On the positivity of semigroups of operators

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Abstract. In a Banach space E, let $U(t)$ $(t > 0)$ be a C_0-semigroup with generating operator A. For a cone $K \subseteq E$ with non-empty interior we show: $(\star)$ $U(t)[K] \subseteq K$ $(t > 0)$ holds if and only if A is quasimonotone increasing with respect to K. On the other hand, if A is not continuous, then there exists a regular cone $K \subseteq E$ such that A is quasimonotone increasing, but $(\star)$ does not hold.

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1. Introduction

In Section 2 below we shall prove the result mentioned in the first two phrases of the abstract, and this in the more general context of a Hausdorff topological vector space E: By a *wedge* we mean a non-empty, closed, convex set K in E satisfying $\lambda K \subseteq K$ for $\lambda \geq 0$. Then $\theta \in K$ follows, θ denoting the zero-element of E. The wedge K is called a *cone*, if

$$K \cap (-K) = \{\theta\}. \tag{1}$$

In any case, for $x, y \in E$ we set

$$x \leq y \Longleftrightarrow y - x \in K;\; x \ll y \Longleftrightarrow y - x \in \operatorname{Int} K. \tag{2}$$

Further notations are $E^\star$ for the topological dual of E and

$$K^\star = \{\varphi | \varphi \in E^\star,\ \varphi(x) \geq 0\ (x \in K)\}.$$

Here E is supposed to be a real space, which is not a serious restriction: If E is a complex space, we consider $E_\mathbb{R}$ (i.e. we restrict the scalars to $\mathbb{R}$), and we use the formula

$$(E_\mathbb{R})^\star = \{\operatorname{Re} \varphi | \varphi \in E^\star\}.$$

Now let D be a linear subspace of E and let $A : D \to E$ be linear. This operator is called *quasimonotone increasing* with respect to the wedge $K \subseteq E$ (cf. [10]), if the following holds true:

$$x \in D \cap K,\ \varphi \in K^\star,\ \varphi(x) = 0 \implies \varphi(Ax) \geq 0. \tag{3}$$

In Section 3 we consider ordered Banach spaces E, where the order cone K is normal (in the sense of M. Kreĭn [8]) and solid (i.e., $\operatorname{Int} K \neq \emptyset$). In the final Section 4 we construct counter-examples: Look at (3) with a cone K in a Banach space E. If $\varphi \neq 0$, then $x \in D$ is a support-point of K. Therefore, if K has no support-points $x \neq 0$ in D, then (3) holds for arbitrary linear operators $A : D \to E$, i.e., any such operator is quasimonotone increasing with respect to K.

To carry out our construction, we were searching in an incomplete normed space D for a bounded, closed, convex set $C \neq \emptyset$ without support-points. In 1985, Borwein and Tingley [3] conjectured that such a C exists in every incomplete D. So we asked Professor Borwein by e-mail on recent progress on this conjecture. He answered *immediately* that Fonf [4] had given a positive solution. We *highly appreciate* Professor Borwein's quick reaction.

There exists an extensive literature on positive semigroups of operators; cf., e.g., Arendt [1] or Arendt et al. [2]. Concerning recent research in this direction we refer to [5]. For some notions occurring in the present paper, cf. also the books of Krasnosel'skiĭ [7] and S. Kreĭn [9], respectively.

2. Considerations in topological vector spaces

Let E be a Hausdorff topological vector space, and let K be a wedge in E; the relations $\leq$ and $\ll$ are defined by (2). Furthermore, let $A : D \to E$ be a linear operator, where $D \subseteq E$. If $x \in D$, we consider the initial value problem

$$u(0) = x, \; u' = Au \tag{4}$$

for differentiable functions

$$u : [0, T) \to D \tag{5}$$

($0 < T \leq \infty$).

Theorem 1. (A) *For any $x \in D \cap K$ suppose* (4) *to have a solution*

$$u : [0, T) \to K \tag{6}$$

(*where $T > 0$ may depend upon x*). *Then A is quasimonotone increasing.*
(B) *If*

$$D \cap \operatorname{Int} K \neq \emptyset, \tag{7}$$

A is quasimonotone increasing, and $x \in D \cap K$, then (6) *is true for any solution* (5) *of* (4).

