Gehring theory for time-discrete hyperbolic differential equations

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Abstract. This paper is concerned with extending Gehring theory to be applicable to Rothe's approximate solutions to hyperbolic differential equations.

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1. Introduction

M. Giaquinta and G. Modica have extended Gehring's result [2] of the higher integrability through reverse Hölder inequalities, which enables to have L^p-estimates $(p > 2)$ for the gradients of weak solutions to nonlinear elliptic-systems and of minimizers to variational problems. In addition, M. Giaquinta and M. Struwe [8] have applied the theory to nonlinear parabolic systems, so that there have been established L^p-estimates $(p > 2)$ for the spatial gradients of weak solutions.

On the other hand, the paper [15] with [9] and [10] has adopted the theory to accomplish the higher integrability of the gradients of solutions to difference partial differential equations of Rothe's approximation to parabolic partial differential equations. In [11], [12] and [13], furthermore, there has been made an attempt to extend such scheme to be applicable to Rothe's approximation systems to hyperbolic partial differential equations.

The objective of this paper is to accomplish Gehring theory due to Giaquinta-Modica in order to have applications for time-discrete difference hyperbolic partial differential equations.

The investigation of Gehring theory follows the idea of the proof due to Giaquinta-Modica. However, when we are concerned with discrete problems, we have to make some careful treatments of local cubes with centers restricted to discrete time levels.

We shall use the following notation: For $R > 0$ and $x_0 \in \mathbf{R}^m$, put

$$D_R(x_0) = \{x \in \mathbf{R}^m : \max_{1\le i\le m} |x^i - x_0^i| < R\},$$

and

$$\Delta = D_{3/2}(0).$$

We denote by $Q_R(z_0)$ a cube in $\mathbf{R}^{m+1}$ with center at $z_0 = (t_0, x_0)$:

$$Q_R(z_0) = Q_R(t_0, x_0) = \{t \in \mathbf{R}^1 : |t_0 - t| < R\} \times D_R(x_0)$$

and set

$$\begin{aligned} \Sigma &= Q_{3/2}(0) = (-3/2,\ 3/2) \times \Delta, \\ C_0 &= Q_{1/2}(0), \\ C_k &= \{z \in \Sigma :\ 2^{-k} < \operatorname{dist}(z, \partial\Sigma) \le 2^{1-k}\} \quad (k = 1, 2, \cdots), \end{aligned}$$

where dist is the Euclidean distance and $\partial\Sigma$ is the boundary of Σ.

The symbol $⨍_A$ stands for an average over the domain A:

$$⨍_A = \frac{1}{|A|}\int_A \quad \text{for a measurable subset} \quad A \subset \mathbf{R}^{m+1} \ \text{ or } \ A \subset \mathbf{R}^m,$$

where $|A|$ is the Lebesgue measure of A. For any real-valued function $\mathcal{F}$ defined in Σ and any real number s, we use the notation

$$E(\mathcal{F}, s) \ = \ \{z \in \Sigma : \mathcal{F}(z) > s\}.$$

Let $h > 0$ be a positive number. We set

$$t_n = nh \ \ (n \in \mathbf{Z}), \qquad L_h = \sum_{n \in \mathbf{Z}} \big(\{t_n\} \times \Delta\big).$$

By an h-time step function, we mean a function defined in $\mathbf{R}^1 \times \Delta$ and of the form

$$\mathcal{F}(t, x) = \mathcal{F}(nh, x) \ \text{ for } \ (t, x) \in ((n-1)h, nh] \times \Delta \ \ (n \in \mathbf{Z}).$$

Our main result is stated as follows.

Theorem. *Let $g \in L^q(\Sigma)$ and $f \in L^r(\Sigma)$, $r > q > 1$, be nonnegative h-time step functions. Suppose that there exist two constants θ and b with $0 \le \theta < 1$, $b > 1$ such that*

$$(1.1) \quad ⨍_{Q_{R/2}(z_0)} g^q\, dz \ \le \ b\Big\{\Big(⨍_{Q_R(z_0)} g\, dz\Big)^q + ⨍_{Q_R(z_0)} f^q\, dz\Big\} + \theta ⨍_{Q_R(z_0)} g^q\, dz$$

