Differentiability for minimizers of anisotropic integrals

P. Cavaliere, A. D'Ottavio, F. Leonetti, M. Longobardi

Abstract. We consider a function $u : \Omega \to \mathbb{R}^N$, $\Omega \subset \mathbb{R}^n$, minimizing the integral $\int_\Omega (|D_1 u|^2 + \cdots + |D_{n-1}u|^2 + |D_n u|^p)\,dx$, $2(n+1)/(n+3) \le p < 2$, where $D_i u = \partial u/\partial x_i$, or some more general functional with the same behaviour; we prove the existence of second weak derivatives $D(D_1 u), \ldots, D(D_{n-1}u) \in L^2$ and $D(D_n u) \in L^p$.

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0. Introduction

We consider the integral functional

$$I(u) = \int_\Omega F(Du(x))\,dx, \tag{0.1}$$

where Ω is bounded open subset of $\mathbb{R}^n$, $n \ge 2$, and $u : \Omega \to \mathbb{R}^N$, $N \ge 1$. F satisfies the following growth condition

$$a \sum_{i=1}^{n} |\xi_i|^{q_i} - b \le F(\xi) \le c \sum_{i=1}^{n} |\xi_i|^{q_i} + d, \qquad \forall\, \xi \in \mathbb{R}^{nN},$$

with a, b, c, d positive constants and $1 < q_i$, $i = 1, \ldots, n$. The isotropic case, i.e. $q_i = q\ \forall\, i$, has been deeply studied, see, for example, [G]. In this paper we study the anisotropic case, in which at least one of the q_i's differs from the others. We recall that in the anisotropic case, minimizers of (0.1) may be singular when no restriction is assumed on the q_i's ([G1], [M]). On the other hand, if the q_i's are close enough, there are regularity results, among them, [M1], [FS], [FS1] deal with scalar minimizers $u : \Omega \to \mathbb{R}$ of (0.1) and [L], [BL], [BL1], [D] consider (possibly) vector valued minimizers $u : \Omega \to \mathbb{R}^N$. In the present paper we improve on the differentiability result for minimizers of (0.1) contained in [BL1]. As there, the prototype for (0.1) is

$$I(u) = \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{n-1} |D_i u|^2 + \frac{1}{p} |D_n u|^p \right) dx, \tag{0.2}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $Du = (D_1 u, \ldots, D_n u)$, $D_i u = \partial u/\partial x_i$, $1 < p < 2$.

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1. Notation and main results

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, u be a (possibly) vector-valued function, $u : \Omega \to \mathbb{R}^N$, $N \geq 1$; we consider integrals

$$I(u) = \int_\Omega F(Du(x))\, dx, \tag{1.1}$$

where $F : \mathbb{R}^{nN} \to \mathbb{R}$ is in $C^1(\mathbb{R}^{nN})$ and satisfies, for some positive constants c and m,

$$|F(\xi)| \leq c(1 + \sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^p), \tag{1.2}$$

$$|\frac{\partial F}{\partial \xi_i^\alpha}(\xi)| \leq c(1 + |\xi_i|) \qquad \text{if } \ i = 1, \dots, n-1, \tag{1.3}$$

$$|\frac{\partial F}{\partial \xi_n^\alpha}(\xi)| \leq c(1 + |\xi_n|^{p-1}) \tag{1.4}$$

and

$$\begin{aligned} \sum_{j=1}^{n} \sum_{\beta=1}^{N} &\left(\frac{\partial F}{\partial \xi_j^\beta}(\nu) - \frac{\partial F}{\partial \xi_j^\beta}(\lambda) \right) (\nu_j^\beta - \lambda_j^\beta) \\ &\geq m \sum_{j=1}^{n-1} |\nu_j - \lambda_j|^2 + m \left(1 + |\nu_n|^2 + |\lambda_n|^2 \right)^{(p-2)/2} |\nu_n - \lambda_n|^2, \end{aligned} \tag{1.5}$$

for every $\lambda, \nu, \xi \in \mathbb{R}^{nN}$, $\alpha = 1, \dots, N$. Here, $\lambda = \{\lambda_i^\alpha\}$, $\xi = \{\xi_i^\alpha\}$, $|\lambda_i|^2 = \sum_{\alpha=1}^N |\lambda_i^\alpha|^2$. About p, we assume that

$$1 < p < 2. \tag{1.6}$$

We point out that (0.2) verifies (1.2)–(1.5). We say that u minimizes the integral (1.1) if $u : \Omega \to \mathbb{R}^N$, $u \in W^{1,p}(\Omega)$ with $D_i u \in L^2(\Omega)$ for $i = 1, \dots, n-1$, and

$$I(u) \leq I(u + \phi),$$

for every $\phi : \Omega \to \mathbb{R}^N$ with $\phi \in W_0^{1,p}(\Omega)$ and $D_i \phi \in L^2(\Omega)$ for $i = 1, \dots, n-1$. We first prove the following differentiability result for Du:

Theorem 1. *Let $u : \Omega \to \mathbb{R}^N$ satisfy $u \in W^{1,p}(\Omega)$ with $D_i u \in L^2(\Omega)$ for $i = 1, \dots, n-1$. If F satisfies* (1.2)–(1.5), (1.6) *and u minimizes the integral* (1.1)*, then for $s = 1, \dots, n-1$*

$$D_s(D_i u) \in L^2_{\text{loc}}(\Omega), \quad \forall\, i = 1, \dots, n-1, \tag{1.7}$$

$$D_s(D_n u) \in L^p_{\text{loc}}(\Omega), \tag{1.8}$$

$$D_s \left((1 + |D_n u|^2)^{(p-2)/4} D_n u \right) \in L^2_{\text{loc}}(\Omega). \tag{1.9}$$

This differentiability result allows us to improve on the integrability of first $n-1$ components $D_1 u, \dots, D_{n-1} u$ of the gradient:

Corollary 1. *Under the assumptions of Theorem 1 we have*

$$D_s u \in L^{\bar{p}^*}_{\mathrm{loc}}(\Omega), \quad s = 1, \dots, n-1,$$

where

$$\bar{p}^* = \frac{2pn}{p(n-3)+2} > 2.$$

So, by the improved integrability, we can get the existence of second weak derivatives with respect to x_n:

Theorem 2. *Under the assumptions of Theorem 1, if p verifies the additional restriction*

$$2\frac{n+1}{n+3} \leq p < 2, \tag{1.10}$$

then

$$D_n(D_i u) \in L^2_{\mathrm{loc}}(\Omega), \quad \forall\, i = 1, \dots, n-1,$$
$$D_n(D_n u) \in L^p_{\mathrm{loc}}(\Omega),$$
$$D_n\Big((1 + |D_n u|^2)^{(p-2)/4} D_n u\Big) \in L^2_{\mathrm{loc}}(\Omega).$$

Using Sobolev imbedding theorem we get Hölder continuity for u in dimension 2 and 3:

Corollary 2. *Under the assumptions of Theorem 2, we have*

$$u \in C^{0,\beta}_{\mathrm{loc}}(\Omega), \quad \forall\, \beta < 1, \quad \text{when } n = 2,$$
$$u \in C^{0,1-1/p}_{\mathrm{loc}}(\Omega), \quad \text{when } n = 3.$$

Remark. The higher differentiability contained in Theorem 1 and 2 was proved in [BL1] under the stronger assumption $2 - 2/(n+1) < p < 2$.

2. Known results

For a vector-valued function $f(x)$, define the difference

$$\tau_{s,h} f(x) = f(x + he_s) - f(x),$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction, and $s = 1, 2, \dots, n$. For $x_0 \in \mathbb{R}^n$, let $B_R = B_R(x_0)$ be the ball centered at x_0 with radius R. We now state several lemmas that we need later. In the following $f : \Omega \to \mathbb{R}^k$, $k \geq 1$; B_ρ, B_R, $B_{2\rho}$ and B_{2R} are concentric balls.

Lemma 1. *If $0 < \rho < R$, $|h| < R - \rho$, $1 \le t < \infty$, $s \in \{1, \dots, n\}$, f, $D_s f \in L^t(B_R)$, then*

$$\int_{B_\rho} |\tau_{s,h} f(x)|^t \, dx \le |h|^t \int_{B_R} |D_s f(x)|^t \, dx.$$

(See [G, p. 45], [C, p. 28].)

Lemma 2. *Let $f \in L^t(B_{2\rho})$, $1 < t < \infty$, $s \in \{1, \dots, n\}$; if there exists a positive constant C such that*

$$\int_{B_\rho} |\tau_{s,h} f(x)|^t \, dx \le C|h|^t,$$

for every h with $|h| < \rho$, then there exists $D_s f \in L^t(B_\rho)$. (See [G, p. 45], [C, p. 26].)