Proof: (A) As in (3), suppose

$$x \in D \cap K, \; \varphi \in K^\star, \; \varphi(x) = 0.$$

To show

$$\varphi(Ax) \geq 0,$$

take a solution (6) of (4). Then

$$\varphi(Ax) = \varphi(Au(0)) = \varphi(u'(0)) = \lim_{t\downarrow 0} \frac{\varphi(u(t)) - \varphi(u(0))}{t}$$
$$= \lim_{t\downarrow 0} \frac{\varphi(u(t)) - \varphi(x)}{t} = \lim_{t\downarrow 0} \frac{1}{t}\varphi(u(t)) \geq 0,$$

the last inequality being a consequence of (6).

(B) Assume (7) to hold, and let A be quasimonotone increasing. Choose $p \in D \cap \operatorname{Int} K$, and choose $\lambda > 0$ such that

$$Ap \ll \lambda p. \tag{8}$$

Suppose $x \in D \cap K$, and let the function (5) be a solution of (4). Our aim is to show

$$u(t) \in K \ (0 \leq t < T). \tag{9}$$

For $\varepsilon > 0$ put

$$w_\varepsilon(t) = u(t) + \varepsilon e^{\lambda t} p \qquad (0 \leq t < T). \tag{10}$$

Then $w_\varepsilon(0) = u(0) + \varepsilon p \in \operatorname{Int} K$, hence

$$\theta \ll w_\varepsilon(0). \tag{11}$$

Furthermore,

$$w_\varepsilon'(t) - Aw_\varepsilon(t) = u'(t) + \lambda\varepsilon e^{\lambda t} p - Au(t) - \varepsilon e^{\lambda t} Ap$$
$$= \varepsilon e^{\lambda t}(\lambda p - Ap),$$

and therefore (8) implies

$$\theta \ll w_\varepsilon'(t) - Aw_\varepsilon(t) \ (0 \leq t < T). \tag{12}$$

A being quasimonotone increasing, the inequalities (11), (12) imply that w_ε can be estimated from below by the trivial solution $v(t) \equiv \theta$ of the differential equation in (4) (cf. [10]):

$$\theta \ll w_\varepsilon(t) \ (0 \leq t < T).$$

We substitute for $w_\varepsilon(t)$ by (10); then $\varepsilon \downarrow 0$ gives (9). □

Remark 1. If K is a cone, then in case (B) of Theorem 1 the initial value problem (4) has at most one solution (for arbitrary $x \in D$): Consider a solution $u : [0, T) \to D$ of (4) with $x = \theta$; (B) implies $u(t) \in K$ and $-u(t) \in K$ for $0 \leq t < T$, hence $u(t) \equiv \theta$ because of (1).

Remark 2. If E is a Banach space and $A : E \to E$ is linear, continuous (so $D = E$), then (B) also is true without the hypothesis (7); i.e., the wedge K need not to be solid in this case (cf. [11], [12]).

3. Considerations in Banach spaces

We start with an example: Let $E = \mathbb{R}^3$ be ordered by means of the cone

$$K = \left\{(\xi, \eta, \zeta) | \zeta \geq \sqrt{\xi^2 + \eta^2}\right\}.$$

The natural identification of $E^\star$ with E yields $K^\star = K$, and then it is easy to show that

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

defines a quasimonotone increasing operator $A : \mathbb{R}^3 \to \mathbb{R}^3$. With I denoting the identity on $\mathbb{R}^3$, the inclusion

$$(A + \lambda I)(K) \subseteq K \tag{13}$$

holds for no real λ. On the other hand, linear operators fulfilling (13) (for at least one λ) are always quasimonotone increasing.

Now let E be an arbitrary Banach space, and let $A : D \to E$ be linear, D being dense in E. Concerning the initial value problem (4), we formulate three conditions (H_0), (H_1), (H_2) (cf. S. Kreĭn [9]):

(H_0) For any $x \in D$, (4) has a solution $u : [0, \infty) \to D$.

(H_1) For any $x \in D$, (4) has a unique solution

$$u(\cdot) = U(\cdot)x \ : [0, \infty) \to D.$$

(H_2) Condition (H_1) holds, and

$$x_n \to \theta \ \text{ in } \ D \Longrightarrow U(t)x_n \to \theta \ (t > 0). \tag{14}$$

If (H_1) holds, then the operators

$$U(t) : D \to D \ (t > 0)$$

are linear. Under condition (H_2) they are also continuous, hence there is a unique linear, continuous continuation

$$U(t) : E \to E \ (t > 0) \tag{15}$$

of them. If (H_2) holds with (14) uniformly satisfied on each finite interval $(0, T]$, then the operators (15) form a C_0-semigroup (cf. S. Kreĭn, loc. cit.).

