define $T : K^{\mathbb{N}} \times \ell^1 \to C(K)^*$ by

$$T((k_i),(a_i)) = \sum_{i\in\mathbb{N}} a_i \delta_{k_i},$$

where δ_x is the Dirac measure in the point $x \in K$. It is well known and easy to prove that the mapping T is surjective as K is scattered.

We shall show that T is continuous if we consider $K^{\mathbb{N}}$ with the product topology, ℓ^1 with the norm topology and $C(K)^*$ with the w^*–topology. Let $\mu = \sum_{i\in\mathbb{N}} a_i\delta_{k_i} \in C(K)^*$, where $a = (a_i)_{i\in\mathbb{N}} \in l^1$ and $k = (k_i)_{i\in\mathbb{N}} \in K^{\mathbb{N}}$. Further let $f \in C(K)$, $\|f\| = 1$ and $\varepsilon > 0$.

There is a number $n \in \mathbb{N}$ such that $\sum_{i>n} |a_i| < \frac{\varepsilon}{5}$. There is an open set $U \subset K^{\mathbb{N}}$ containing k such that

$$(\forall\, (l_i)_{i\in\mathbb{N}} \in U) \;\; (\forall\, i \le n) \;\; (|f(k_i) - f(l_i)|)\|a\| < \frac{\varepsilon}{5}\,.$$

For $((l_i),(b_i)) \in U \times B(a, \frac{\varepsilon}{5})$, where $B(a,r)$ denotes the open ball with center a and radius r, we have

$$\begin{aligned}|(\mu - T((l_i),(b_i)))(f)| &\le \sum_{i\le n} |a_i f(k_i) - b_i f(l_i)| + \sum_{i>n} [|a_i||f(k_i)| + |b_i||f(l_i)|] \le \\ &\le \sum_{i\le n} [(|a_i f(k_i) - a_i f(l_i)| + |a_i f(l_i) - b_i f(l_i)|)] + \sum_{i>n} (|a_i| + |b_i|) \le \\ &\le \sup_{i\le n} |f(k_i) - f(l_i)|\|a\| + \|a-b\|\|f\| + \sum_{i>n} |a_i| + \sum_{i>n} (|a_i| + |b_i - a_i|) < \varepsilon.\end{aligned}$$

Thus T is continuous.

The product $K^{\mathbb{N}}$ is hereditarily separable because each K^n is. Namely, let A be any subset of $K^{\mathbb{N}}$. For any $n \in \mathbb{N}$ we choose a countable set $T_n \subset K^n$ which is dense in the projection $p_n(A)$ of A to the "first n coordinates". Let S_n be an arbitrary at most countable subset of A which projects onto T_n by p_n. Put $S = \bigcup_{n\in\mathbb{N}} S_n$. The set S is obviously at most countable and dense in A.

As ℓ^1 is a separable metric space and $K^{\mathbb{N}}$ is hereditarily separable, $K^{\mathbb{N}} \times \ell^1$ is also hereditarily separable. Namely, let $A \subset K^{\mathbb{N}} \times l^1$ be arbitrary. Let $(B_n)_{n\in\mathbb{N}}$ be a countable base of open sets of ℓ^1 and choose an at most countable dense subset L_n of $\pi_{K^{\mathbb{N}}}((K^{\mathbb{N}} \times B_n) \cap A)$ and $S_n \subset (K^{\mathbb{N}} \times B_n) \cap A$ an at most countable set such that $\pi_{K^{\mathbb{N}}} S_n = L_n$. The set $S = \bigcup_{n\in\mathbb{N}} S_n$ is at most countable and dense in A. So $K^{\mathbb{N}} \times \ell^1$ is hereditarily separable and hence $C(K)^*$ is also w^*–hereditarily separable due to the continuity of T.

Thus there is no system with property (Z) in $(C(K))^*$ by the corollary of Proposition 3. □

We may notice that the space ℓ^∞, which can be identified with the space of continuous functions on the separable compact space $K = \beta\mathbb{N}$, is an example of a Banach space X for which (X^*, w^*) is separable, but a system of functionals in X^* with property (Z) still exists (see Example 1). In this example, K is not hereditarily separable. The next example shows that even the heredity of the separability of K cannot help to exclude the existence of a system with property (Z) in $C(K)^*$.