holds for every $Q_R(z_0) \subset \Sigma$ with $z_0 \in L_h$. Then $g \in L^p_{loc}(\Sigma)$ for $p \in [q, q+\varepsilon)$ and

$$(1.2) \quad \Big(⨍_{Q_{R/2}(z_0)} g^p\, dz\Big)^{1/p} \le \ C\Big\{\Big(⨍_{Q_R(z_0)} g^q\, dz\Big)^{1/q} + \Big(⨍_{Q_R(z_0)} f^p\, dz\Big)^{1/p}\Big\}$$

holds for every $Q_R(z_0) \subset \Sigma$ *with* $z_0 \in L_h$*, where* C *and* ε *are positive constants depending only on* b, θ, q, r, m.

Remark 1. Note that each domain of integration in inequalities (1.1) and (1.2) is restricted to a cube with its center z_0 on some level of L_h. To prove the Theorem by using the approach of [3] based on the covering argument due to Calderón-Zygmund ([1], [3]), we have to handle also general cubes with centers not always restricted to L_h. Thus we must treat the cubes carefully, some of them with centers in L_h and the others with centers elsewhere.

Remark 2. The constants C and ε in the Theorem are independent of h; the Theorem is invariant by translations and dilatations. Hence it suffices to verify the inequality (1.2) with $Q_R(z_0)$ and $Q_{2R}(z_0)$ replaced, respectively, by C_0 and Σ:

$$\Big(\fint_{C_0} g^p \, dz \Big)^{1/p} \le C \Big\{ \Big(\fint_{\Sigma} g^q \, dz \Big)^{1/q} + \Big(\fint_{\Sigma} f^p \, dz \Big)^{1/p} \Big\}. \tag{1.3}$$

Inequality (1.3) is trivially valid if $g = 0$ *a.e.* in Σ. Hence by multiplying g and f by a suitably chosen constant, we may assume without loss of generality that

$$\fint_{\Sigma} g^q \, dz = 1. \tag{1.4}$$

Following the idea of Giaquinta-Modica ([3]; see also [4]), we introduce a positive-valued function

$$\Phi(z) = \sum_{k=0}^{\infty} (2^{m+1})^{-k/q} \chi_{C_k}(z), \quad z \in \Sigma, \tag{1.5}$$

where χ_{C_k} is the characteristic function of C_k.

According to the definition of Φ, it is sufficient to show

$$\Big(\fint_{\Sigma} (g\Phi)^p \, dz \Big)^{1/p} \le C \Big\{ \Big(\fint_{\Sigma} (g\Phi)^q \, dz \Big)^{1/q} + \Big(\fint_{\Sigma} (f\Phi)^p \, dz \Big)^{1/p} \Big\} \tag{1.6}$$

instead of proving (1.3).

As in [7, Proposition 5.1], we prove the Theorem in two steps. In the first step, we show that the inequality

$$\int_{E(g\Phi, \nu t)} (g\Phi)^q \, dz \le a \Big\{ t^{q-1} \int_{E(g\Phi, t)} g\Phi \, dz + \int_{E(f\Phi, t)} (f\Phi)^q \, dz \Big\} \tag{1.7}$$

holds for any $t \in [1, \infty)$, where ν and a are positive constants depending on m, q, b, θ (see [3], [7]). The second step, where we derive (1.6) from (1.7), follows entirely the idea of [7, Proposition 5.1].

In the sequel, therefore, we devote ourselves to showing the inequality (1.7).

2. Proof of the Theorem

In what follows, g, f, θ and b are the quantities given in the Theorem.

To prove the Theorem, we shall show that (1.7) is valid under the assumption (1.4). We shall state Calderón-Zygmund's subdivision lemma of the form established in [7].

