Remark to dynamic contact problems for bodies with a singular memory

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Abstract. The existence of a solution to the dynamic contact of a body having a singular memory with a rigid undeformable support is proved under some weaker assumption than that in [3].

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In [3] the existence of a weak solution to a nonlinear dynamic boundary contact problem for a body with a singular memory and a rigid obstacle was proved. However, the assumption on the relation between the dimension and the "penalty parameter" α introduced in (1) was redundantly restrictive there, particularly for the three-dimensional case. Here, we relax this relation exploiting the dual estimate of the acceleration (8) like it was done e.g. in [4] and [5]. This leads to a result independent of the dimension of the problem which makes a difference with the case of the domain contact ([2]), where such a dual estimate cannot be applied.

We consider a bounded domain $\Omega \subset \mathbb{R}^N$ occupied by the body made of an elastic material with the memory for which the stress tensor $\widetilde{\sigma} \equiv (\sigma_{ij})$ has for a displacement u the form

$$\sigma_{ij}(u) = \sigma^I_{ij}(u) + \sigma^M_{ij}(u) \quad \text{with} \quad \sigma^I_{ij}(u) = \frac{\partial W}{\partial e_{ij}}(\cdot, \widetilde{e}(u)),$$

$$\sigma^M_{ij}(u)(\tau,\cdot) = \int_0^\tau K(\tau-\ell)\frac{\partial V}{\partial e_{ij}}(\cdot, \widetilde{e}(u(\tau,\cdot)) - \widetilde{e}(u(\ell,\cdot)))\,\mathrm{d}\ell.$$

The small strain tensor $\widetilde{e}(u)$ has components $e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and for $Z = V, W$ the stored energy function $Z\colon \mathbb{R}^{N+N^2} \to \mathbb{R}$ is assumed to be C_2-smooth and to have the partial Hesse matrix strongly elliptic and bounded, i.e.

there are $\beta_0^Z, \beta_1^Z > 0$ such that

$$\beta_0^Z \xi_{ij}\xi_{ij} \le \frac{\partial^2 Z}{\partial e_{ij}\partial e_{kl}}(x,\omega)\,\xi_{ij}\xi_{kl} \quad \text{and}$$
$$\frac{\partial^2 Z}{\partial e_{ij}\partial e_{kl}}(x,\omega)\xi_{ij}\zeta_{kl} \le \beta_1^Z \sqrt{\xi_{ij}\xi_{ij}}\sqrt{\zeta_{kl}\zeta_{kl}}$$

for all symmetric $N\times N$ matrices ξ, ζ, ω and all $x\in\Omega$. Here and in the sequel, the summation convention is consequently used. For $Z = V, W$ we assume, moreover, $Z(\cdot,0)=0$ and $\frac{\partial Z}{\partial e_{ij}}(\cdot,0)=0$ on Ω, $i,j=1,\dots,N$. The kernel K is supposed to have the form

$$\begin{aligned} &K(\tau) = \tau^{-2\alpha}a(\tau) + r(\tau),\ \tau\in\mathbb{R}_+ \equiv \langle 0,+\infty)\ \text{ with }\ \alpha\in(0,1),\\ &K(\tau)=0,\ \tau\le 0. \end{aligned} \tag{1}$$

Both a and r are sufficiently smooth, nonnegative and decreasing functions on $\mathbb{R}_+$ with $a(\tau)>0$ for τ from a neighbourhood of the origin. The boundary Γ of Ω is $C_{1,1}$-smooth and divided into three measurable pairwisely disjoint parts: a contact part Γ_C, where no friction occurs, and the remaining parts Γ_T and Γ_U. We shall solve the problem

$$\begin{aligned} &\ddot u - \frac{\partial}{\partial x_j}\sigma_{ij}(u) = f_i,\ i=1,\dots,N, \quad \text{on}\ \ Q\equiv I\times\Omega,\ I\equiv(0,\mathscr{T}),\\ &u_n\le 0,\ T_n(u)\le 0,\ T_n(u)\,u_n = 0,\quad T_t(u)=0\ \ \text{on}\ \ S_C\equiv I\times\Gamma_C,\\ &T = T_0\ \ \text{on}\ \ S_T\equiv I\times\Gamma_T,\qquad u=U\ \ \text{on}\ \ S_U\equiv I\times\Gamma_U\\ &\text{and}\qquad u(0,\cdot)=u_0,\ \dot u(0,\cdot)=u_1\ \ \text{on}\ \ \Omega. \end{aligned} \tag{2}$$

Here T denotes the boundary stress vector ($T_i(u)\equiv\sigma_{ij}(u)n_j$, $i=1,\dots,N$, where n is the unit outer normal vector). For a vector function w: $\Gamma\to\mathbb{R}^N$ we denote $w_n\equiv w_in_i$ its normal component and $w_t\equiv w-w_nn$ its tangential component. The time will be denoted by τ and the appropriate time derivatives are denoted by dots.