Example 6. Let K be the "double-arrow" space (see [18], $K = \{x \mid x \in (0,1]\} \cup \{x^+ \mid x \in [0,1)\}$). Then K is a hereditarily separable compact space and there is a system with property (Z) in $(C(K))^*$ (in particular, $(C(K)^*, w^*)$ is not hereditarily separable).

The space $((C(K))^*, w^*)$ is separable since the signed measures with finite support in some countable dense subset of K form a dense subset.

We denote by $\mathcal{F}$ the set $\{\mu_x \mid x \in (0,1)\}$ of all measures $\mu_x = \frac{1}{2}\delta_{x^+} - \frac{1}{2}\delta_x$. The set $\mathcal{F}$ has cardinality of continuum and it has property (Z) because $(f(x^+) - f(x))_{x\in(0,1)} \in c_0((0,1))$ for every $f \in C(K)$. The last statement is a well known and easy fact. We can get the system with property (Z) in $(C(K))^*$ also by Proposition 2 and Example 2 due to the fact that the WCG space $c_0([0,1])$ is a quotient space of $C(K)$.

In our only example of a nonseparable space X with no system of continuous functionals with property (Z), (X^*, w^*) is hereditarily separable. We do not know whether the existence of a system with property (Z) in X^* can be characterized by the existence of a (special) w^*-nonseparable subset of X^*.

Question 3. *Does a system of functionals with property* (Z) *exist in every dual space* (X^*, w^*) *which contains a weak-star closed nonseparable sequentially compact* (*or a weak-star nonseparable compact*) *subset?*

Lemma 4. *Let X be a Banach space, Y a closed subspace of X and $S = X/Y$. Let S be separable* (*or (S^*, w^*) be hereditarily separable*) *and let $\mathcal{F}_X \subset X^*$ be a system with property* (*Z*)*. Then there is a system $\mathcal{F}_Y \subset Y^*$ with property* (*Z*) *and the same cardinality as $\mathcal{F}_X$.*

Proof: It is not difficult to verify that the separability of S implies that (S^*, w^*) is hereditarily separable. Thus we suppose the latter condition in what follows.

We consider the set $\mathcal{F}_Y$ of restrictions of elements of $\mathcal{F}_X$ to Y. It is obvious that it is a system with property (Z_0) and thus, by Remark 1, the only fact to prove is that the cardinalities of $\mathcal{F}_X$ and of $\mathcal{F}_Y$ are equal.

Let $R : X^* \to Y^*$ be the restriction operator.

Let $f \in \mathcal{F}_Y$ be fixed and non-zero. We consider the set $R^{-1}(f)$. It is finite by property (Z).

Every element $g \in R^{-1}(0)$ can be expressed in the form $g = h_g \circ L$, with $h_g \in S^*$ and $L : X \to S$ the quotient map, in a unique form. As $\mathcal{F}_X$ has property (Z), $\mathcal{F}_S = \{h_g \mid g \in \mathcal{F}_X \cap R^{-1}(0)\}$ has (Z_0). Thus, by Remark 1 and the corollary of Proposition 3, $\mathcal{F}_S$ is at most countable and the same is true for $\mathcal{F}_X \cap R^{-1}(0)$.

Hence the (uncountable) cardinalities of $\mathcal{F}_X$ and of $\mathcal{F}_Y$ coincide and the proof is finished. □

Remark 4. In particular, we may apply Lemma 4 when $X = Y \oplus S$, Y, S are closed subspaces of X and S is separable.

The existence of a system of functionals with property (Z) on a nonseparable space seems to be related to the deep Josefson-Niessenzweig theorem, see e.g. [2, Chapter XII]. We are going to give another sufficient condition for the existence of a system with property (Z) which is closely related to a part of the proof of Josefson-Niessenzweig theorem in [2, p. 223].

We formulate first an easy lemma which gives a characterization of Banach spaces X for which a system $\mathcal{F} \subset X^*$ with property (Z) exists.

