# On closure of the pre-images of families of mappings

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*Abstract.* The closures of the pre-images associated with families of mappings in different topologies of normed spaces are considered. The question of finding a description of these closures by means of families of the same kind as original ones is studied. It is shown that for the case of the weak topology this question may be reduced to finding an appropriate closure of a given family. There are discussed various situations when the description may be obtained for the case of the strong topology. An example of a family is constructed which shows that it is, in general, impossible to find such a description for this case.

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## 1. Introduction

Let $\mathfrak{A} = \{A_\chi\}_{\chi\in\Lambda}$ be a family of continuous mappings $A_\chi : X \to Y$ where $X, Y$ are normed spaces. For $f \in Y$ we set $Z(\mathfrak{A}, f) = \{u \in X \mid A_\chi u = f, A_\chi \in \mathfrak{A}\}$. We are interested in the question: Does there exist a family $\mathfrak{B} = \{A_\theta\}_{\theta\in\Gamma}$ of continuous mappings $A_\theta : X \to Y$ such that for every $f \in Y$ the set $Z(\mathfrak{B}, f)$ coincides with the closure of $Z(\mathfrak{A}, f)$ in the strong (weak) topology of $X$?

For $u \in X$ let us denote by $F(\mathfrak{A}, u)$ the set $\{A_\chi u \mid A_\chi \in \mathfrak{A}\}$. For the case of the strong topology, assuming that each $A_\chi$ maps $X$ onto $Y$ and there exist constants $\nu_1, \nu_2 > 0$ such that

$$\nu_1 \|u - v\|_X \le \|A_\chi u - A_\chi v\|_Y \le \nu_2 \|u - v\|_X \quad \text{for all } \ u, v \in X, \quad \text{for all } \ \chi \in \Lambda,$$

the above stated question becomes to: Is it possible to find such a family $\mathfrak{B} \subset C(X, Y)$ that for every $u \in X$ the set $F(\mathfrak{B}, u)$ is equal to the closure of $F(\mathfrak{A}, u)$ in the strong topology of $Y$?

The prototype of this question is the problem of extension of optimal control problems with cost functionals which do not depend on controls. Indeed, an extension of the problem

$$\begin{aligned} &J(u) \to \min \\ &Au = f, u \in X, A \in \mathfrak{A} \end{aligned} \tag{1}$$

is often searched in the form

$$\begin{aligned} &J(u) \to \min \\ &Bu = f, u \in X, B \in \mathfrak{B}, \end{aligned}$$

where $\mathfrak{B}$ is a larger set of operators (we refer to [7] for a general definition of extension of an abstract variational problem). The necessity to be in agreement with the continuity properties of the cost functional $J$ causes special demands on the properties of $\mathfrak{B}$. If $J$ is weakly continuous, then one must seek the closure of $Z(\mathfrak{A}, f)$ in the weak topology of $X$ and the extension of (1) is simply the passage from the set $\mathfrak{A}$ to the $G$-closure of the set $\mathfrak{A}$. If $J$ is not weakly continuous, then in the process of extension it is necessary to preserve the strong closure of the set of feasible states $Z(\mathfrak{A}, f)$ in the above mentioned sense.

There are many results on extensions of optimal control problems. See, for instance, [7] and [16] for the case of ordinary differential equations and [10], [11], [12], [14], [15] for the case of elliptic equations. We also refer to [1], [2], [5], [13] and references therein for related questions.

It is clear that most of these successful extensions have, as a basis, some abstract properties of the involved families of mappings (defined by state equations and sets of admissible controls). The purpose of the present paper is to discuss and systematize these properties for various situations.

Since the source of the problem studied in this paper lies in the field of optimal control problems for ODEs and PDEs, we will deal mainly with families of mappings between Banach spaces (Sections 4, 5) and we will illustrate our results by examples of families which are encountered in this field. A part of our results depends only on some general properties of families of mappings and we formulate them in the setting of topological spaces (Section 3).

## 2. Basic notations

The letters $\mathfrak{X}$, $\mathfrak{Y}$ will denote topological spaces. The symbol cl means the closure operation in $\mathfrak{X}$. By $C(\mathfrak{X}, \mathfrak{Y})$ we denote the set of all continuous mappings of $\mathfrak{X}$ into $\mathfrak{Y}$. The regularity of a topological space $\mathfrak{X}$ is meant in the sense of [4]. Let $\mathfrak{A} \subset C(\mathfrak{X}, \mathfrak{Y})$. For $\tilde{x} \in \mathfrak{X}, \tilde{y} \in \mathfrak{Y}$ we define $F(\mathfrak{A}, \tilde{x}) = \{A\tilde{x} \mid A \in \mathfrak{A}\}$ and $Z(\mathfrak{A}, \tilde{y}) = \{x \in \mathfrak{X} \mid Ax = \tilde{y}, A \in \mathfrak{A}\}$.

