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The periodic problem for semilinear differential inclusions in Banach spaces

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Abstract. Sufficient conditions on the existence of periodic solutions for semilinear differential inclusions are given in general Banach space. In our approach we apply the technique of the translation operator along trajectories. Due to recent results it is possible to show that this operator is a so-called decomposable map and thus admissible for certain fixed point index theories for set-valued maps. Compactness conditions are formulated in terms of the Hausdorff measure of noncompactness.

Keywords: periodic solutions, translation operator along trajectories, set-valued maps, C_0-semigroup, R_δ-sets

Classification: 34A60, 34C25, 47H11

Introduction

In the paper we will be concerned with the existence of mild periodic solutions for a differential inclusion in Banach space X of the type

$$x'(t) \in Ax(t) + F(t, x(t)), \tag{1}$$

where A is the infinitesimal generator of a semigroup $\{U(t)\}_{t \geq 0}$ of class C_0 and F is a set-valued map with convex values.

This problem and other boundary value problems more general than the initial value problem were studied under various conditions on A and F by several authors. Here we mention the works [18], where A depends in addition on a time variable t and generates a strongly continuous evolution system and F may be nonconvex valued, [11] for a functional-differential inclusion of the form (1) (see also the references given there) and [17] for the quasi-linear case, i.e. the operator A depends on t and x. In all these papers the Banach space X is assumed to be separable.

The approach proposed in the present note is to study the periodic problem by the method of the translation operator along trajectories (also called the Poincaré operator). This well-known technique was developed in [14] in the classical case, i.e. where we have unique solvability of the initial value problem.

In the case where we do not have unique solvability and in particular for differential inclusions the method was adopted in [9] in the finite dimensional setting

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and the topological degree theory for the so-called decomposable mappings (see Section 2 for the definition) was applied. Due to recent results on the topological structure of the set of mild solution (see [6], [5]) it is possible to extend the method also to differential inclusions of the form (1). However, the extension is complicated by the fact that the translation operator is compact (or more generally condensing) only in particular cases (e.g. for the trivial system $x' = 0$ the translation operator is the identity I on the space X and thus it is condensing iff X is finite dimensional). Thus we impose compactness assumptions on both, the semigroup and the multivalued perturbation and we will see in conclusion that periodic solutions can be obtained if we strengthen the compactness assumptions on $\{U(t)\}_{t\geq 0}$ and weaken them on F and vice versa.

The attitude presented here was also used recently in [13]. However, in contrast to this paper, we include in our discussion the case where the Banach space is nonseparable. In addition, weaker compactness assumptions on the semigroup will be considered (see Section 6).

1. Preliminaries

In the sequel, X will always denote a Banach space over $\mathbb{C}$ or $\mathbb{R}$ with the norm $|\cdot|$, $B_\alpha(x)$ is the open ball in X with center x and radius α and $\overline{B}_\alpha(x)$ is its closure. If A is a linear operator in X, we denote by $\varrho(A)$ the resolvent set and by $\sigma(A)$ the spectrum of A. By $\sigma_P(A)$ we mean the point spectrum of the operator A. The Banach space of all bounded linear operators in X with the operator norm $|\cdot|_0$ is denoted by $\mathcal{L}(X)$. Given reals $a < b$ we let $C([a,b],X)$ be the Banach space of continuous $y:[a,b]\to X$ equipped with the maximum norm $|\cdot|_\infty$ and by $L^1([a,b],X)$ we mean the space of all strongly measurable maps $f:[a,b]\to X$ such that $|f|_1 := \int_a^b |f(\tau)|\,d\tau$ is finite; $L^1([a,b],X)$ is a Banach space w.r.t. the norm $|\cdot|_1$.

Recall that on the nonempty, bounded subsets of X the Hausdorff measure of noncompactness (MNC) χ is given by

$$\chi(\Omega) := \inf\Big\{\alpha > 0 \mid \text{ there are finitely many points } x_1,\dots,x_n \in X \text{ with } \Omega \subset \bigcup_{i=1}^n B_\alpha(x_i)\Big\}.$$

It is well known that χ is monotone, i.e. $\Omega \subset \tilde\Omega$ implies $\chi(\Omega) \le \chi(\tilde\Omega)$ and algebraically semiadditive, i.e. $\chi(\Omega + \tilde\Omega) \le \chi(\Omega) + \chi(\tilde\Omega)$ for bounded Ω, $\tilde\Omega \subset X$.