Lemma 5. *Let X be a Banach space. Let Γ be an uncountable set with cardinality κ. Then there is a subset of X^* with cardinality κ and with property (Z) if and only if there is a continuous linear operator $T : X \to c_0(\Gamma)$ such that $\operatorname{dens} T(X) = \kappa$.*

Proof: Let $\mathcal{F} \subset X^*$ be a family of elements of X^* which has property (Z) and let $\operatorname{card} \mathcal{F} = \kappa$. We define the operator T by $T(x) = \left(f(x)\right)_f \in \mathcal{F}$, for $x \in X$. As $\mathcal{F}$ has the property (Z), we easily obtain that $T(x) \in c_0(\mathcal{F})$ for every $x \in X$ and that $T : X \to c_0(\mathcal{F})$ is a bounded operator. Now let γ be the least ordinal of cardinality κ. For every $f \in \mathcal{F}$ we choose $x_f \in X$ such that $f(x_f) = 1$. By property (Z), the set $A_f = \{g \in \mathcal{F} \mid g(x_f) \neq 0\}$ is obviously at most countable. Now we will construct by induction a transfinite sequence $(f_\alpha)_{\alpha<\gamma} \subset \mathcal{F}$ such that $f_\beta(x_{f_\alpha}) = 0$ for $0 \le \alpha < \beta < \gamma$. We choose an arbitrary $f_0 \in \mathcal{F}$. Now let $\beta < \gamma$ and suppose that $f_\alpha, \alpha < \beta$ are already chosen. Since $\operatorname{card}\left(\bigcup_{\alpha<\beta} A_{f_\alpha}\right) \le \aleph_0 \operatorname{card} \beta \le \max(\aleph_0, \operatorname{card} \beta) < \operatorname{card} \gamma$, we can choose $f_\beta \in \mathcal{F} \setminus \bigcup_{\alpha<\beta} A_{f_\alpha}$. Clearly $f_\beta(x_{f_\alpha}) = 0$ for $0 \le \alpha < \beta < \gamma$. For $0 \le \alpha < \beta < \gamma$ we have $\|T(x_{f_\alpha}) - T(x_{f_\beta})\| \ge \|f_\beta(x_{f_\alpha}) - f_\beta(x_{f_\beta})\| = 1$. Thus $\operatorname{dens} T(X) \ge \operatorname{card} \gamma = \kappa$. Since clearly $\operatorname{dens} T(X) \le \operatorname{dens} c_0(\Gamma) = \kappa$, as $\operatorname{card} \Gamma$ is infinite, we have $\operatorname{dens} T(X) = \kappa$. Identifying Γ and $\mathcal{F}$, we obtain the first implication.

Now suppose that there exists a continuous linear mapping $T : X \to c_0(\Gamma)$ such that $\operatorname{dens} T(X) = \kappa$. Without loss of generality we may suppose that $\|T\| \le 1$. For each $\gamma \in \Gamma$ put $f_\gamma(x) = (T(x))_\gamma$ and $\mathcal{F} = \{f_\gamma \mid \gamma \in \Gamma\} \setminus \{0\}$. The family $\mathcal{F}$ has clearly the property (Z_0) and $\|f\| \le 1$ for every $f \in \mathcal{F}$. We choose $\Gamma^* \subset \Gamma$

such that $f_{\gamma_1} \neq f_{\gamma_2}$ for $\gamma_1 \neq \gamma_2, \gamma_1, \gamma_2 \in \Gamma^*$ and $\mathcal{F} = \{f_\gamma \mid \gamma \in \Gamma^*\}$. Thus the canonical "restriction" mapping $R : c_0(\Gamma) \to c_0(\Gamma^*)$ is clearly an isometry when restricted to $T(X)$ and consequently $\aleph_0 \operatorname{card} \Gamma^* \geq \operatorname{dens} c_0(\Gamma^*) \geq \operatorname{dens}(R \circ T(X)) = \operatorname{dens} T(X) = \kappa$. It implies that $\kappa \geq \operatorname{card} \mathcal{F} = \operatorname{card} \Gamma^* \geq \kappa$, because κ is uncountable, and we are done. □

Proposition 5. *Let X be a Banach space and let $\ell^1(\Gamma)$ be isomorphic to a subspace of X, where Γ is an arbitrary infinite set. Then there is a linear continuous operator $T : X \to c_0(\Gamma)$ such that $\operatorname{dens} T(X) = \operatorname{card} \Gamma$. In particular, if Γ is uncountable, there is a system $\mathcal{F} \subset X^*$ with property (Z) and the same cardinality as Γ.*

Proof: By Lemma 5, it suffices to prove the existence of T with the mentioned properties.

We consider an Abelian group of cardinality $\operatorname{card} \Gamma$ (e.g. the free Abelian group generated by Γ) endowed with the discrete topology. We shall denote it also Γ further on.

In all what follows we mean by 'linear' the respective property of a space or a map related to the field of reals also when we are considering the spaces of complex-valued functions and operators among them.