The letters $X, Y$ will always denote real normed spaces. The symbol "$\to$" ("$\rightharpoonup$" and "$\overset{*}{\rightharpoonup}$") means to be "strongly convergent to" ("weakly convergent to" and "weakly $*$ convergent to", respectively). For $M \subset X$, $\mathrm{cl}_s M$ ($\mathrm{cl}_w M$) stands for the strong (weak) closure of $M$ in $X$. Denote by $B(X, Y)$ ($C(X,Y)$) the set of all continuous linear (continuous) mappings of $X$ with values in $Y$.

For a multivalued mapping $\mathcal{F} : X \rightsquigarrow Y$ its inverse $\mathcal{F}^{-1} : Y \rightsquigarrow X$ is defined in the following way: $x \in \mathcal{F}^{-1}(y)$ if and only if $y \in \mathcal{F}(x)$. The symbol $\mathrm{cl}_s \mathcal{F}$ ($\mathrm{cl}_w \mathcal{F}$) denotes the multivalued mapping defined by $(\mathrm{cl}_s \mathcal{F})(x) = \mathrm{cl}_s \mathcal{F}(x)$ ($(\mathrm{cl}_w \mathcal{F})(x) = \mathrm{cl}_w \mathcal{F}(x)$) for $x \in X$.

Let $\mathfrak{A}$ be a subset of $C(X, Y)$. The symbol $\mathrm{cl}_u \mathfrak{A}$ means the closure of $\mathfrak{A}$ in the metric topology of $C(X, Y)$. The symbol $\mathrm{cl}_s \mathfrak{A}$ ($\mathrm{cl}_w \mathfrak{A}$) denotes the sequential closure of $\mathfrak{A}$ in the topology of point-wise convergence in $C(X, Y)$ when $Y$ is endowed with strong (weak) topology.

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space and $\Omega$ a bounded open subset of $\mathbb{R}^n$. $C_0^\infty(\Omega)$ is the set of all functions with compact support in $\Omega$ having all derivatives of arbitrary order continuous in $\Omega$. $L^p(\Omega)$, $1 \le p \le \infty$ are the usual Lebesgue spaces of measurable functions $f$ on $\Omega$. $H^1(\Omega)$ is the Sobolev space of functions $f \in L^2(\Omega)$ such that $\partial f/\partial x_i \in L^2(\Omega)$, $1 \le i \le n$, with the usual norm. $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. For $f \in L^1(\Omega)$, $\operatorname{supp} f$ denotes the support of $f$.

## 3. The abstract case

Let $K \subset C(\mathfrak{X}, \mathfrak{Y})$ be a fixed set and let $\mathfrak{A} \subset K$. A family $\mathfrak{B} \subset C(\mathfrak{X}, \mathfrak{Y})$ is said to be *a $K$-extension of* $\mathfrak{A}$ if

$$\begin{aligned} &\operatorname{cl} Z(\mathfrak{A}, y) = Z(\mathfrak{B}, y) \quad \text{for all} \quad y \in \mathfrak{Y}, \\ &\mathfrak{A} \subset \mathfrak{B} \subset K. \end{aligned} \tag{2}$$

The role of the set $K$ is to select from all continuous mappings those which are of interest. The mappings from $K$ usually describe a concrete physical process and according to the physical laws processes of this type are described by equations of specific kind. So, from the practical point of view the possible extensions must be subsets of $K$.

In this section, we present an abstract result concerning existence of $K$-extensions of families of mappings. This result becomes possible due to the relative compactness of the families in the topology of point-wise convergence in $C(\mathfrak{X}, \mathfrak{Y})$.

Let us recall a useful definition. A family $\mathfrak{A}$ is said to be *uniformly continuous* if for every $x \in \mathfrak{X}$, and every $y \in \mathfrak{Y}$, for any neighborhood $V$ of $y$, there exist neighborhoods $U$ and $W$ of $x$ and $y$, respectively, such that whenever $A \in \mathfrak{A}$ and $Ax \in W$ then $AU \subset V$ (cf. [4]).

**Proposition 3.1.** *Let $\mathfrak{Y}$ be a regular topological space. Let $\mathfrak{A}$ be a uniformly continuous family. Let $\mathfrak{B}$ be a closure of $\mathfrak{A}$ in the topology of point-wise convergence in $C(\mathfrak{X}, \mathfrak{Y})$.*

(i) *If $\mathfrak{A}$ is such that for every $x \in \mathfrak{X}$ the closure of the set $F(\mathfrak{A}, x)$ is compact in $\mathfrak{Y}$, then*

$$\operatorname{cl} Z(\mathfrak{A}, y) \subset Z(\mathfrak{B}, y) \quad \textit{for all} \quad y \in \mathfrak{Y}.$$