Lemma 2.1. *For every $s \geq |\Sigma|^{1/q}$, there exists a family, denoted by $\mathfrak{Q}$, of cubes with their sides parallel to the axes and with disjoint interiors such that*

$$(2.1)\qquad \begin{aligned} &Q \subset C_k \ \text{ for some } \ k = 0, 1, 2, \cdots, \\ &s^q < \fint_Q (g\Phi)^q \, dz \leq 3^{m+1} \cdot s^q, \\ &g\Phi \leq s \ \text{ a.e. on } \ \Sigma \setminus \bigcup_{Q \in \mathfrak{Q}} Q. \end{aligned}$$

PROOF: We have only to employ the argument expanded in [7], making the first decomposition into 3×3^m cubes and all the others into 2×2^m cubes of the equal size. □

Remark 3. The family $\mathfrak{Q}$ constructed depends on a quantity s, which will be chosen as $s = \nu t$, $t \geq 1$. In fact, we have by Lemma 2.1 that

$$(2.2)\qquad \int_{E(g\Phi, s)} (g\Phi)^q \, dz \leq 3^{m+1} \cdot s^q \cdot \sum_{Q \in \mathfrak{Q}} |Q|.$$

If $s = \nu t$, the left-hand side of (2.2) becomes that of the desired inequality (1.7). In the sequel, we shall estimate the right-hand side of (2.2). The positive constant ν will be fixed later.

We estimate the quantity $\sum_{Q \in \mathfrak{Q}} |Q|$. According to the location of cubes, we first classify cubes of $\mathfrak{Q}$, defined in Lemma 2.1, into three types:

$$\begin{aligned} \mathfrak{Q}_A &= \{Q \in \mathfrak{Q} : \ Q \cap L_h \neq \varnothing\}, \\ \mathfrak{Q}_B^+ &= \{Q \in \mathfrak{Q} : \ Q \cap L_h = \varnothing, \ Q \subset \Sigma^+\}, \\ \mathfrak{Q}_B^- &= \{Q \in \mathfrak{Q} : \ Q \cap L_h = \varnothing, \ Q \subset \Sigma^-\}, \end{aligned}$$

where $\Sigma^+ = \{(t, x) \in \Sigma : t \geq 0\}$ and $\Sigma^- = \{(t, x) \in \Sigma : t \leq 0\}$.

Under this setting, we introduce a family of cubes which plays a role for controlling cubes with the centers on L_h. With each $Q \in \mathfrak{Q}$, we associate a cube $\widehat{Q}$ and a family $\mathfrak{P}(Q)$ in the following way: For Q of the form

$$Q = Q_\rho(\tau, \xi) \ \text{ with } \ t_{n_\star - 1} < \tau \leq t_{n_\star},$$

we put

$$\widehat{Q} = \begin{cases} Q & \text{for } \ Q \in \mathfrak{Q}_A, \\ Q_\rho(t_{n_\star - 1} + \rho, \xi) & \text{for } \ Q \in \mathfrak{Q}_B^+, \\ Q_\rho(t_{n_\star} - \rho, \xi) & \text{for } \ Q \in \mathfrak{Q}_B^- \end{cases}$$

and set

$$\mathfrak{P}(Q) = \left\{ Q_r(z_0) : \ z_0 \in L_h, \ \widehat{Q} \subset Q_r(z_0) \subset \Sigma \right\}.$$

Lemma 2.2. *For each $Q \in \mathfrak{Q}$, $Q \subset C_k$ $(k = 0, 1, 2, \cdots)$, it holds that*

$$(2^{m+1})^{k-1} s^q \ < \ \sup_{P \in \mathfrak{P}(Q)} ⨍_P g^q \, dz.$$

Proof: By the assumption on Q, we have from (2.1) that

$$(2^{m+1})^k s^q < ⨍_Q g^q \, dz = ⨍_{\widehat{Q}} g^q \, dz.$$

Here, the equality is due to the definition of $\widehat{Q}$ and the fact that g is an h-time step function. Since there exists a cube $P \in \mathfrak{P}(Q)$ satisfying $P \supset \widehat{Q}$ with $|P| \leq 2^{m+1} |\widehat{Q}|$, we immediately obtain Lemma 2.2. □

Lemma 2.3. *Suppose that $s \geq \left(2 \cdot (2^{m+1})^2 \cdot |\Sigma|\right)^{1/q}$. Then, for each $Q \in \mathfrak{Q}$ included in C_k $(k = 0, 1, 2, \cdots)$, there exists a cube $P_Q \in \mathfrak{P}(Q)$ satisfying*

$$|P_Q| \ < \ (2^{m+1})^{-(k-2)} \cdot s^{-q} \cdot 2|\Sigma| \tag{2.3}$$

and

$$(2^{m+1})^{k-1} s^q \ < \ \frac{2b}{1-\theta} \left\{ \left(⨍_{P_Q} g \, dz \right)^q \ + \ ⨍_{P_Q} f^q \, dz \right\}. \tag{2.4}$$