We need several basic facts from the harmonic analysis which can be found in [17] and [11]. Let G be the dual group to Γ, i.e. the group of all characters of the group Γ. As Γ is discrete, G is compact and Abelian and so there is a Haar probability measure ν on G.

Let F be the Fourier transform on Γ, i.e. $F : f \mapsto \hat{f}$. By [17, Theorem 1.2.4 (d)], $F : \ell^1(\Gamma, \mathbb{C}) \to C_0(G, \mathbb{C})$ is continuous. We denote by $\tilde{F}$ the inverse Fourier transform to F, i.e. $\tilde{F} : f \mapsto \check{f}$ (see [11, 3.1.2]). It follows from [17, Theorem 1.2.4 (d)] that $\tilde{F} : L^1(G, \mathbb{C}, \nu) \to c_0(\Gamma, \mathbb{C})$ is continuous. Since $C_0(G, \mathbb{C}) = C(G, \mathbb{C})$ embeds naturally to $L^1(G, \mathbb{C}, \nu)$ as ν is finite, we get by the inverse formula [11, 31.44 (b), p. 241] that $\tilde{F} \circ F$ is the identity on $\ell^1(\Gamma, \mathbb{C})$.

We identify $\ell^1(\Gamma)$ with its isomorphic copy in X. Let $R : c_0(\Gamma, \mathbb{C}) \to c_0(\Gamma)$ be the operator taking every complex-valued function from $c_0(\Gamma, \mathbb{C})$ to its real part.

We use without mentioning it explicitly the natural embeddings of $\ell^1(\Gamma)$ to $\ell^1(\Gamma, \mathbb{C})$, of $C(G, \mathbb{C})$ to $L^\infty(G, \mathbb{C}, \nu)$, and of $L^\infty(G, \mathbb{C}, \nu)$ to $L^1(G, \mathbb{C}, \nu)$.

Since it is well known that $L^\infty(G, \nu)$ is "injective" in the sense that every bounded linear map from a subspace X_0 of a Banach space X into $L^\infty(G, \nu)$ can be extended to a bounded linear map of X into $L^\infty(G, \nu)$ (see [2, p. 223]), we can find a bounded linear operator $L : X \to L^\infty(G, \mathbb{C}, \nu)$ which extends F by extending the real and imaginary parts of $F : \ell^1(\Gamma) \to L^\infty(G, \mathbb{C}, \nu)$ separately.

We put now $T = R \circ \tilde{F} \circ L$. Notice that $T(\ell^1(\gamma)) = \ell^1(\gamma)$ because of the inversion formula mentioned above, i.e. due to the fact that $\tilde{F} \circ F$ is the identity map on $\ell^1(\Gamma, \mathbb{C})$. It follows that the density of $T(X)$ in $c_0(\Gamma)$ is the cardinality of Γ. It is not smaller because $T(X)$ contains $\ell^1(\Gamma)$ and it is not greater because the cardinality of Γ is infinite and so the density of $c_0(\Gamma)$ equals the cardinality of Γ. □

Remark 5.

(i) Suppose that an infinite set $\mathcal{T}$ of nonzero continuous linear mappings $F : X \to Y$ "having property (Z_0)" is given, i.e.

(Z_0) $\lim_{n\to\infty} F_n(x) = 0$ for every injective sequence $(F_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ and every $x \in X$.

It is easy to see that there exists a system $\mathcal{F} \subset X^*$ which has the property (Z_0) and $\operatorname{card}\mathcal{F} = \operatorname{card}\mathcal{T}$. In fact, we can choose for each $F \in \mathcal{T}$ a $g_F \in X^*$ such that $\|g_F\| = 1$ and $f_F = g_F \circ F \neq 0$. Then clearly $\mathcal{F} = \{f_F \mid F \in \mathcal{T}\}$ has property (Z_0). Suppose that $\operatorname{card}\mathcal{F} < \operatorname{card}\mathcal{T}$. Then there necessarily exists an $f \in \mathcal{F}$ such that $f = f_{F_n}$ for some injective sequence $(F_n)_{n\in\mathbb{N}} \subset \mathcal{T}$. Then, for every $x \in X$, $|f(x)| = |g_{F_n}(F_n(x))| \leq \|F_n(x)\| \to 0$. Consequently, $f = 0$ which is a contradiction.