Next, given a subspace $Y \subset X$ and $\Omega_0 \subset Y$ bounded we write $\chi_Y(\Omega_0)$ for the Hausdorff MNC relative to Y, i.e. we require that the centers of the covering balls are chosen from Y instead of X. It is easy to see that $\chi(\Omega_0) \le \chi_Y(\Omega_0) \le 2\chi(\Omega_0)$ for $\Omega_0 \subset Y$ bounded.

We will also use the sequential MNC $\tilde\chi_0$ generated by χ, i.e.

$$\tilde\chi_0(\Omega) := \sup\{\chi(\{x_n : n \ge 1\}) : (x_n) \text{ is a sequence in } \Omega\},$$

with $\Omega \subset X$ bounded. Then we have

$$\tilde{\chi}_0(\Omega) \leq \chi(\Omega) \leq 2\tilde{\chi}_0(\Omega) \tag{2}$$

and $\chi(\Omega) = \tilde{\chi}_0(\Omega)$ in case X is separable.

Given an operator $T \in \mathcal{L}(X)$, its χ-norm is given by

$$|T|^{(\chi)} := \chi(T\overline{B}_1(0)).$$

It is known that $|T|^{(\chi)} \leq |T|_0$ and that $\chi(T\Omega) \leq |T|^{(\chi)}\chi(\Omega)$ for any $\Omega \subset X$ bounded.

For all the above given results on the Hausdorff MNC and its related concepts we refer to [1].

The following interchange rule of χ and integration is shown in [12, Theorem 2.2.3].

Proposition 1. *Let a sequence of functions $(f_n) \subset L^1([a,b],X)$ be such that there exists $\nu \in L^1([a,b],\mathbb{R})$ with $|f_n(t)| \leq \nu(t)$ a.e. on $[a,b]$ and every $n = 1,2,3,\ldots$, and that there exists $q \in L^1([a,b],\mathbb{R})$ such that $\chi(\{f_n(t) : n \geq 1\}) \leq q(t)$ a.e. on $[a,b]$. Then for any $t \in [a,b]$ the following estimations hold:*

(i) $\chi\left(\left\{\int_a^t f_n(\tau)\,d\tau : n \geq 1\right\}\right) \leq 2\int_a^t q(\tau)\,d\tau$ *if X is an arbitrary Banach space,*

(ii) $\chi\left(\left\{\int_a^t f_n(\tau)\,d\tau : n \geq 1\right\}\right) \leq \int_a^t q(\tau)\,d\tau$ *if X is a separable Banach space.*

A Banach space X is called a weakly compactly generated Banach space (wcg space), if there exists a weakly compact set $K \subset X$ such that the linear span of K is dense in X (see [8], [16]).

Evidently every reflexive Banach space is a wcg space. Furthermore every separable Banach space is wcg, even compactly generated — consider $\{x_n/n : n \geq 1\}$ where (x_n) is a dense sequence in the unit ball of the separable Banach space.

Given two Banach spaces X, Y, by a set-valued map $\varphi : X \to Y$ we mean a transformation from X into the nonempty closed subsets of Y. We say that φ is upper semi-continuous (u.s.c.), if $\varphi^{-1}(V) := \{x \in X : \varphi(x) \subset V\}$ is open whenever $V \subset Y$ is open. The set-valued map $\varphi : X \to X$ is called condensing, if φ is u.s.c. and for every bounded $\Omega \subset X$ the set $\varphi(\Omega) \subset X$ is bounded and $\chi(\varphi(\Omega)) < \chi(\Omega)$ provided that $\chi(\Omega) \neq 0$. For instance, an operator $T \in \mathcal{L}(X)$ is condensing if $|T|^{(\chi)} < 1$. Finally, a point $x \in X$ such that $x \in \varphi(x)$ is called a fixed point of φ.

2. The class of decomposable mappings

In our considerations we will use fixed point results which are available for the so-called decomposable mappings (see [9]). Recall that a set-valued mapping

$\Phi : X \to X$ is decomposable if there is a Banach space Y and two maps $\varphi : X \to Y$ and $f : Y \to X$ where f is a continuous (single-valued) map, φ is a u.s.c. map with values being R_δ-sets (i.e. for each $x \in X$ the set $\varphi(x)$ can be represented as an intersection of a decreasing sequence of compact, contractible sets, see [10]) and $\Phi = f \circ \varphi$.

A decomposable map is easiest denoted by a diagram

$$X \xrightarrow{\varphi} Y \xrightarrow{f} X$$

called the decomposition of Φ.