For $M=\Omega$, I and Q we denote by $H^\alpha(M)$, $\alpha\ge 0$, the usual Sobolev (Sobolev-Slobodeckii for α noninteger) spaces of the Hilbert type. Those with bold $\boldsymbol{H}$ contains functions with ranges in $\mathbb{R}^N$. (This notation is analogously used for $\boldsymbol{L}$-spaces.) Their duals are denoted by $H^{\alpha*}(M)$, $\boldsymbol{H}^{\alpha*}(M)$. Furthermore, $\mathring{\boldsymbol{H}}^1(\Omega):=\{w\in\boldsymbol{H}^1(\Omega);\ w=0 \text{ on } \Gamma\}$ and $\boldsymbol{H}^{-1}(\Omega)\equiv\left(\mathring{\boldsymbol{H}}^1(\Omega)\right)^*$. We exploit the usual Bochner-type spaces and denote for a Banach space X and an interval $I\subset\mathbb{R}$ by $B_0(I;X)$ the space of bounded functions $I\to X$ equipped with the sup-norm. Moreover, $\boldsymbol{H}^\alpha_w(\Omega)=\{v\in\boldsymbol{H}^\alpha(\Omega);\ v=w \text{ on } \Gamma_U\}$ for a suitable w. Then the *variational formulation* of the problem follows:

A weak solution to (2) will be a function $u \in B_0\big(I; \boldsymbol{H}^1(\Omega)\big) \cap \mathscr{C}$ such that $\dot u \in L_\infty(I;\, \boldsymbol{L}_2(\Omega))$, $\dot u(\mathscr{T},\cdot) \in \boldsymbol{L}_2(\Omega)$ and for all $v \in \boldsymbol{H}^1(Q) \cap \mathscr{C}$ the following inequality holds:

$$\text{(3)}\quad \int_Q \sigma_{ij}(u)e_{ij}(v-u) - \dot u_i(\dot v_i - \dot u_i)\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega (\dot u_i(v_i-u_i))\,(\mathscr{T},\cdot)\,\mathrm{d}x \\ \geq \int_\Omega (u_1)_i(v_i(0,\cdot) - (u_0)_i)\,\mathrm{d}x + \int_Q f_i(v_i-u_i)\,\mathrm{d}x\,\mathrm{d}\tau + \int_{S_T} T_{0,i}(v_i-u_i)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Here, the cone $\mathscr{C} := \{v \in L_2\big(I; \boldsymbol{H}^1_U(\Omega)\big)\,;\; v_n \le 0 \text{ a.e. in } S_C\}$.

As in [3] we solve this problem penalizing the contact condition. u_ε will be the weak solution of the penalized problem, if $u_\varepsilon \in B_0\big(I; \boldsymbol{H}^1_U(\Omega)\big)$ for which $\dot u_\varepsilon \in B_0(I;\, \boldsymbol{L}_2(\Omega))$ and $\ddot u_\varepsilon \in L_2\big(I; \boldsymbol{H}^{1*}_0(\Omega)\big)$, the initial condition in (2) is satisfied and the following equation

$$\text{(4)}\quad \int_Q (\ddot u_\varepsilon)_i v_i + \sigma_{ij}(u_\varepsilon)e_{ij}(v)\,\mathrm{d}x\,\mathrm{d}\tau + \int_{S_C} \frac{1}{\varepsilon}(u_\varepsilon)_n^+ v_n\,\mathrm{d}x\,\mathrm{d}\tau \\ = \int_Q f_i v_i\,\mathrm{d}x\,\mathrm{d}\tau + \int_{S_T} T_{0,i}v_i\,\mathrm{d}x\,\mathrm{d}\tau$$

holds for all $v \in L_2\big(I; \boldsymbol{H}^1_0(\Omega)\big)$. In fact, the penalization consists in replacing the Signorini boundary value condition on S_C in (2) by the condition $T_n(u_\varepsilon) = -\frac{1}{\varepsilon}(u_\varepsilon)_n^+$.