Theorem 2. *Suppose the Banach space E to be ordered by a solid, normal cone K, and let $A : D \to E$ ($\overline{D} = E$) be a linear, quasimonotone increasing operator fulfilling* (H_0). *Then* (H_2) *is true, and* (14) *holds uniformly on each finite interval* $(0, T]$.

Proof: $\overline{D} = E$ and $\operatorname{Int} K \neq \emptyset$ imply (7). Then Remark 1 implies (H_1), and (B) of Theorem 1 implies

$$U(t)[D \cap K] \subseteq K \ (t > 0). \tag{16}$$

We choose $p \in D \cap \operatorname{Int} K$. The normality of K implies the boundedness (in norm) of the order-interval

$$[-p, p] = \{x | x \in E, -p \leq x \leq p\}.$$

This set is also closed, convex, symmetric, and we have $\theta \in \operatorname{Int}[-p, p]$. Therefore (after equivalent renorming of E, if necessary) we can assume that $[-p, p]$ is the closed unit ball of E:

$$[-p, p] = S(\theta; 1) = \{x | x \in E, \|x\| \leq 1\}. \tag{17}$$

For $0 < T < \infty$ the sets $\{U(t)p | 0 < t \leq T\}$ are bounded, so there are numbers $R = R(T) > 0$ such that

$$U(t)p \in S(\theta; R) = [-Rp, Rp] \qquad (0 < t \leq T). \tag{18}$$

Then

$$\|U(t)x\| \leq R \qquad (x \in D,\ \|x\| \leq 1,\ 0 < t \leq T), \tag{19}$$

and therefore (14) holds uniformly on $(0, T]$. To show (19), consider $x \in D$, $\|x\| \leq 1$; (17) implies

$$-p \leq x \leq p,$$

then (16) yields

$$-U(t)p \leq U(t)x \leq U(t)p \qquad (t > 0),$$

and because of (18) we get (19). □

Remark 3. For the operators (15) we can write (16) in the following form:

$$U(t)[K] \subseteq K \ (t > 0).$$

4. Construction of counter-examples

Again let E be a Banach space, and let $A : D \to E$ be linear, where

$$D \neq \overline{D} = E, \tag{20}$$

$$A \neq \lambda I|_D \ (\lambda \in \mathbb{R}). \tag{21}$$

We suppose (H_0) to be satisfied.
We shall construct a cone $K \subseteq E$ having the following two properties:

(I) A is quasimonotone increasing with respect to K;

(II) there is a solution $u : [0, \infty) \to D$ of (4) satisfying $u(0) \in K$, but such that the inclusion $\{u(t) | t \geq 0\} \subseteq K$ does not hold.

Observe that from (H_0) and (21) we get the existence of a solution $u : [0, \infty) \to D$ of (4), such that for (at least) one $t > 0$

$$a = u(0) \ \text{ and } \ b = u(t)$$

are linear independent elements of D. If some cone K satisfies

$$a \in K, \ b \notin K, \tag{22}$$

then (II) holds.

(20) implies D to be an incomplete normed space. Let C be a nonvoid, bounded, closed, convex subset of D without support-points (cf. Fonf [4]). The points a, b of D being linearly independent, we can suppose

$$a \in C, \ C \cap \mathbb{R}b = \emptyset. \tag{23}$$

Denote by $\overline{C}$ the closure of C in E. Then

$$K = \bigcup_{\lambda \geq 0} \lambda \overline{C} \tag{24}$$

is a cone in E (which is regular in the sense of Krasnosel'skiĭ [6]), and because of (23) we have (22), hence (II). Property (I), i.e. the quasimonotonicity of A with respect to the cone (24), follows from the considerations in Section 1.

Remark 4. Let E be a Banach space, and suppose $A : D \to E$ to be a densely defined closed, linear operator, which generates a C_0-semigroup. There are two possibilities:

1. $D \neq \overline{D}$: Then A is not continuous and (20), (21), (H_0) hold, hence there exists a cone $K \subseteq E$ having the properties (I), (II).

2. $D = \overline{D}$: Then $A : E \to E$ is continuous, and there is no wedge $K \subseteq E$ having the properties (I), (II) (cf. Remark 2 above).

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