For condensing, decomposable mappings a topological fixed point index theory can be constructed (see [4], [2]). Applying this fixed point index theory it is possible to obtain the following results (see [3] for the proofs), which will be useful for us in the sequel. First we have

Theorem 2 (Leray-Schauder alternative)**.** *Let $\Phi : X \to X$ be a condensing, decomposable mapping. Then either the set*

$$G := \{x \in X \mid x \in \lambda\Phi(x) \quad \text{for} \quad \lambda \in (0,1)\}$$

is unbounded or Φ has a fixed point.

Our next result is the generalization of a fixed point theorem due to Lasota and Opial for the decomposable mappings. In [15] this result was shown for compact maps with convex values and was applied in the Nicoletti boundary value problem.

Theorem 3. *Let $\Phi : X \to X$ be a condensing, decomposable mapping and let $\varphi_0 : X \to X$ be a condensing map with closed convex values. Assume also that φ_0 is homogeneous and that $x \in \varphi_0(x)$ implies $x = 0$. Suppose that there exists $\alpha > 0$ such that*

$$\Phi(x) \subset \varphi_0(x) + \overline{B}_\alpha(0) \quad \text{for each} \quad x \in X.$$

Then Φ has a fixed point.

3. The periodic problem for semilinear differential inclusions

In the paper we consider the periodic problem

$$\begin{cases} x'(t) \in Ax(t) + F(t, x(t)), \\ x(0) = x(T) \end{cases} \tag{3}$$

under the following assumptions:

(A1) A is a closed linear (unbounded) operator on X, which is the infinitesimal generator of a semigroup $\{U(t)\}_{t\geq 0}$ of class C_0;

(A2) there is a constant $\omega \in \mathbb{R}$ such that $|U(t)|^{(X)} \leq \exp(\omega t)$ for each $t \geq 0$.

Recall that given any semigroup $\{U(t)\}_{t\geq 0}$ there are constants $C \geq 0$ and $\omega \in \mathbb{R}$ such that $|U(t)|_0 \leq C\exp(\omega t)$ for $0 \leq t < \infty$. Moreover, renorming the Banach space X in an appropriate way it is possible to choose the constant C equal to 1. Hence, since $|\cdot|^{(\chi)} \leq |\cdot|_0$ we see that assumption (A2) is always fulfilled for some $\omega \in \mathbb{R}$. In the sequel we will impose further restrictions on ω.

We will suppose that the set-valued map $F : [0,T] \times X \to X$ has closed convex values and satisfies

- (F1) for every $x \in X$ the set-valued map $F(\cdot, x) : [0,T] \to X$ has a strongly measurable selection,
- (F2) for a.e. $t \in [0,T]$ the set-valued map $F(t,\cdot) : X \to X$ is u.s.c.,
- (F3) $\|F(t,x)\| := \sup\{|z| : z \in F(t,x)\} \leq \gamma(t)(1+|x|)$ a.e. on $[0,T]$ and every $x \in X$ with $\gamma \in L^1([0,T],\mathbb{R})$,
- (F4) for every nonempty bounded set $B \subset X$ we have $\chi(F(t,B)) \leq k(t)\chi(B)$ a.e. on $[0,T]$ with $k \in L^1([0,T],\mathbb{R})$.

Given $x_0 \in X$, we denote by $S(x_0) \subset C([0,T],X)$ the set of all mild solutions of the initial value problem

$$\begin{cases} x'(t) \in Ax(t) + F(t,x(t)), \\ x(0) = x_0. \end{cases} \tag{4}$$

Recall that a continuous mapping $x : [0,T] \to X$ is called a mild solution of the initial value problem (4) if x satisfies the integral equation

$$x(t) = U(t)x_0 + \int_0^t U(t-s)f(s)\,ds \quad \text{on} \quad [0,T], \tag{5}$$

where $f \in N_F(x) := \{g \in L^1([0,T],X) : g(t) \in F(t,x(t)) \text{ a.e. on } [0,T]\}$. Note that under the above conditions (F1)–(F3) it is well known that for each $y \in C([0,T],X)$ the set $N_F(y)$ is nonempty (see [7]).

A solution of the periodic problem (3) is a map $x \in C([0,T],X)$ such that equation (5) is fulfilled and $x(0) = x(T)$.

4. The operator of translation along trajectories

In order to apply the results on decomposable mappings from Section 2 on the periodic problem (3) we introduce the operator of translation along trajectories P_t, $0 \leq t \leq T$, given by the diagram

$$X \xrightarrow{S} C([0,T],X) \xrightarrow{e_t} X, \tag{6}$$

i.e. $P_t := e_t \circ S$. Here S is the solution operator as in Section 3 and e_t denotes the evaluation mapping $e_t(f) = f(t)$.