We solve our problems under the assumption

$$\text{(5)}\quad \begin{aligned} f \in L_2\Big(I; \boldsymbol{H}^{1*}(\Omega)\Big),\ T_0 \in H^1\Big(I; \boldsymbol{H}^{\frac12 *}(\mathit{\Gamma}_T)\Big),\ u_0 \in \boldsymbol{H}^1(\Omega) \\ \text{with } (u_0)_n \le 0 \text{ on } \mathit{\Gamma}_C, \\ u_1 \in \boldsymbol{L}_2(\Omega),\ \text{and } U \in \boldsymbol{H}^2(Q) \text{ such that } U(0,\cdot) = u_0, \\ \text{on } \mathit{\Gamma}_U \text{ and } U \equiv 0 \text{ on } S_C. \end{aligned}$$

We start from the following existence lemma for the penalized problem proved in [3, p. 586]:

Lemma. *Let the assumptions about Ω, its boundary, V, W, the assumptions* (5) *and*

$$\text{(6)}\quad 2\beta_1^V \int_{\mathbb{R}_+} K(s)\,ds < \beta_0^W$$

hold. Then there exists a solution to the penalized problem (4).

This solution, moreover, satisfies the following a priori estimate independent of $\varepsilon > 0$:

$$\sup_{\tau\in I}\Big(\|\dot u_\varepsilon(\tau,\cdot)\|^2_{\boldsymbol{L}_2(\Omega)}+\|\nabla u_\varepsilon(\tau,\cdot)\|^2_{L_2\left(\Omega;\,\mathbb{R}^{N^2}\right)}+\frac{1}{\varepsilon}\|(u_\varepsilon)_n^+(\tau,\cdot)\|^2_{L_2(\Gamma_C)}\Big)$$

$$(7)\quad \|\nabla\pi_Y u_\varepsilon\|^2_{H^\alpha\left(I;\,L_2(\Omega;\mathbb{R}^{N^2})\right)}\le c_0\equiv c_0(\mathscr{J})\ \text{ with }\ \mathscr{J}\equiv\Big[\beta_0^W,\beta_1^W,\beta_0^V,\beta_1^V,$$

$$\|u_0\|_{\boldsymbol{H}^1(\Omega)},\|u_1\|_{\boldsymbol{L}_2(\Omega)},\|f\|_{L_2(I;\boldsymbol{H}^{1*}(\Omega))},\|T_0\|_{H^1\left(I;\,\boldsymbol{H}^{1/2*}(\Gamma_T)\right)},\|U\|_{\boldsymbol{H}^2(Q)},\Big].$$

Here, Y denotes the L_2-orthogonal complement to the algebraic kernel $\mathscr{R}$ of the operator $\widetilde{e}$. It is well known that all elements of $\mathscr{R}$ are affine functions (sums of linear functions and constants). We remark that the assumption on the volume force f is, in fact, somewhat stronger in [3] ($f \in \boldsymbol{L}_2(\Omega)$). Checking the a-priori estimates and other procedures there easily shows that such a stronger assumption (which usually serves to the performance of the technique of local shifts in arguments unemployed in our context) were never used there and therefore is redundant.

The further proof of the existence theorem in [3] was based on an L_1-estimate of the penalty term. We improve that procedure as follows:

Let us put an arbitrary $v \in L_2\Big(I;\ \mathring{\boldsymbol{H}}^1(\Omega)\Big)$ into (4). From the assumptions (5) and the a priori estimate (7) the following dual estimate follows straightforwardly from (4)

$$(8)\qquad \|\ddot u_\varepsilon\|_{L_2(I;\,\boldsymbol{H}^{-1}(\Omega))}\le c_1\equiv c_1(\mathscr{J}).$$

Interpolating this result with the fractional-derivative norm from (7) with the help of well-known methods from [1] combined with the Fourier transformation used for suitable extensions of u_ε (for details cf. Theorem 8.1 and its proof in [1] or [4]), we obtain the following result:

$$(9)\qquad \{u_\varepsilon;\,\varepsilon>0\}\ \text{is bounded in}\ H^{1+\frac{\alpha}{2}}(I;\,\boldsymbol{L}_2(\Omega))\,.$$