(ii) *If $\mathfrak{A}$ is such that for any generalized sequence $\{y_\gamma\}_{\gamma \in \Theta} \subset \mathfrak{Y}$, for every $x \in \mathfrak{X}$ such that $y_\gamma = A_\gamma x$, $A_\gamma \in \mathfrak{A}$ for all $\gamma \in \Theta$ and $y_\gamma \to y$ in $\mathfrak{Y}$, there is a generalized sequence $\{x_\beta\}_{\beta \in \Xi} \subset \mathfrak{X}$ for which $y = A_\beta x_\beta$, $A_\beta \in \mathfrak{A}$ for all $\beta \in \Xi$ and $x_\beta \to x$ in $\mathfrak{X}$, then*

$$Z(\mathfrak{B}, y) \subset \operatorname{cl} Z(\mathfrak{A}, y) \quad \textit{for all} \quad y \in \mathfrak{Y}.$$

Proof: Let $x \in \operatorname{cl} Z(\mathfrak{A}, y)$. Then there are generalized sequences $\{A_\gamma\}_{\gamma\in\Theta} \subset \mathfrak{A}$ and $\{x_\gamma\}_{\gamma\in\Theta}$ such that $x_\gamma \to x$ in $\mathfrak{X}$ and $y = A_\gamma x_\gamma$ for all $\gamma \in \Theta$. Put $\mathcal{G}(x) = \operatorname{cl} F(\mathfrak{A}, x)$ for any $x \in \mathfrak{X}$. The set $M = \prod_{x\in\mathfrak{X}} \mathcal{G}(x)$ is compact in the space $\mathfrak{Z} = \prod_{x\in\mathfrak{X}} \mathfrak{Y}_x$ endowed with the Tychonoff topology ($\mathfrak{Y}_x = \mathfrak{Y}, x \in \mathfrak{X}$). Hence the generalized sequence $\{z_\gamma\}, z_\gamma = \{A_\gamma x\}_{x\in\mathfrak{X}}$ contains a subsequence $\{z_\beta\}$ which converges to some $z \in M$. It is clear that $z = \{Ax\}_{x\in\mathfrak{X}}$ for some mapping $A : \mathfrak{X} \to \mathfrak{Y}$. We have also that $Ax = y$. It may be seen that $A \in C(\mathfrak{X}, \mathfrak{Y})$. Indeed, let $x \in \mathfrak{X}$ and $V \subset \mathfrak{Y}$ be a neighborhood of $Ax$. Since $\mathfrak{Y}$ is regular, there is an open set $V_1 \subset \mathfrak{Y}$ such that $Ax \in V_1 \subset \overline{V_1} \subset V$. Since $\mathfrak{A}$ is uniformly continuous, there exists a neighborhood $U$ of $x$ such that $AU \subset V$.

Let $x \in Z(\mathfrak{B}, y)$. Then $Ax = y$ for some $A \in \mathfrak{B}$. Since $\mathfrak{B}$ is a closure of $\mathfrak{A}$ in the topology of point-wise convergence in $C(\mathfrak{X}, \mathfrak{Y})$, there exists a generalized sequence $\{A_\gamma\}_{\gamma\in\Theta} \subset \mathfrak{A}$ such that $A_\gamma x \to y$ in $\mathfrak{Y}$. From the corresponding assumption it follows that there exists a generalized sequence $\{x_\beta\}_{\beta\in\Xi}$ such that $x_\beta \to x$ in $\mathfrak{X}$ and $y = A_\beta x_\beta, A_\beta \in \mathfrak{A}$ for all $\beta \in \Xi$. Hence $x \in \operatorname{cl} Z(\mathfrak{A}, y)$. □

The next theorem is an immediate consequence of Proposition 3.1.

**Theorem 3.1.** *Let $\mathfrak{Y}$ be a regular topological space. Let $K$ be a non-empty subset of $C(\mathfrak{X}, \mathfrak{Y})$ which is closed in $C(\mathfrak{X}, \mathfrak{Y})$ equipped with the topology of point-wise convergence. Let $\mathfrak{A} \subset K$ be a uniformly continuous family. Suppose that $\mathfrak{A}$ satisfies assumptions from* (i) *and* (ii) *of the previous proposition. Then the closure of $\mathfrak{A}$ in $C(\mathfrak{X}, \mathfrak{Y})$ equipped with the topology of point-wise convergence is the $K$-extension of $\mathfrak{A}$.*

Here we would like to point out that the situation described in this theorem corresponds to a great part of optimal control problems for ODEs and for PDEs where controls appear in boundary conditions and on the right-hand side ([9], [16]). Such problems may be very often described in the following way.