We have the following

Lemma 4. *Let the conditions* (A1), (A2), (F1)–(F4) *be fulfilled. Then the operator of translation along trajectories P_t, $0 \le t \le T$, is decomposable.*

Proof: By means of standard techniques occurring in the existence theory of (4) it is known that the mapping S is u.s.c. Moreover, it was shown in [5] (see Section 6 in this paper, compare also [6]) that for each $x_0 \in X$ the set $S(x_0) \subset C([0,T],X)$ is an R_δ-set. It follows that diagram (6) defines indeed a decomposable map. □

The following result is crucial in our further investigations.

Lemma 5. *Let $\Gamma \subset L^1([a,b],X)$. Assume that Γ is integrably bounded, i.e. there exists $\nu \in L^1([a,b],\mathbb{R})$ such that $\|\{f(t) : f \in \Gamma\}\| \le \nu(t)$ a.e. on $[a,b]$, and that there exists $q \in L^1([a,b],\mathbb{R})$ such that $\chi(\{f(t) : f \in \Gamma\}) \le q(t)$ a.e. on $[a,b]$. Then for any $t \in [a,b]$ the following estimations hold:*

(i) $\chi\left(\left\{\int_a^t f(\tau)\,d\tau : f \in \Gamma\right\}\right) \le 4\int_a^t q(\tau)\,d\tau$ *if X is an arbitrary Banach space,*

(ii) $\chi\left(\left\{\int_a^t f(\tau)\,d\tau : f \in \Gamma\right\}\right) \le 2\int_a^t q(\tau)\,d\tau$ *if X is a wcg space,*

(iii) $\chi\left(\left\{\int_a^t f(\tau)\,d\tau : f \in \Gamma\right\}\right) \le \int_a^t q(\tau)\,d\tau$ *if X is a separable Banach space.*

Proof: Let (f_n) be an arbitrary sequence in Γ and consider $\chi\left(\left\{\int_a^t f_n(\tau)\,d\tau : n \ge 1\right\}\right)$. Then by Proposition 1(i), we have that $\chi\left(\left\{\int_a^t f_n(\tau)\,d\tau : n \ge 1\right\}\right) \le 2\int_a^t q(\tau)\,d\tau$. Further, using (2), we see that

$$\chi\left(\left\{\int_a^t f(\tau)\,d\tau : f \in \Gamma\right\}\right) \le 2\tilde{\chi}_0\left(\left\{\int_a^t f(\tau)\,d\tau : f \in \Gamma\right\}\right) \tag{7}$$

and we obtain (i).

Now let X be a wcg space and let again (f_n) be a sequence in Γ. Since all the f_n are strongly measurable, we may assume that $Y_0 := \overline{\operatorname{span}}\bigcup_{n\ge 1} f_n([a,b])$ is separable. Since X is a wcg space, it is known that there exists a closed separable subspace Y such that $Y_0 \subset Y \subset X$ and a continuous linear projection $P : X \to Y$ with $|P|_0 = 1$ (see [8, p. 149]). Therefore it is clear that for any bounded $A \subset Y$ we have $\chi_Y(A) = \chi(A)$ and we conclude from Proposition 1(ii) that $\chi\left(\left\{\int_a^t f_n(\tau)\,d\tau : n \ge 1\right\}\right) = \chi_Y\left(\left\{\int_a^t f_n(\tau)\,d\tau : n \ge 1\right\}\right) \le \int_a^t q(\tau)\,d\tau$. Now, again applying (7), we get the estimation in (ii).

In case X is separable we know that the coefficient 2 in the estimation (7) may be omitted. Thus the claim in (iii) follows again by an application of Proposition 1(ii).

□

Theorem 6. *Let $\Omega \subset X$ bounded and let the conditions* (A1), (A2), (F1)–(F4) *be fulfilled. Moreover assume*

(A3) $\quad t \to U(t)$ *is continuous with respect to the norm in $\mathcal{L}(X)$ for $t > 0$.*

Then the following estimations hold:

(i) $\chi(P_T(\Omega)) \le \exp\left(\omega T + 4|k|_1\right)\ \chi(\Omega)$ *if X is an arbitrary Banach space,*
(ii) $\chi(P_T(\Omega)) \le \exp\left(\omega T + 2|k|_1\right)\ \chi(\Omega)$ *if X is a wcg space.*

Proof: We start with the proof showing that the set $S(\Omega) \subset C([0,T],X)$ is bounded and equicontinuous at every $t > 0$.