Such a result for $\pi_Y u_\varepsilon$ is a direct consequence of Theorem 12.4 in Chapter 1 of [1], if we use the described extension and the Fourier transformation in time to such prolonged functions. The differentiability order of the employed spaces and the possibility to apply the local straightening or any other local regularization of the boundary without changing the mentioned order and such that the appropriate norms remain uniformly bounded show that the assumption about the high smoothness there need not be here required. On the other hand, for $\pi_{\mathscr{R}} u_\varepsilon$ we exploit the first two terms estimated in (7) together with the fact that the derivatives $D^\beta w = 0$ for any $w \in \mathscr{R}$ and any β with $|\beta| \ge 2$, i.e. $\{\pi_{\mathscr{R}} u_\varepsilon;\,\varepsilon > 0\}$

is bounded in $L_2(I; \boldsymbol{H}^s(\Omega))$ for every $s > 0$. By interpolation of this with (8) we derive an ε-independent estimate for $\|\nabla \pi_{\mathscr{R}} u_\varepsilon\|^2_{H^\alpha(I;\, \boldsymbol{L}_2(\Omega))}$, too. Thus the projection π_Y in the appropriate term can be dropped both in (7) and in the above described interpolation procedure. Reinterpolating the just proved relation (9) with such a new fractional-derivative estimate of u_ε, we obtain

$$\{\dot u_\varepsilon;\, \varepsilon > 0\} \text{ is bounded in } L_2\Big(I; \boldsymbol{H}^{\frac{\alpha}{2-\alpha}}(\Omega)\Big). \tag{10}$$

Let us denote

$$\boldsymbol{H}^{\beta_1,\beta_2}(Q) \equiv H^{\beta_1}(I;\, \boldsymbol{L}_2(\Omega)) \cap L_2\left(I; \boldsymbol{H}^{\beta_2}(\Omega)\right).$$

The results (9) and (10) yield that there is a sequence $\varepsilon_k \to 0$ and a function u such that $u_{\varepsilon_k} \rightharpoonup u$ (weakly) in $L_2\big(I; \boldsymbol{H}^1(\Omega)\big)$ and for any $\delta \in (0,\alpha)$ the strong convergence $\dot u_{\varepsilon_k} \to \dot u$ in $\mathscr{H}^\delta \equiv \boldsymbol{H}^{\frac{\delta}{2},\frac{\delta}{2-\delta}}(Q)$ holds. The last convergence follows from the obvious weak convergence via the compact imbedding theorem $\mathscr{H}^{\alpha_1} \hookrightarrow \mathscr{H}^{\alpha_2}$ valid for $2 > \alpha_1 > \alpha_2 \geq 0$. The strong convergence of velocities in $\boldsymbol{L}_2(Q)$ is a particular case of this fact. It is crucial due to the "bad" sign of the velocities which occurs after performing the necessary integration by parts in the acceleration in (4). Due to the nonlinearities in the strain-stress relation and to the presence of the memory term, however, we need also the strong L_2-convergence of the space gradients. Its proof is standardly based on the strong monotonicity of the strain-stress relation ensured by the assumption (6). We put $v = u$ into (4) and add $\int_Q \sigma_{ij}(u) e_{ij}(u - u_{\varepsilon_k})\,\mathrm{d}x\,\mathrm{d}t$ to both sides of (4). All above mentioned convergences and the mentioned monotonicity yield the strong convergence and ensures that $\sigma_{ij}(u_{\varepsilon_k}) \xrightarrow{L_2} \sigma_{ij}(u)$, $i,j = 1,\dots,N$. Thus we have proved

Theorem. *Under the assumptions of Lemma there exists a weak solution u to the contact problem* (2) *such that $u \in \boldsymbol{H}^{1+\frac{\alpha}{2},1}(\Omega)$ and $\dot u \in \boldsymbol{H}^{\frac{\alpha}{2},\frac{\alpha}{2-\alpha}}(Q)$.*

Remark. 1. The space "regularity" of the velocity yields that $u \in \boldsymbol{H}^{1+\frac{\alpha}{2},1+\frac{\alpha}{2-\alpha}}(I \times \tilde\Omega)$ for any domain $\tilde\Omega$ having its closure inside Ω. Such a result can be proved via the shift technique, cf. Corollary in [3]. In the interior of separate parts of the boundary this technique yields the regularity of u in the tangential directions.

2. To be able to include the friction e.g. as in [4], we need at least L_2-traces of the velocities, i.e. we should prove that $\dot u \in L_2(I; \boldsymbol{H}^\gamma(\Omega))$ for some $\gamma > \frac{1}{2}$ which needs $\alpha > \frac{2}{3}$, cf. (10). However, the assumption (6) restricts α to $\big(0,\frac{1}{2}\big)$.

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