Let $X, Y, V$ be separable normed spaces and $L_1 : V \to Y$ compact linear operator. Assume that a family $\mathfrak{C}$ of operators of $X$ into $V$ is bounded in the normed space $B(X, V)$ and $L_2 \in B(X, Y)$ is an isomorphism. Consider the family

$$\mathfrak{A} = \{L_2 + L_1 C \mid C \in \mathfrak{C}\}.$$

If there exists $0 < q < 1$ such that $\left\| L_2^{-1} L_1 C \right\| \leq q$ for any $C \in \mathfrak{C}$, then the family $\mathfrak{B}$ which is a closure of $\mathfrak{A}$ in the strong operator topology of $B(X, Y)$ is a $K$-extension of $\mathfrak{A}$ with $K = B(X, Y)$.

## 4. Weak and strong closures

First of all, let us investigate the case of weak closure. This can be done by using the theory of $G$-convergence of abstract operators ([17]).

Let $X$ be a separable reflexive Banach space and $X^*$ be its conjugate. Let $\nu_1, \nu_2 > 0$. Denote by $M(\nu_1, \nu_2)$ the class of mappings $A : X \to X^*$ satisfying the conditions

$$(3) \qquad \begin{aligned} &(Ax - Ay, x - y) \geq \nu_1 \|x - y\|_X^2 , \\ &\|Ax - Ay\|_{X^*} \leq \nu_2 \|x - y\|_X , \end{aligned}$$

where $(\cdot,\cdot)$ is duality between $X^*$ and $X$. We recall that by Browder-Minty theorem ([6]) each $A \in M(\nu_1,\nu_2)$ has continuous inverse $A^{-1} : X^* \to X$.

A mapping $A : X \to X^*$ satisfying condition (3) is said to be *a G-limit* of the sequence $\{A_k\} \subset M(\nu_1,\tilde{\nu}_2)$ and the sequence $\{A_k\}$ *G-converges* to $A$ if for every $x^* \in X^*$

$$A_k^{-1}x^* \rightharpoonup A^{-1}x^* \quad \text{in} \quad X.$$

By using the diagonal process, one can prove

**Proposition 4.1.** *Let $\mathfrak{A} \subset M(\nu_1,\nu_2)$ for some $\nu_1,\nu_2 > 0$. Suppose that for $x = 0$ the set $F(\mathfrak{A},x)$ is bounded in $X^*$.*

(i) *If $\mathfrak{B}$ is a set of all G-limits of G-convergent sequences from $\mathfrak{A}$, then $\mathfrak{B} \subset M\left(\nu_1,\nu_1^{-1}\nu_2^2\right)$ and*

$$\mathrm{cl}_w\, Z(\mathfrak{A},x^*) = Z(\mathfrak{B},x^*) \quad \textit{for all} \quad x^* \in X^*.$$

(ii) *If $\mathfrak{C} = \mathrm{cl}_w\,\mathfrak{A}$, then $\mathfrak{C} \subset M(\nu_1,\nu_2)$ and*

$$\mathrm{cl}_w\, F(\mathfrak{A},x) = F(\mathfrak{C},x) \quad \textit{for all} \quad x \in X.$$

Further, for a family $\mathfrak{A} \subset M(\nu_1,\nu_2)$ we shall denote by $\mathrm{cl}_G\,\mathfrak{A}$ the set of all $G$-limits of $G$-convergent sequences from $\mathfrak{A}$. It should be noted that for some classes of differential operators there are many results which give an explicit description (by means of coefficients of a differential operator) of a $G$-closure (see, for instance, [3], [11], [17] and references therein).

Now let us consider the problem of the strong closure. For this, it is useful to admit the following definition. Let $K \subset C(X,Y)$ be a fixed set and let $\mathfrak{A} \subset K$. A family $\mathfrak{B} \subset C(X,Y)$ is called *a strong $K$-extension of* $\mathfrak{A}$ if (2) is fulfilled with $\mathrm{cl}_s$ instead of cl.

Denote by $L(\nu_1,\nu_2)$ the class of mappings $A : X \to Y$ satisfying the conditions

$$\begin{aligned} &\nu_1 \|x_1 - x_2\|_X \leq \|Ax_1 - Ax_2\|_Y \leq \nu_2 \|x_1 - x_2\|_X \quad \text{for all} \quad x_1,x_2 \in X, \\ &A \text{ maps } X \text{ onto } Y. \end{aligned}$$

As was mentioned in Introduction, the problem of finding a description of strong closures of sets $F(\mathfrak{A},x)$, $x \in X$ for family $\mathfrak{A}$ from $L(\nu_1,\nu_2)$ is equivalent to the analogous problem for sets $Z(\mathfrak{A},y)$, $y \in Y$. It is a consequence of the following proposition.

**Proposition 4.2.** *Let $\mathfrak{A} \subset L(\nu_1, \nu_2)$ for some $\nu_1, \nu_2 > 0$. Let $\mathcal{F} : X \rightsquigarrow Y$ be a multivalued mapping defined as $\mathcal{F}(x) = F(\mathfrak{A}, x)$ for all $x \in X$. Then* $\operatorname{cl}_s \mathcal{F}^{-1} = (\operatorname{cl}_s \mathcal{F})^{-1}$.