Proof: The condition $\theta < 1$ allows us to find a cube $Q_{R/2}(z_0) \in \mathfrak{P}(Q)$, $z_0 \in L_h$, such that

$$\frac{1 + \sqrt{\theta}}{2} \sup_{P \in \mathfrak{P}(Q)} ⨍_P g^q \, dz \ \leq \ ⨍_{Q_{R/2}(z_0)} g^q \, dz. \tag{2.5}$$

From Lemma 2.2, it follows that

$$(2.6) \qquad (2^{m+1})^{k-1} s^q < \frac{2}{1+\sqrt{\theta}} \fint_{Q_{R/2}(z_0)} g^q \, dz,$$

which implies

$$|Q_R(z_0)| \; < \; (2^{m+1})^{-(k-2)} s^{-q} \frac{2|\Sigma|}{1+\sqrt{\theta}} \fint_{\Sigma} g^q \, dz.$$

Hence, we have by assumption (1.4) that (2.3) holds for $P_Q = Q_R(z_0)$.

The estimate (2.3) together with the assumption on s yield

$$(2.7) \qquad |P_Q| < (2^{-k})^{m+1},$$

so that each side of P_Q is less than 2^{-k} in length. Since $\widehat{Q} \subset P_Q$ and $\widehat{Q} \subset C_0 \cup \cdots \cup C_k$, the estimate

$$\mathrm{dist}\,(P_Q, \partial\Sigma) > \mathrm{dist}\,(\widehat{Q}, \partial\Sigma) - 2^{-k} \geq 0$$

holds, which implies $P_Q \subset \Sigma$. Thereby, $P_Q = Q_R(z_0) \in \mathfrak{P}(Q)$. On the other hand, since $\theta < 1$, we have by (2.5)

$$\sqrt{\theta} \fint_{Q_R(z_0)} g^q \, dz \;\leq\; \frac{1+\sqrt{\theta}}{2} \sup_{P \in \mathfrak{P}(Q)} \fint_P g^q \, dz \;\leq\; \fint_{Q_{R/2}(z_0)} g^q \, dz.$$

By this inequality and (1.1), we obtain

$$(2.8) \qquad \fint_{Q_{R/2}(z_0)} g^q \, dz \;\leq\; \frac{b}{1-\sqrt{\theta}} \left\{ \left(\fint_{Q_R(z_0)} g \, dz \right)^q + \fint_{Q_R(z_0)} f^q \, dz \right\}.$$

Consequently, the combination of (2.6) with the estimate (2.8) with $P_Q = Q_R(z_0)$ yields the estimate (2.4). □

Let $P_Q \in \mathfrak{P}(Q)$ be a cube in Lemma 2.3. For $Q \in \mathfrak{Q}_A$, the cube P_Q includes $Q (= \widehat{Q})$. For $Q \in \mathfrak{Q}_B^+ \cup \mathfrak{Q}_B^-$, however, P_Q does not necessarily include $Q (\neq \widehat{Q})$. To surmount the complication, we make another classification of $\mathfrak{Q}$ in accordance with the size of cubes compared with h. We introduce two disjoint subclasses of $\mathfrak{Q}$:

$$\mathfrak{Q}_{[A]} = \{Q \in \mathfrak{Q} : \; P_Q \supset Q \;\text{ or }\; P_Q \equiv Q_R(z_0) \supset Q_{h/2}(z_0)\},$$
$$\mathfrak{Q}_{[B]} = \{Q \in \mathfrak{Q} : \; P_Q \not\supset Q \;\text{ and }\; P_Q \equiv Q_R(z_0) \subsetneqq Q_{h/2}(z_0)\}$$

and put

$$\mathfrak{Q}_{[B]}^{\pm} = \mathfrak{Q}_{[B]} \cap \mathfrak{Q}_B^{\pm} = \{Q \in \mathfrak{Q}_{[B]} : Q \subset \Sigma^{\pm}\}.$$

Obviously, $\mathfrak{Q}_A \subset \mathfrak{Q}_{[A]}$ and $\mathfrak{Q}_{[B]} \subset \mathfrak{Q}_B^+ \cup \mathfrak{Q}_B^-$.