Lemma 3. *For every $\gamma \in (-1/2, 0)$ we have*

$$(2\gamma + 1)|a - b| \le \frac{|(1 + |a|^2)^\gamma \, a - (1 + |b|^2)^\gamma \, b|}{(1 + |a|^2 + |b|^2)^\gamma} \le \frac{c(k)}{2\gamma + 1}|a - b|,$$

for all $a, b \in \mathbb{R}^k$. (See [AF].)

Lemma 4. *Let Q be an open cube of $\mathbb{R}^n$, $f \in W^{1,1}(Q)$, with $D_i f \in L^{p_i}(Q)$, $p_i \ge 1$, $i = 1, \dots, n$ and*

$$\frac{1}{\bar{p}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}.$$

If $\bar{p} < n$ and $p_i < \bar{p}^* = \bar{p}n/(n - \bar{p})$ $\forall\, i = 1, \dots, n$, then $f \in L^{\bar{p}^*}(Q)$. (See [T], [AF1].)

3. Proof of Theorem 1

Since u minimizes the integral (1.1) with growth conditions as in (1.2)–(1.4), u solves the Euler equation

$$\int_\Omega \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial F}{\partial \xi_i^\alpha}(Du(x)) D_i \phi^\alpha(x) \, dx = 0, \tag{3.1}$$

for all functions $\phi : \Omega \to \mathbb{R}^N$, with $\phi \in W_0^{1,p}(\Omega)$ and $D_1\phi, \dots, D_{n-1}\phi \in L^2(\Omega)$. Let $R > 0$ be such that $\overline{B_{4R}} \subset \Omega$ and let B_ρ and B_R be concentric balls with $0 < \rho < R \le 1$. Fix s, take $0 < |h| < R$ and let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a "cut off" function in $C_0^2(B_R)$ with $0 \le \eta \le 1$ in $\mathbb{R}^n$ and $\eta \equiv 1$ on B_ρ. Using $\phi = \tau_{s,-h}(\eta^2 \tau_{s,h} u)$ in (3.1) we get, as usual,

$$0 = \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \int \frac{\partial F}{\partial \xi_i^\alpha}(Du) \tau_{s,-h} \left(D_i(\eta^2 \tau_{s,h} u^\alpha) \right) dx$$
$$= \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \int \tau_{s,h} \left(\frac{\partial F}{\partial \xi_i^\alpha}(Du) \right) (2\eta D_i \eta \tau_{s,h} u^\alpha + \eta^2 \tau_{s,h} D_i u^\alpha)\, dx,$$

so that

$$(I) = \int_{B_R} \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \tau_{s,h} \left(\frac{\partial F}{\partial \xi_i^\alpha}(Du) \right) \tau_{s,h} D_i u^\alpha \eta^2 \, dx \tag{3.2}$$
$$= - \int_{B_R} \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \tau_{s,h} \left(\frac{\partial F}{\partial \xi_i^\alpha}(Du) \right) 2\eta D_i \eta \tau_{s,h} u^\alpha \, dx = (II).$$

We apply (1.5) so that

$$m \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u(x)|^2 \eta^2(x)\, dx$$
$$+ m \int_{B_R} (1 + |D_n u(x)|^2 + |D_n u(x + h e_s)|^2)^{(p-2)/2} |\tau_{s,h} D_n u(x)|^2 \eta^2(x)\, dx \le (I).$$

Set

$$V(\xi_n) = (1 + |\xi_n|^2)^{(p-2)/4}\, \xi_n, \quad \forall\, \xi \in \mathbb{R}^{nN}. \tag{3.3}$$

Using Lemma 3 we find

$$C_2 |\tau_{s,h} D_n u(x)| \le \frac{|\tau_{s,h} V(D_n u(x))|}{(1 + |D_n u(x)|^2 + |D_n u(x + h e_s)|^2)^{(p-2)/4}} \tag{3.4}$$
$$\le C_3 |\tau_{s,h} D_n u(x)|,$$

for some positive constants C_2, C_3 depending only on N and p. Then

$$m \int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2 \, dx + m \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2 \, dx \le C_4 (I), \tag{3.5}$$

for some positive constant C_4, depending only on N and p. We use the left-hand side of (3.4), Hölder's inequality with $2/(2-p)$ and $2/p$ in order to get

$$\int_{B_R} |\tau_{s,h} D_n u(x)|^p \eta^p(x)\,dx$$
$$\le C_2^{-p} \int_{B_R} (1+|D_n u(x)|^2+|D_n u(x+he_s)|^2)^{p(2-p)/4} |\tau_{s,h} V(D_n u(x))|^p \eta^p(x)\,dx$$
$$\le C_2^{-p} \Big(\int_{B_R} (1+|D_n u(x)|^2+|D_n u(x+he_s)|^2)^{p/2}\,dx \Big)^{(2-p)/2} \times$$
$$\times \Big(\int_{B_R} |\tau_{s,h} V(D_n u(x))|^2 \eta^2(x)\,dx \Big)^{p/2}.$$