Proof: The proof is straightforward. □

We have

**Proposition 4.3.** *Let $\mathfrak{A} \subset L(\nu_1, \nu_2)$ for some $\nu_1, \nu_2 > 0$. If $\mathfrak{A}$ is sequentially compact in the topology of point-wise convergence in $C(X, Y)$, then*

$$\operatorname{cl}_s Z(\mathfrak{A}, y) = Z(\operatorname{cl}_s \mathfrak{A}, y) \quad \textit{for all} \ \ y \in Y.$$

*If $\mathfrak{A}$ is sequentially compact in the topology of uniform convergence in $C(X, Y)$, then*

$$\operatorname{cl}_s Z(\mathfrak{A}, y) = Z(\operatorname{cl}_u \mathfrak{A}, y) \quad \textit{for all} \ \ y \in Y.$$

Proof: Let $x \in \operatorname{cl}_s Z(\mathfrak{A}, y)$. Then there exist sequences $\{x_k\}$ and $\{A_k\} \subset \mathfrak{A}$ such that $A_k x_k = y$ and $x_k \to x$ in $X$. Since $\mathfrak{A}$ is sequentially compact in the topology of point-wise convergence in $C(X, Y)$, there is a subsequence of $\{A_k\}$ (still denoted $\{A_k\}$) which point-wisely converges to some $A \in C(X, Y)$. By means of the inequality

$$\|y - Ax\|_Y \leq \|A_k x_k - A_k x\|_Y + \|A_k x - Ax\|_Y ,$$

we get that $Ax = y$. Hence $x \in Z(\operatorname{cl}_s \mathfrak{A}, y)$.

Let $x \in Z(\operatorname{cl}_s \mathfrak{A}, y)$. Then for some $A \in \operatorname{cl}_s \mathfrak{A}$ we have $Ax = y$. Since $A \in \operatorname{cl}_s \mathfrak{A}$, there is a sequence $\{A_k\} \subset \mathfrak{A}$ which point-wisely converges to $A$. Define the sequence $\{x_k\}$ by the equations $A_k x_k = y$. Now from the inequality

$$\nu_1 \|x_k - x\|_X \leq \|A_k x_k - A_k x\|_Y = \|y - A_k x\|_Y \tag{4}$$

it follows that $x_k \to x$ in $X$, i.e. $x \in \operatorname{cl}_s Z(\mathfrak{A}, y)$.

The proof of the second statement is similar. □

It is easy to see that if $X$ is separable and a family $\mathfrak{A} \subset L(\nu_1, \nu_2)$ is such that the closure of the set $F(\mathfrak{A}, x)$ is compact in $Y$ for all $x \in X$, then $\mathfrak{A}$ is sequentially compact in the topology of point-wise convergence in $C(X, Y)$. To illustrate this situation, let us consider an example.

Let $X = H_0^1(\Omega)$ and $Y = H^{-1}(\Omega)$. Let $a$ be an $n \times n$-matrix with entries from $L^\infty(\Omega)$ such that

$$\langle a(x)\xi, \xi\rangle \geq \nu \xi^2 \ \text{ a.e. } \ x \in \Omega, \quad \text{for all} \ \ \xi \in \mathbb{R}^n \ (\nu > 0).$$

Let $\{b_i^\alpha\}_{\alpha\in\Lambda} \subset L^\infty(\Omega)$, $1 \leq i \leq n$ and let $\{c^\alpha\}_{\alpha\in\Lambda} \subset L^\infty(\Omega)$ be bounded sets of functions. Consider a family $\mathfrak{A}$ consisting of operators $A_\alpha : X \to Y$, $\alpha \in \Lambda$ defined by

$$A_\alpha u = -\operatorname{div} a\nabla u + b_i^\alpha u_{x_i} + c^\alpha u \ \text{ for all } \ u \in X,$$

where summation over repeated indices is assumed. Let us denote by $K$ the set of operators of the same kind as $A_\alpha$ (linear second order elliptic operators in divergence form). Let $\mu_2$ be a constant such that $\mu_2 \le c^\alpha(x)$ a.e. $x \in \Omega$, for all $\alpha \in \Lambda$. Then one can see that under some obvious assumptions on $\Omega$ and on the constant $\mu_2$ the family $\mathfrak{A}$ belongs to $L(\nu_1, \nu_2)$ for some $\nu_1, \nu_2 > 0$. It is also clear that the closure of the set $F(\mathfrak{A}, u)$ is compact in $Y$ for all $u \in X$. Hence the strong $K$-extension of the family $\mathfrak{A}$ may be chosen as its sequential closure in the strong operator topology of $B(X, Y)$.

The same reasoning is valid for cases of general second order linear elliptic systems with the fixed second order terms.