Lemma 2.4. *Suppose that* $s \geq (2 \cdot (2^{m+1})^2 \cdot |\Sigma|)^{1/q}$ *and* $Q \in \mathfrak{Q}_{[B]}$. *Let* $P_Q \in \mathfrak{P}(Q)$ *be a cube from Lemma 2.3. Then there exists a translation* $P_Q^\dagger$ *of* P_Q *toward the t-axis such that*

$$Q \subset P_Q^\dagger \subset \Sigma \ \text{ and } \ P_Q^\dagger \cap L_h = \varnothing. \tag{2.9}$$

Proof: Suppose $Q \subset C_k$ $(k = 0, 1, 2, \cdots)$. We give here the proof only for $Q \in \mathfrak{Q}_{[B]}^+$; cubes from $\mathfrak{Q}_{[B]}^-$ can be treated in the same way.

Note that $Q \equiv Q_\rho(\tau, \xi) \in \mathfrak{Q}_{[B]}^+$ is contained in only one strip $(t_{n_\star-1}, t_{n_\star}] \times \Delta$ $(0 \leq t_{n_\star-1} < 3/2)$. By the definition of $\mathfrak{Q}_{[B]}$, $P_Q \equiv Q_R(t_{n_0}, x_0) \in \mathfrak{P}(Q)$ satisfies $R < h/2$. Hence, the inclusion

$$\widehat{Q} = Q_\rho(t_{n_\star-1} + \rho, \xi) \subset P_Q = Q_R(t_{n_0}, x_0)$$

implies $t_{n_0} = t_{n_\star-1}$ and $\rho \leq R < h/2$.

If $t_{n_\star} \leq 3/2$, the relation $\rho \leq R < h/2$ assures the existence of a translation $P_Q^\dagger = Q_R(t^\dagger, x_0)$ of P_Q satisfying

$$Q \subset P_Q^\dagger \subset (t_{n_\star-1}, t_{n_\star}) \times \Delta \subset \Sigma.$$

In the other case $t_{n_\star} > 3/2$ $(\geq t_{n_\star-1})$, we show that there is also a translation $P_Q^\dagger$ of P_Q such that

$$Q \subset P_Q^\dagger \subset (t_{n_\star-1}, 3/2) \times \Delta. \tag{2.10}$$

Note that C_k including Q intersects $(t_{n_\star-1}, 3/2) \times \Delta$ $(\supset Q)$ and hence the distance of the Σ's top surface $\{3/2\} \times \Delta$ to C_k is less than to the level $\{t_{n_\star-1}\} \times \Delta$, that is, $2^{-k} < 3/2 - t_{n_\star-1}$. Besides, the estimate (2.7) still holds and implies $2R < 2^{-k}$. Combining these inequalities, we obtain the relation $2R < 3/2 - t_{n_\star-1}$, which enables us to find $P_Q^\dagger$ satisfying (2.10). □

Let P_Q be a cube from Lemma 2.3 and $P_Q^\dagger$ a cube from Lemma 2.4. We introduce the following sub-families of $\mathfrak{Q}_{[B]}$:

$$\mathfrak{B}_1 = \{Q \in \mathfrak{Q}_{[B]} : \ \mathfrak{J}(P_Q^\dagger) \geq \mathfrak{J}(P_Q)\},$$
$$\mathfrak{B}_2 = \{Q \in \mathfrak{Q}_{[B]} : \ \mathfrak{J}(P_Q^\dagger) < \mathfrak{J}(P_Q)\},$$

where

$$\mathfrak{J}(P) = \Big(\fint_P g \, dz \Big)^q + \fint_P f^q \, dz.$$

Lemma 2.5. *Let* $s \geq \big(2 \cdot (2^{m+1})^4 \cdot |\Sigma|\big)^{1/q}$. *Then there exists a family* $\{R_Q : Q \in \mathfrak{Q}\}$ *of cubes such that*

$$s \;<\; C_* \Big\{ \fint_{R_Q} g\Phi\, dz + \Big(\fint_{R_Q} (f\Phi)^q\, dz \Big)^{1/q} \Big\}, \qquad Q \in \mathfrak{Q}, \tag{2.11}$$