Now, splitting the integral and changing variables yield

$$C_2^{-p} \Big(\int_{B_R} (1+|D_n u(x)|^2+|D_n u(x+he_s)|^2)^{p/2}\,dx \Big)^{(2-p)/2}$$
$$\le C_5 \Big(\int_{B_{2R}} (1+|D_n u(y)|^p)\,dy \Big)^{(2-p)/2} = C_6,$$

for some positive constants C_5 and C_6, independent of h, so that

$$C_6^{-2/p} \Big(\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p\,dx \Big)^{2/p} \le \int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2\,dx, \tag{3.6}$$

then, using (3.6), (3.5) and (3.2) we arrive at

$$\frac{m}{2} C_6^{-2/p} \Big(\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p\,dx \Big)^{2/p} + \frac{m}{2} \int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2\,dx$$
$$+ m \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2\,dx \le C_4(I) = C_4(II). \tag{3.7}$$

We recall that, from (3.2)

$$(II) = -\int \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \tau_{s,h} \Big(\frac{\partial F}{\partial \xi_i^\alpha}(Du) \Big) 2\eta D_i \eta \tau_{s,h} u^\alpha\,dx;$$

now we shift the difference operator $\tau_{s,h}$ from $\left(\partial F/\partial \xi_i^\alpha\right)(Du)$ to $2\eta D_i \eta \tau_{s,h} u^\alpha$ ([N]):

$$(3.8)\quad (II) = -\int \sum_{i=1}^{n}\sum_{\alpha=1}^{N} \tau_{s,h}\left(\frac{\partial F}{\partial \xi_i^\alpha}(Du)\right) 2\eta D_i \eta \tau_{s,h} u^\alpha \, dx$$
$$= -\int \sum_{i=1}^{n}\sum_{\alpha=1}^{N} \frac{\partial F}{\partial \xi_i^\alpha}(Du)\tau_{s,-h}\left(2\eta D_i \eta \tau_{s,h} u^\alpha\right) dx.$$

We use the growth conditions (1.3), (1.4) and Cauchy-Schwartz's inequality in (3.8) in order to get

$$(3.9)\quad C_4(II) \le C_7 \left(\int_{B_{2R}} (1+\sum_{i=1}^{n-1}|D_i u|^2 + |D_n u|^{2p-2})\, dx\right)^{1/2} \times$$
$$\times \left(\int_{B_{2R}} |\tau_{s,-h}\left(2\eta D\eta \tau_{s,h} u\right)|^2 \, dx\right)^{1/2},$$

for some positive constant C_7 independent of h. Since $0 < 2p-2 < p$,

$$(3.10)\qquad \left(\int_{B_{2R}} (1+\sum_{i=1}^{n-1}|D_i u|^2 + |D_n u|^{2p-2})\, dx\right)^{1/2} = C_8 < \infty.$$

Now we apply Lemma 1:

$$(3.11)\quad \left(\int_{B_{2R}} |\tau_{s,-h}\left(2\eta D\eta \tau_{s,h} u\right)|^2 \, dx\right)^{1/2}$$
$$\le |h| \left(\int_{B_{3R}} |D_s\left(2\eta D\eta \tau_{s,h} u\right)|^2 \, dx\right)^{1/2} = |h| \left(\int_{B_{R}} |D_s\left(2\eta D\eta \tau_{s,h} u\right)|^2 \, dx\right)^{1/2},$$

since $\eta = 0$ outside B_R. Taking into account (3.7), (3.9), (3.10) and (3.11), we arrive at