In the case when $X$ and $Y$ are separable reflexive Banach spaces we have

**Proposition 4.4.** *Let $\mathfrak{A} \subset L(\nu_1, \nu_2)$ for some $\nu_1, \nu_2 > 0$. Suppose that for $x = 0$ the set $F(\mathfrak{A}, x)$ is bounded in $Y$. Assume also that*

$$\mathrm{cl}_w F(\mathfrak{A}, x) = \mathrm{cl}_s F(\mathfrak{A}, x) \quad \textit{for all} \;\; x \in X. \tag{5}$$

*Then*

$$\mathrm{cl}_s Z(\mathfrak{A}, y) = Z(\mathrm{cl}_w \mathfrak{A}, y) \quad \textit{for all} \;\; y \in Y.$$

Proof: Let $x \in \mathrm{cl}_s Z(\mathfrak{A}, y)$. Then there exist sequences $\{x_k\}$ and $\{A_k\} \subset \mathfrak{A}$ such that $A_k x_k = y$ and $x_k \to x$ in $X$. By assumption of the proposition, the set $F(\mathfrak{A}, x)$ is bounded in $Y$ for all $x \in X$. By the diagonal process, we can extract a subsequence of $\{A_k\}$ (still denoted $\{A_k\}$) which point-wisely converges in $C(X, Y)$ where $Y$ is considered with the weak topology to some mapping $A$. It is also clear that $A$ satisfies

$$\|Ax_1 - Ax_2\|_Y \le \nu_2 \|x_1 - x_2\|_X \quad \text{for all} \;\; x_1, x_2 \in X.$$

By using the inequality

$$\|A_k x - A_k x_k\|_Y \le \nu_2 \|x - x_k\|_X ,$$

we obtain that $Ax = y$. This gives that $x \in Z(\mathrm{cl}_w \mathfrak{A}, y)$.

Let $x \in Z(\mathrm{cl}_w \mathfrak{A}, y)$. Then there is $A \in \mathrm{cl}_w \mathfrak{A}$ such that $Ax = y$. By the condition (5), there exists a sequence $\{A_k\} \subset \mathfrak{A}$ such that $A_k x \to y$ in $Y$. Define the sequence $\{x_k\}$ by the equations $A_k x_k = y$. It follows from the inequality (4) that $x_k \to x$ in $X$. In other words, $x \in \mathrm{cl}_s Z(\mathfrak{A}, y)$. □

Condition (5) of this proposition is satisfied if $\mathrm{cl}_s F(\mathfrak{A}, x)$ is convex in $Y$ for all $x \in X$. This situation often occurs in optimal control problems for distributed parameter systems described by the second order elliptic (or parabolic) equations when controls appear only in the first order terms and the set of admissible controls is decomposable. There are analogous results when controls appear in the second order terms and the set of admissible controls is decomposable (see [11], [12], [15]).

We remind that a set $M \subset L^1(\Omega)$ is called *decomposable* if for all $f, g \in M$ and for any characteristic function $\chi$ of a measurable subset of $\Omega$, the set $M$ contains the element $\chi f + (1-\chi) g$.

The other possible situation is when all sets $F(\mathfrak{A}, x), x \in X$ are closed in $Y$. If this is the case, $\mathfrak{A}$ is the strong $\mathfrak{A}$-extension of itself, provided that $\mathfrak{A} \subset L(\nu_1, \nu_2)$ for some $\nu_1, \nu_2 > 0$. For instance, if $\{a_\chi\}_{\chi \in \Lambda} \subset L^\infty(\Omega)$ is decomposable and closed in $L^2(\Omega)$ and there exist $\nu_1, \nu_2 > 0$ such that $\nu_1 \leq a_\chi(x) \leq \nu_2$ a.e. $x \in \Omega$, for all $\chi \in \Lambda$, then the set

$$\bigcup_{\chi \in \Lambda} \left\{ \frac{d}{dx} \left( a_\chi \frac{du}{dx} \right) \right\}$$

is closed in $H^{-1}(\Omega)$ for any $u \in H_0^1(\Omega)$.

The following example shows that a strong $K$-extension of a given family $\mathfrak{A} \subset K$ for some natural $K$ may not exist.

Let $\Omega = (0,1)$ and $a^\varepsilon(x) = (1+\varepsilon)\chi + \lambda(\varepsilon^{-1}x)(1-\chi)$, $1 \geq \varepsilon > 0$ where $\chi$ is the characteristic function of the interval $(0, 1/2)$ and $\lambda(x) = \frac{1}{10}\sin x + 2$. It is clear that $a^\varepsilon \overset{*}{\rightharpoonup} a^0 = 2 - \chi$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$. Define the operators $A^\varepsilon : H_0^1(\Omega) \to H^{-1}(\Omega)$ as