Let $y \in S(\Omega)$. Then there is $x_0 \in \Omega$ and $f \in N_F(y)$ such that

$$y(t) = U(t)x_0 + \int_0^t U(t-\tau)f(\tau)\,d\tau, \quad t \in [0,T]. \tag{8}$$

It follows by (F3) that

$$\begin{aligned}
|y(t)| &\le |U(t)|_0|x_0| + \int_0^t |U(t-\tau)f(\tau)|\,d\tau \\
&\le M\,R + M\int_0^t \gamma(\tau)(1+|y(\tau)|)\,d\tau \\
&\le M\,R + M\int_0^T \gamma(\tau)\,d\tau + M\int_0^t \gamma(\tau)|y(\tau)|\,d\tau,
\end{aligned}$$

with $|U(s)|_0 \le M$ for $s \in [0,T]$ and $\|\Omega\| \le R$. Applying the Gronwall inequality gives

$$|y(t)| \le M\left(R + \int_0^T \gamma(\tau)\,d\tau\right)\exp\left(M\int_0^T \gamma(\tau)\,d\tau\right) =: \tilde{R} \ \text{ for every } \ t \in [0,T]$$

i.e. $S(\Omega)$ is bounded w.r.t. $|\cdot|_\infty$ by $\tilde{R}$. Moreover, again by (F3)

$$\|\,\{f(t) : f \in N_F(S(\Omega))\}\,\| \le \gamma(t)(1+\tilde{R}) \ \text{ a.e. on } \ [0,T]. \tag{9}$$

In order to prove the equicontinuity of $S(\Omega)$, observe that by (8) we have

$$\begin{aligned}
y(t) - y(s) = U(t)x_0 - U(s)x_0 &+ \int_0^s U(t-\tau)f(\tau) - U(s-\tau)f(\tau)\,d\tau \\
&+ \int_s^t U(t-\tau)f(\tau)\,d\tau
\end{aligned}$$

for $T \geq t > s \geq 0$. Thus

$$
\begin{aligned}
|y(t) - y(s)| &\leq |U(t) - U(s)|_0 |x_0| + \int_0^s |U(t-\tau) - U(s-\tau)|_0 |f(\tau)|\, d\tau \\
&\quad + \int_s^t |U(t-\tau)|_0 |f(\tau)|\, d\tau \\
&\leq |U(t) - U(s)|_0\, R + (1 + \tilde{R}) \Bigg[\int_0^s |U(t-\tau) - U(s-\tau)|_0 \gamma(\tau)\, d\tau \\
&\quad + M \int_s^t \gamma(\tau)\, d\tau \Bigg],
\end{aligned}
$$

where the last expression is independent of the particular $y \in S(\Omega)$ and tends to zero as $s \longrightarrow t$ as a consequence of the assumption (A3). Thus we see that $S(\Omega)$ is equicontinuous at $t > 0$.

Further, it is clear that we have the inclusion

$$P_t(\Omega) \subset U(t)\Omega + \left\{ \int_0^t U(t-\tau) f(\tau)\, d\tau : f \in N_F(S(\Omega)) \right\} \quad \text{on} \quad [0, T].$$

Defining the function $\psi(t) := \chi(P_t(\Omega))$, $0 \leq t \leq T$, we thus obtain by the properties of χ

$$\psi(t) \leq \chi(U(t)\Omega) + \chi\left(\left\{ \int_0^t U(t-\tau) f(\tau)\, d\tau : f \in N_F(S(\Omega)) \right\} \right). \tag{10}$$

Using (9) we see that

$$\| \{ U(t-\tau) f(\tau) : f \in N_F(S(\Omega)) \} \| \leq M \gamma(\tau)(1 + \tilde{R}) \quad \text{for a.e.} \quad \tau \in [0, t].$$

Moreover, since $\{U(t-\tau) f(\tau) : f \in N_F(S(\Omega))\} \subset U(t-\tau) F(\tau, P_\tau(\Omega))$ a.e. we get by (F4)

$$\chi(\{ U(t-\tau) f(\tau)\, d\tau : f \in N_F(S(\Omega)) \}) \leq \exp(\omega(t-\tau)) k(\tau) \psi(\tau) \quad \text{for a.e.} \quad \tau \in [0, t].$$

Here the right hand side of the estimation is integrable since in particular the function ψ is continuous at every $t > 0$. This follows from

$$\psi(t) - \psi(s) \leq \chi(\{ y(t) - y(s) : y \in S(\Omega) \}) \leq \sup\{ |y(t) - y(s)| : y \in S(\Omega) \}$$

and the fact proven above that $S(\Omega)$ is equicontinuous in $t > 0$.