(3.12)
$$\left(\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p \, dx\right)^{2/p} + \int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2 \, dx + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2 \, dx$$
$$\le C_9 |h| \left(\int_{B_R} |D_s\left(2\eta D\eta \tau_{s,h} u\right)|^2 \, dx\right)^{1/2} = (III),$$

for some positive constant C_9, independent of h. Now, using the Young's inequality, for every $\epsilon > 0$ we have

$$(III) \le \frac{C_9^2 |h|^2}{\epsilon} + \epsilon \int_{B_R} |D_s (2\eta D\eta \tau_{s,h} u)|^2 \, dx. \tag{3.13}$$

The integral in the previous inequality is dealt with as follows:

$$\int_{B_R} |D_s (2\eta D\eta \tau_{s,h} u)|^2 \, dx \le 2 \int_{B_R} |D_s (2\eta D\eta) \tau_{s,h} u|^2 \, dx + 2 \int_{B_R} |2\eta D\eta \tau_{s,h} D_s u|^2 \, dx = (A) + (B). \tag{3.14}$$

Now Lemma 4 allows us to use Lemma 1 to get for some positive constants C_{10} and C_{11}, independent of h,

$$(A) \le C_{10} |h|^2 \int_{B_{2R}} |D_s u|^2 \, dx = C_{11} |h|^2, \tag{3.15}$$

which holds true just for $s = 1, \ldots, n-1$, since $D_1 u, \ldots, D_{n-1} u \in L^2$ but $D_n u \in L^p$, $p < 2$. On the other hand, we have, for $s = 1, \ldots, n-1$,

$$(B) \le C_{12} \int_{B_R} |\tau_{s,h} D_s u|^2 \eta^2 \, dx \le C_{12} \sum_{i=1}^{n-1} \int_{B_R} |\tau_{s,h} D_i u|^2 \eta^2 \, dx \tag{3.16}$$

for a positive constant C_{12}, independent of h. We insert (3.15) and (3.16) into (3.14), use the resulting inequality in (3.13) and keep in mind (3.12). Then

$$\left(\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p \, dx \right)^{2/p} + \int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2 \, dx + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2 \, dx$$
$$\le \frac{C_{13} |h|^2}{\epsilon} + \epsilon C_{13} \left(|h|^2 + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2 \, dx \right),$$

for some positive constant C_{13}, independent of h and ϵ, so taking $\epsilon = 1/(2C_{13})$, we finally get

$$\int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2 \, dx + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2 \, dx \le C_{14} |h|^2,$$

$$\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p \, dx \le C_{14}^{p/2} |h|^p,$$

for some positive constant C_{14}, independent of h. Since $\eta = 1$ on $B_\rho \subset B_R$, we can apply Lemma 2 and, after recalling (3.3) for the definition of $V(D_n u)$, we get (1.7), (1.8), (1.9), so we end the proof. □

4. Proof of Corollary 1

Since we can change the order for distributional derivatives, so $D_i D_s u = D_s D_i u$, using the result of Theorem 1 we get

$$\begin{gathered} D_i D_s u \in L^2_{\rm loc}(\Omega), \quad i = 1, \dots, n-1, \\ D_n D_s u \in L^p_{\rm loc}(\Omega) \end{gathered}$$

for every $s \in \{1, \dots, n-1\}$. Applying Lemma 4 with $p_1 = \cdots = p_{n-1} = 2$, $p_n = p$ we obtain $\bar{p} = (2pn)/\left[p(n-1)+2\right] < n$ thus $\bar{p}^* = (2pn)/\left[p(n-3)+2\right]$ and

$$D_s u \in L^{\bar{p}^*}_{\rm loc}(\Omega) \quad \forall\, s = 1, \dots, n-1.$$

This ends the proof. □

5. Proof of Theorem 2

Corollary 1 guarantees that

$$D_1 u, \dots, D_{n-1} u \in L^{\bar{p}*}_{\rm loc}(\Omega).$$

Moreover the additional restriction (1.10) implies that $\bar{p}^* \geq p/(p-1)$, thus

$$D_1 u, \dots, D_{n-1} u \in L^{p/(p-1)}_{\rm loc}(\Omega). \tag{5.1}$$

Now we proceed as in the proof of Theorem 1 until (3.8). Then, using the growth conditions (1.3), (1.4) and the Hölder's inequality with $p/(p-1)$ and p, we get

$$C_4(II) \leq C_{15} \Bigg(\int\limits_{B_{2R}} \left(1 + \sum_{i=1}^{n-1} |D_i u|^{p/(p-1)} + |D_n u|^p \right) dx \Bigg)^{(p-1)/p} \times \\ \times \Bigg(\int\limits_{B_{2R}} \left| \tau_{s,-h} \left(2\eta D\eta \tau_{s,h} u \right) \right|^p dx \Bigg)^{1/p},$$

for some positive constant C_{15} independent of h. The previous inequality is exactly (5.5) in [BL1] and from now the proof goes on as there. For the convenience of reader we quote the main steps. We use the higher integrability result stated in (5.1):

$$\Bigg(\int\limits_{B_{2R}} (1 + \sum_{i=1}^{n-1} |D_i u|^{p/(p-1)} + |D_n u|^p)\, dx \Bigg)^{(p-1)/p} = C_{16} < \infty.$$