$$A^\varepsilon = \frac{d}{dx}\left( a^\varepsilon \frac{d}{dx} \right).$$

Set $\mathfrak{A} = \{A^\varepsilon\}_{\varepsilon \in (0,1]}$. For $u \in H_0^1(\Omega)$ let us denote by $\mathcal{G}(u)$ the strong closure of $F(\mathfrak{A}, u)$ in $H^{-1}(\Omega)$. If $u \in H_0^1(\Omega)$ is such that the set $\operatorname{supp} u' \cap (1/2, 1)$ is non-empty, then $\mathcal{G}(u) = \left\{ \left(a^\varepsilon u'\right)' \mid 0 < \varepsilon \leq 1 \right\}$. It is also clear that $\left(a^0 u'\right)' \in \mathcal{G}(u)$ if $\operatorname{supp} u' \subset [0, 1/2]$. Let us assume that there exists the operator

$$A = \frac{d}{dx}\left( b \frac{d}{dx} \right), b \in L^\infty(\Omega), \tag{6}$$

which is a selection of the multivalued mapping $\mathcal{G}$ and $Av = \left(a^0 v'\right)'$ where $v$ belongs to $C_0^\infty(0, 1/2)$ and the set $\operatorname{supp} v$ is non-empty. Put $\varphi = v + \psi$ for $\psi \in C_0^\infty(1/2, 1)$ with non-empty set support. From the equalities $A\varphi = \left(a^\varepsilon \varphi'\right)'$, $\varepsilon = \varepsilon(\varphi)$ and $Av = \left(a^0 v'\right)'$ in $H^{-1}(\Omega)$ it follows that $\left(b - a^0\right) v' = c_1 = \text{const}$, $(b - a^\varepsilon)\varphi' = c_2 = \text{const}$ in $L^2(\Omega)$ and $\left(a^\varepsilon(x) - a^0(x)\right) v'(x) = c_3 = \text{const}$ a.e. $x \in (0, 1/2)$. Hence $\varepsilon v'(x) = \text{const}$ a.e. $x \in (0, 1/2)$. But this is impossible since the support of the function $v$ is non-empty. Consequently, we conclude that there does not exist any family $\mathfrak{B}$ of the differential operators of the divergence type such that

$$\mathcal{G}(u) = F(\mathfrak{B}, u) \quad \text{for all} \quad u \in H_0^1(\Omega).$$

Thus, if $K$ is a class of operators of the form (6) for which $1 \leq b(x) \leq 3$ a.e. $x \in \Omega$, there does not exist a strong $K$-extension of $\mathfrak{A}$.

## 5. Other closures

For a given family $\mathfrak{A}$ of continuous mappings of $X$ into $Y$ let us consider various closures of the graph $\operatorname{gr}\mathcal{F}$ of the multivalued mapping $\mathcal{F} : X \rightsquigarrow Y$ defined as $\mathcal{F}(x) = F(\mathfrak{A}, x)$ for all $x \in X$.

First of all, we shall construct a family $\mathfrak{A}$ for which the closure of $\operatorname{gr}\mathcal{F}$ in the weak topology of $X \times Y$ coincides with the whole space $X \times Y$.

Let $\Omega = (0,1)$ and $X = H_0^1(\Omega)$, $Y = H^{-1}(\Omega)$. Let $\mathfrak{A} = \{A_\chi\}_{\chi\in\Lambda}$ where $\Lambda$ is the set of all characteristic functions of measurable subsets of $\Omega$ and

$$A_\chi = \frac{d}{dx}\left((1+\chi)\frac{d}{dx}\right),\ \chi \in \Lambda.$$

Let $\lambda : \mathbb{R} \to \{-1, 1\}$ be the 1-periodic function

$$\lambda(t) = \begin{cases} 1 & \text{for } t \in [0, 1/2), \\ -1 & \text{for } t \in [1/2, 1). \end{cases}$$

Set $\lambda^n(x) = \lambda(nx)$, $\chi^n = (1+\lambda^n)/2$ and $u^n(x) = \int_0^x \lambda^n(t)\,dt$, $n = 1, 2, \ldots$. It is clear that $u^n \rightharpoonup 0$ in $H_0^1(\Omega)$. On the other hand, $(1+\chi^n)\frac{du^n}{dx} \rightharpoonup \frac{1}{2}$ in $L^2(\Omega)$. Denote by $\omega_{m,k}$ the intervals $\left(\frac{k}{m}, \frac{k+1}{m}\right)$, $k = 0, 1, \ldots, m-1$; $m = 1, 2, \ldots$. Put $v^n_{m,k} = \chi_{m,k}u^{l(m,n)}$, $k = 0, 1, \ldots, m-1$; $m, n = 1, 2, \ldots$ where $l(m,n) = mn$ and $\chi_{m,k}$ is the characteristic function of $\omega_{m,k}$. Hence $v^n_{m,k} \rightharpoonup 0$ in $H_0^1(\Omega)$ and $(1+\chi^n)\frac{dv^n_{m,k}}{dx} \rightharpoonup \frac{1}{2}\chi_{m,k}$ in $L^2(\Omega)$ as $n \to +\infty$. Thus, we see that the element $(0, f)$ belongs to the weak closure of $\operatorname{gr}\mathcal{F}$ for $f = \frac{dg}{dx}$ when $g$ is equal to a simple function constructed by means of the intervals $\omega_{m,k}$. Since the strong closure of the set consisting of such $f$ coincides with $H^{-1}(\Omega)$, the weak closure of $\operatorname{gr}\mathcal{F}$ coincides with $X \times Y$.