We are now in the position to apply Lemma 5 from which we obtain

$$
\begin{aligned}
&\chi\left(\left\{ \int_0^t U(t-\tau) f(\tau)\, d\tau : f \in N_F(S(\Omega)) \right\} \right) \\
&\qquad \leq c_i \int_0^t \exp(\omega(t-\tau)) k(\tau) \psi(\tau)\, d\tau,
\end{aligned} \tag{11}
$$

where $c_1 = 4$ and $c_2 = 2$ for the cases (i) and (ii), respectively. Hence, we get from (10) and (A2)

$$\psi(t) \leq \exp(\omega t)\chi(\Omega) + c_i \int_0^t \exp(\omega(t-\tau))k(\tau)\psi(\tau)\,d\tau \quad \text{on} \quad [0,T].$$

Denoting the right hand side of the above inequality by $\varrho(t)$ we see that

$$\varrho'(t) = \omega\varrho(t) + c_i k(t)\psi(t) \leq (\omega + c_i k(t))\varrho(t) \quad \text{a.e. on} \quad [0,T].$$

It follows

$$\psi(t) \leq \varrho(t) \leq \exp\left(\omega t + c_i \int_0^t k(\tau)\,d\tau\right) \chi(\Omega) \quad \text{on} \quad [0,T]$$

and we are done, since $\psi(T) = \chi(P_T(\Omega))$. □

In the case where X is a separable Banach space, the space $C([0,T],X)$ is separable and thus the set $S(\Omega) \subset C([0,T],X)$ is also separable. Hence, there is sequence $(x_n) \subset S(\Omega)$ such that $S(\Omega) \subset \overline{(x_n)}$. Then $\chi(P_t(\Omega)) = \chi(\{x_n(t) : n \geq 1\})$ and it follows that $\chi(P_t(\Omega))$ is measurable (see [16, p. 157]). Therefore we have

Corollary 7 (see [13]). *Let X be a separable Banach space, $\Omega \subset X$ bounded and let the conditions* (A1), (A2), (F1)–(F4) *be fulfilled. Then*

$$\chi(P_T(\Omega)) \leq \exp\left(\omega T + |k|_1\right) \chi(\Omega).$$

Proof: The proof follows from the above remark and the fact that due to Lemma 5(iii) the coefficient c_i on the right hand side of the estimation (11) is 1. □

In the general case we do not know how to prove measurability of $\chi(P_t(\Omega))$. In the above Theorem 6 we used assumption (A3) to get continuity of $\chi(P_t(\Omega))$.

5. Periodic solutions of semilinear differential inclusions

In this section we will present sufficient conditions for the existence of solutions to problem (3).

Theorem 8. *Suppose that the conditions* (A1), (A2), (F1), (F2) *and* (F4) *are fulfilled. Assume also that*

(F3′) $\|F(t,x)\| \leq \delta(t)$ *a.e. on* $[0,T]$ *and every* $x \in X$ *with* $\delta \in L^1([0,T],\mathbb{R})$

and $1 \notin \sigma_P(U(T))$. *Then the periodic problem* (3) *has a solution in each of the following cases:*

(i) (A3) *is fulfilled and* $\omega T + 4|k|_1 < 0$ *if* X *is an arbitrary Banach space,*
(ii) (A3) *is fulfilled and* $\omega T + 2|k|_1 < 0$ *if* X *is a wcg space,*
(iii) $\omega T + |k|_1 < 0$ *if* X *is separable.*

Proof: We will apply the fixed point Theorem 3 with P_T instead of Φ and $U(T)$ instead of φ_0. By Lemma 4, Theorem 6 and Corollary 7 we know that P_T is a condensing, decomposable mapping. Since $U(T)$ is linear, it is homogeneous, condensing and since $1 \notin \sigma_P(U(T))$ we see that $x = 0$ is the only solution of $x = U(T)x$. Next, we have for each $f \in N_F(y)$, where $y \in C([0,T], X)$,

$$\begin{aligned}\left|\int_0^T U(T-\tau)f(\tau)\,d\tau\right| &\le \int_0^T |U(T-\tau)f(\tau)|\,d\tau \\ &\le M\int_0^T |f(\tau)|\,d\tau \le \ M\int_0^T \delta(\tau)\,d\tau =: \alpha\end{aligned}$$

by (F3$'$). It follows $P_T(x) \subset U(T)x + \overline{B}_\alpha(0)$ and we see that all the assumptions of Theorem 3 are fulfilled. Hence, P_T has a fixed point and the periodic problem (3) has a solution. □

As a first corollary we will consider the case where the semigroup $\{U(t)\}_{t\ge 0}$ is compact (i.e. $U(t)$ is a compact map for each $t > 0$).