Applying Lemma 1 with $t = p$

$$\left(\int_{B_{2R}} |\tau_{s,-h}\left(2\eta D\eta\tau_{s,h}u\right)|^p\,dx \right)^{1/p} \leq |h| \left(\int_{B_{3R}} |D_s\left(2\eta D\eta\tau_{s,h}u\right)|^p\,dx \right)^{1/p}$$
$$\leq |h| \left(\int_{B_R} |D_s\left(2\eta D\eta\right)\tau_{s,h}u|^p\,dx \right)^{1/p} + |h| \left(\int_{B_R} |2\eta D\eta\tau_{s,h}D_s u|^p\,dx \right)^{1/p}$$
$$= |h| \Big\{ (A) + (B) \Big\}.$$

Using again Lemma 1, we get

$$(A) \leq C_{17} \left(\int_{B_{2R}} |D_s u|^p\,dx \right)^{1/p} |h| = C_{18}|h|,$$

for some positive constants C_{17} and C_{18}, independent of h. On the other hand, using Hölder's inequality, we have

$$(B) \leq C_{19} \left(\int_{B_R} |\tau_{s,h}D_s u|^p \eta^p\,dx \right)^{1/p} \leq C_{19} \left(\sum_{i=1}^{n} \int_{B_R} |\tau_{s,h}D_i u|^p \eta^p\,dx \right)^{1/p}$$
$$\leq C_{20} \left(\sum_{i=1}^{n-1} \int_{B_R} |\tau_{s,h}D_i u|^p \eta^p\,dx \right)^{1/p} + C_{20} \left(\int_{B_R} |\tau_{s,h}D_n u|^p \eta^p\,dx \right)^{1/p}$$
$$\leq C_{21} \left(\sum_{i=1}^{n-1} \int_{B_R} |\tau_{s,h}D_i u|^2 \eta^2\,dx \right)^{1/2} + C_{20} \left(\int_{B_R} |\tau_{s,h}D_n u|^p \eta^p\,dx \right)^{1/p},$$

for some positive constants C_{19}, C_{20} and C_{21}, independent of h. Eventually, we get

$$\left(\int_{B_R} |\tau_{s,h}D_n u|^p \eta^p\,dx \right)^{2/p} + \int_{B_R} |\tau_{s,h}V(D_n u)|^2 \eta^2\,dx + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h}D_i u|^2 \eta^2\,dx$$
$$\leq \frac{C_{22}|h|^2}{\epsilon} + \epsilon C_{22} \left(|h|^2 + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h}D_i u|^2 \eta^2\,dx + \left(\int_{B_R} |\tau_{s,h}D_n u|^p \eta^p\,dx \right)^{2/p} \right),$$

for some positive constant C_{22}, independent of h and ϵ, so taking $\epsilon = 1/(2C_{22})$, we finally have

$$\int_{B_R} |\tau_{s,h} V(D_n u)|^2 \eta^2 \, dx + \int_{B_R} \sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 \eta^2 \, dx \le C_{23} |h|^2,$$

$$\int_{B_R} |\tau_{s,h} D_n u|^p \eta^p \, dx \le C_{23}^{p/2} |h|^p,$$

for some positive constant C_{23}, independent of h, where s may also assume the value n. Application of Lemma 2 ends the proof. □

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Paola Cavaliere, Maria Longobardi:
Facoltà di Scienze, Università di Salerno, via S. Allende, 84081 Baronissi (SA), Italy

E-mail: cavalier@matna3.dma.unina.it
longob@matna3.dma.unina.it

Anna D'Ottavio, Francesco Leonetti:
Dipartimento di Matematica Pura ed Applicata, Università di L'Aquila, 67100 L'Aquila, Italy

E-mail: dottavio@axscaq.aquila.infn.it
leonetti@univaq.it

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