(ii) Suppose that there exists a projectional resolution of identity $\{P_\alpha \mid \omega_0 \leq \alpha \leq \mu\}$ on a Banach space X (cf. [3, p. 236]) such that the system of operators $\mathcal{T}^* = \{P_{\alpha+1} - P_\alpha \mid \omega_0 \leq \alpha < \mu\}$ is uncountable. By [3, Lemma 1.2 (ii), p. 236], $\mathcal{T} = \mathcal{T}^* \setminus \{0\}$ "has the property (Z_0)" and thus, by (i) and Remark 1, there exists a system $\mathcal{F} \subset X^*$ with property (Z) such that $\operatorname{card}\mathcal{F} = \operatorname{card}\mathcal{T}$.

(iii) However, at present, we do not know any Banach space admitting a system $\mathcal{F} \subset X^*$ with property (Z) to which we could apply the observation (ii) and no other sufficient condition noticed before.

3. The main result

We summarize our results concerning convex functions in the following statement.

Theorem. *Let X, Y be Banach spaces, $L : X \to Y$ be continuous linear and surjective, and one of the following conditions holds:*

(a) *there is a fundamental system of uncountable cardinality κ on Y;*

(b) $\ell^1(\Gamma)$ *is isomorphic to a subspace of Y and the uncountable cardinality of Γ is κ.*

If D is a subset of some one-dimensional subspace R of X with cardinality at most κ, then there is a continuous convex function $f : X \to \mathbb{R}$ such that $R \setminus G(f) = D$.

In particular, there is a continuous convex function $f : X \to \mathbb{R}$ such that $G(f)$ is not Borel. If moreover $\kappa \geq \mathfrak{c}$, then there is a continuous convex function $f : X \to \mathbb{R}$ such that $G(f)$ does not have the Baire property in the restricted sense.

Proof: Whenever Y fulfills (a) or (b), we deduce, from previous propositions, that there is a Z-system of cardinality κ in X^*. We may use Proposition 1 and Proposition 2 if (a) holds, and Proposition 5 and Proposition 2 if (b) is satisfied.

Let Z be a closed subspace of X complemented to R. We use Remark 4 and Lemma 3 to $X = Z \oplus R$ to get a convex continuous function f on X with $G(f) \cap R = R \setminus D$.

To prove the other statements, we may choose any one-dimensional subspace R of X and any subset D of R of cardinality $\aleph_1$ (hence at most κ) which is not Borel. Indeed, if $\aleph_1 = \mathfrak{c}$ we can choose any non-Borel set $D \subset R$ and we may choose any set $D \subset R$ of cardinality $\aleph_1$ if $\aleph_1 < \mathfrak{c}$ (since each uncountable Borel subset of R has cardinality $\mathfrak{c}$). By the preceding we can find a continuous convex function $f : X \to \mathbb{R}$ with $G(f) \cap R = R \setminus D$ which is obviously not Borel.

If moreover $\kappa \geq \mathfrak{c}$, we choose any subset D of the R above such that D, and so also $R \setminus D$, does not have the Baire property in R (so $R \setminus D$ does not have the Baire property in the restricted sense). Now $\operatorname{card} D \leq \kappa$ and we may find f as above. □

Remark 6. We note that Examples 1, 2, 3, 4, 6 describe great number of nonseparable spaces for which Theorem can be applied.

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He also pointed out that the papers [7] and [5] are related to our Section 2.

Indeed, one may check that our Proposition 5 can be proved as follows. Following the first part of the proof of Theorem 4 from [7], we get that there is some quotient space of X which is isomorphic to $\ell^2(\Gamma)$ and thus clearly has a fundamental system of cardinality $\operatorname{card}\Gamma$. Now applying our Proposition 1 and Proposition 2 we get the system $\mathcal{F} \subset X^*$ with property (Z) of cardinality $\operatorname{card}\Gamma$. The mentioned part of the proof of Theorem 4 in [7] is not easy, it is based on [10, Proposition 2.2] and [16, Proposition 1.5].

Recall that Lemma 5 implies that the nonexistence of an (uncountable) system with property (Z) in X^* is equivalent to the fact that the image of X by any continuous linear map T of X to any $c_0(\Gamma)$ is separable. Thus we may notice that our Proposition 4 gives a solution of the first part of the exercise from IV.3 in [5] and that any solution of it gives the nonexistence of a system with property (Z) in X^*, where $X = C(K)$ is the space of continuous functions on the Kunen compact.

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