Corollary 9. *Let* (A1), (F1), (F2), (F3$'$) *and* (F4) *be fulfilled and assume that* $1 \notin \sigma_P(U(T))$. *Moreover, assume that* $\{U(t)\}_{t\ge 0}$ *is a compact semigroup. Then the periodic problem* (3) *has a solution.*

Proof: Since the mappings $U(t)$, $t > 0$, are compact, we may choose the constant $\omega < 0$ in (A2) such that $\omega T + 4|k|_1 < 0$. Next recall that by Pazy's theorem assumption (A3) is fulfilled, since $\{U(t)\}_{t\ge 0}$ is a compact semigroup (see [19, p. 48]). Thus the corollary follows from part (i) in the above theorem. □

Since in Theorem 8 it is sufficient to assume $1 \notin \sigma_P(U(T))$ we are able to express this assumption in terms of the infinitesimal generator A of the semigroup. Recall that in general the spectrum of the operators $U(t)$ cannot be characterized by the spectrum of A (see [19, p. 44], for an example where $\sigma(A) = \emptyset$ but $\sigma(U(t)) \neq \emptyset$ for any $t \ge 0$).

In the following corollary we assume that X is a Banach space over $\mathbb{C}$.

Corollary 10. *Let one of the conditions* (i), (ii) *or* (iii) *of Theorem* 8 *be fulfilled, but instead of* $1 \notin \sigma_P(U(T))$ *assume that*

$$\sigma_P(A) \cap \frac{2\pi i}{T}\mathbb{Z} = \emptyset. \tag{12}$$

Then the periodic problem (3) *has a solution.*

Proof: For the proof it is sufficient to show that $1 \in \sigma_P(U(T))$ is impossible. But, in view of the results from [19, p. 46], we know that if $\exp(\lambda t) \in \sigma_P(U(t))$, then there is $q \in \mathbb{Z}$ such that $\lambda + \frac{2\pi i q}{t} \in \sigma_P(A)$. Thus it is clear that (12) shows that $1 \notin \sigma_P(U(T))$. □

6. The case where $\omega \geq 0$

If we know that $\{U(t)\}_{t\geq 0}$ is a semigroup of contractions, then it is clear that we may choose $\omega = 0$ in the assumption (A2). However, observe that even under the strong compactness assumption $\chi(F(t, B)) = 0$ for $t \in [0, T]$ and bounded $B \subset X$ the estimates from Theorem 6 are not sufficient to guarantee that the operator of translation along trajectories is condensing. This leads to the following considerations.

With the periodic problem (3), where A is the infinitesimal generator of a C_0-semigroup we associate the set-valued map

$$Q : X \to X, \; Q := (I - U(T))^{-1}(P_T - U(T)) \tag{13}$$

provided that $1 \in \varrho(U(T))$.

It is evident that $x_0 \in Q(x_0)$ iff $x_0 \in P_T(x_0)$ and hence we will look in the sequel for fixed points of Q.

Lemma 11. *Let $\Omega \subset X$ be bounded and let the conditions* (A1)–(A3), (F1)–(F4) *be fulfilled. Assume also that $1 \in \varrho(U(T))$. Then the mapping Q defined in* (13) *is decomposable and satisfies the following estimate:*

$$\chi(Q(\Omega)) \leq |(I - U(T))^{-1}|^{(\chi)} \, c \exp\left(\omega T + c|k|_1\right) \; |k|_1 \; \chi(\Omega)$$

where $c = 4$ and $c = 2$ if X is a Banach space and X is wcg space, respectively.