For a multivalued mapping $\mathcal{F}$, it is interesting to consider repeated closures of $\operatorname{gr}\mathcal{F}$, i.e. when, for instance, one first considers weak or strong closure of $\mathcal{F}^{-1}(y)$ for every $y \in Y$ and obtains a multivalued mapping $\mathcal{G}$, then one considers weak or strong closure of $\mathcal{G}(x)$ for every $x \in X$. The results concerning such closures are summarized in Proposition 5.1.

Define multivalued mappings $\mathcal{G}_i, \tilde{\mathcal{G}}_i, 1 \le i \le 4$ setting for every $x \in X$

$$\mathcal{G}_1(x) = \left(\operatorname{cl}_w(\operatorname{cl}_w \mathcal{F})^{-1}\right)^{-1}(x), \tilde{\mathcal{G}}_1(x) = \operatorname{cl}_w\left(\operatorname{cl}_w \mathcal{F}^{-1}\right)^{-1}(x),$$

$$\mathcal{G}_2(x) = \left(\operatorname{cl}_s(\operatorname{cl}_w \mathcal{F})^{-1}\right)^{-1}(x), \tilde{\mathcal{G}}_2(x) = \operatorname{cl}_s\left(\operatorname{cl}_w \mathcal{F}^{-1}\right)^{-1}(x),$$

$$\mathcal{G}_3(x) = \left(\operatorname{cl}_w(\operatorname{cl}_s \mathcal{F})^{-1}\right)^{-1}(x), \tilde{\mathcal{G}}_3(x) = \operatorname{cl}_w\left(\operatorname{cl}_s \mathcal{F}^{-1}\right)^{-1}(x),$$

$$\mathcal{G}_4(x) = \left(\operatorname{cl}_s(\operatorname{cl}_s \mathcal{F})^{-1}\right)^{-1}(x), \tilde{\mathcal{G}}_4(x) = \operatorname{cl}_s\left(\operatorname{cl}_s \mathcal{F}^{-1}\right)^{-1}(x).$$

If $Y = X^*$ and $X$ is a separable reflexive Banach space we have

**Proposition 5.1.** *Let $\mathfrak{A} \subset M(\nu_1, \nu_2)$. Suppose that for $x = 0$ the set $F(\mathfrak{A}, x)$ is bounded in $X^*$. Then for every $x \in X$*

$$\mathcal{G}_1(x) = F(\mathrm{cl}_G \, \mathrm{cl}_w \, \mathfrak{A}, x), \tilde{\mathcal{G}}_1(x) = F(\mathrm{cl}_w \, \mathrm{cl}_G \, \mathfrak{A}, x),$$
$$\mathcal{G}_2(x) = F(\mathrm{cl}_w \, \mathfrak{A}, x), \tilde{\mathcal{G}}_2(x) = F(\mathrm{cl}_G \, \mathfrak{A}, x),$$
$$\mathcal{G}_3(x) = F(\mathrm{cl}_G \, \mathfrak{A}, x), \tilde{\mathcal{G}}_3(x) = F(\mathrm{cl}_w \, \mathfrak{A}, x),$$
$$\mathcal{G}_4(x) = \mathrm{cl}_s \, F(\mathfrak{A}, x), \tilde{\mathcal{G}}_4(x) = \mathrm{cl}_s \, F(\mathfrak{A}, x).$$

Proof: We will prove only that $\mathcal{G}_1(x) = F(\mathrm{cl}_G \, \mathrm{cl}_w \, \mathfrak{A}, x)$ for any $x \in X$. The other equalities can be proved in a similar manner.

Denote by $\mathcal{H}$ the multivalued mapping defined as $\mathcal{H}(x) = F(\mathrm{cl}_w \, \mathfrak{A}, x)$ for all $x \in X$. By Proposition 4.1, $\mathrm{cl}_w \, F(\mathfrak{A}, x) = \mathcal{H}(x)$ for every $x \in X$ and $\mathrm{cl}_w \, \mathcal{H}^{-1}(y) = \mathrm{cl}_w \, Z(\mathrm{cl}_w \, \mathfrak{A}, y) = Z(\mathrm{cl}_G \, \mathrm{cl}_w \, \mathfrak{A}, y)$ for every $y \in Y$. Hence $\mathcal{G}_1(x) = F(\mathrm{cl}_G \, \mathrm{cl}_w \, \mathfrak{A}, x)$ for any $x \in X$. □

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