In case X is separable the constant c can be chosen equal to 1 *and assumption* (A3) *is not necessary.*

Proof: Obviously the following diagram defines a decomposition of Q:

$$X \xrightarrow{S\times I} C([0,T], X) \times X \xrightarrow{(I-U(T))^{-1}(e_T(\cdot)-U(T)\cdot)} X.$$

Next, the inclusion

$$(P_t - U(t))(\Omega) \subset \left\{ \int_0^t U(t-\tau) f(\tau)\, d\tau : f \in N_F(S(\Omega)) \right\} \quad \text{on} \quad [0, T].$$

implies

$$\chi((P_t - U(t))(\Omega)) \leq \chi\left(\left\{ \int_0^t U(t-\tau) f(\tau)\, d\tau : f \in N_F(S(\Omega)) \right\} \right)$$

where the right hand side can be estimated as in the proof of Theorem 6 (see (11)). Hence, it follows by another application of Theorem 6

$$\begin{aligned}\chi((P_t - U(t))(\Omega)) &\le c \int_0^t \exp(\omega(t-\tau))k(\tau)\chi(P_\tau(\Omega))\,d\tau \\ &\le c\exp(\omega t)\int_0^t k(\tau)\exp\left(c\int_0^\tau k(\rho)\,d\rho\right)\,d\tau\;\chi(\Omega) \\ &\le c\exp\left(\omega t + c\int_0^t k(\tau)\,d\tau\right)\int_0^t k(\tau)\,d\tau\;\chi(\Omega)\end{aligned}$$

from which we obtain the lemma. □

We derive the following sufficient conditions for the existence of periodic solutions.

Theorem 12. *Let* (A1), (A2), (F1), (F2), (F3′) *and* (F4) *be fulfilled and assume that* $1 \in \varrho(U(T))$. *Then the periodic problem* (3) *has a solution in each of the following cases:*

- (i) (A3) *is fulfilled and* $|(I-U(T))^{-1}|^{(\chi)}\, 4\exp(\omega T + 4|k|_1)\,|k|_1 < 1$ *if* X *is an arbitrary Banach space,*
- (ii) (A3) *is fulfilled and* $|(I-U(T))^{-1}|^{(\chi)}\, 2\exp(\omega T + 2|k|_1)\,|k|_1 < 1$ *if* X *is a wcg space,*
- (iii) $|(I-U(T))^{-1}|^{(\chi)} \exp(\omega T + |k|_1)\,|k|_1 < 1$ *if* X *is separable.*

Proof: From Lemma 11 it is clear that in all the three cases the mapping Q given by (13) is condensing. Now let $y \in Q(x)$ for some $x \in X$. Then there is $z \in (P_T - U(T))(x)$ such that $y = (I-U(T))^{-1}z$. But we know already from the proof of Theorem 8 that $(P_T - U(T))(X) \subset \overline{B}_\alpha(0)$ with $\alpha = M|\delta|_1$. It follows

$$|y| \le |(I-U(T))^{-1}|_0|z| \le |(I-U(T))^{-1}|_0\,\alpha =: \beta$$

and thus the fixed points of the mapping λQ, $0 \le \lambda \le 1$, are bounded by β. Therefore we can apply the Leray-Schauder principle (Theorem 2) and we see that Q has a fixed point proving the theorem. □

The conditions in the above theorem assuring that the set-valued map Q is condensing can also be fulfilled in the case where $\omega \ge 0$. For example:

Corollary 13. *Let* (A1), (F1), (F2), (F3′) *and* (F4) *with* $k(t) \equiv 0$ *be fulfilled. Assume also that* $1 \in \varrho(U(T))$. *Then the periodic problem* (3) *has a solution.*

Proof: Arguing as in Lemma 11, we see that under the above assumptions the map Q is indeed completely continuous (i.e. Q maps bounded subsets onto relatively compact ones). Then it follows from Theorem 12 that Q has a fixed point and, consequently, the periodic problem (3) has a solution. □

Observe that in the above corollary assumption (A2) is not needed since there are no restrictions on the constant ω.

Remark 14. The results of the paper can be extended to time dependent linear operator $A(t)$, if the family $\{A(t)\}_{t\in[0,T]}$ generates a strongly continuous evolution system $U(t,s)$, $0 \leq s \leq t \leq T$. Here a solution to the periodic problem is a continuous $x : [0,T] \to X$ such that $x(0) = x(T)$ and

$$x(t) = U(t,0)x(0) + \int_0^t U(t,s)f(s)\,ds \quad \text{on} \quad [0,T],$$

where $f \in N_F(x)$.

Now let X be a separable Banach space and assume that $|U(t,s)|_0 \leq \exp(\omega(t-s))$ and $U(t,s)$ depends continuously on t, s while $s < t$. Then, due to results given in [6], the operator of translation along trajectories is decomposable. Hence, conditions from e.g. Theorem 8 are sufficient for the existence of periodic solutions where we replace "$1 \notin \sigma_P(U(T))$" by "$1 \notin \sigma_P(U(T,0))$".

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