and

$$\sum_{Q \in \mathfrak{Q}_{[A]} \cup \mathfrak{B}_1} |Q| \;\leq\; \Big| \bigcup_{Q \in \mathfrak{Q}_{[A]} \cup \mathfrak{B}_1} R_Q \Big|, \qquad \sum_{Q \in \mathfrak{B}_2^{\pm}} |Q| \;\leq\; \Big| \bigcup_{Q \in \mathfrak{B}_2^{\pm}} R_Q \Big| \tag{2.12}$$

hold, where $C_* = (2^{m+1})^{1+2/q}\Big(2b/(1-\theta)\Big)^{1/q}$ *and* $\mathfrak{B}_2^{\pm} = \mathfrak{B}_2 \cap \mathfrak{Q}_{[B]}^{\pm}$.

Proof: Assume that $Q \subset C_k$ $(k = 0, 1, 2, \cdots)$ and that the cube $P_Q \in \mathfrak{P}(Q)$ from Lemma 2.3 is of the form $P_Q = Q_R(z_0)$.

First we shall treat $Q \in \mathfrak{Q}_{[A]} \cup \mathfrak{B}_1$. We define

$$R_Q = \begin{cases} Q_{\overline{R}}(z_0) & \text{if } Q \in \mathfrak{Q}_{[A]}, \\ P_Q^{\dagger} & \text{if } Q \in \mathfrak{B}_1, \end{cases}$$

where $\overline{R} = \min\{r \geq R : Q_r(z_0) \supset Q\}$. Then we have $R_Q \supset Q$. Since cubes from $\mathfrak{Q}$ have mutually disjoint interiors, this inclusion implies the first inequality in (2.12).

We shall show that

$$|R_Q| \;\leq\; 2^{m+1}|P_Q| \tag{2.13}$$

holds for $Q \in \mathfrak{Q}_{[A]}$. In case $Q \in \mathfrak{Q}_A$, $R_Q = P_Q$ and hence (2.13) is valid. In the other case $Q \in \mathfrak{Q}_{[A]} \setminus \mathfrak{Q}_A$ $(\subset \mathfrak{Q}_B^{+} \cup \mathfrak{Q}_B^{-})$, Q and $\widehat{Q}$ are included in one strip $(t_{n_\star - 1}, t_{n_\star}] \times \Delta$, and $P_Q = Q_R(z_0)(\supset \widehat{Q})$ satisfies $R \geq h/2$. Consequently, $Q_{2R}(z_0)$ strides across the strip $(t_{n_\star - 1}, t_{n_\star}] \times \Delta$ and hence includes Q. This means that $\overline{R} \leq 2R$, which results in (2.13).

For $Q \in \mathfrak{Q}_{[A]}$, the inclusion $P_Q \subset R_Q$ and (2.13) gives us

$$\mathfrak{J}(P_Q) \;\leq\; (2^{m+1})^q \mathfrak{J}(R_Q), \tag{2.14}$$

which is also true for $Q \in \mathfrak{B}_1$, because of $\mathfrak{J}(P_Q^{\dagger}) \geq \mathfrak{J}(P_Q)$.

It follows from (2.4) and (2.14) that

$$(2^{m+1})^{k-1} s^q \leq (2^{m+1})^q \frac{2b}{1-\theta} \mathfrak{J}(R_Q). \tag{2.15}$$

Noting the assumption on s, we combine (2.3) with (2.13) to get

$$|R_Q| < (2^{-(k+1)})^{m+1}$$

and hence

$$R_Q \subset C_0 \cup \cdots \cup C_{k+1},$$

which, together with (2.15), give us the desired estimate (2.11).

Next, we deal with $Q \in \mathfrak{B}_2^+$ or $\mathfrak{B}_2^-$. Note that P_Q has the center $z_0 = (t_{n_0}, x_0) \in L_h$ and $P_Q^\dagger = Q_R(t^\dagger, x_0)$ is included in only one strip $(t_{n_0}, t_{n_0+1}] \times \Delta$ or $(t_{n_0-1}, t_{n_0}] \times \Delta$. We introduce the symmetrization $P_Q^\ddagger$ of $P_Q^\dagger$ with respect to the level $\{t_{n_0}\} \times \Delta$:

$$P_Q^\ddagger = Q_R(2t_{n_0} - t^\dagger, x_0).$$

Then the property $\mathfrak{J}(P_Q^\dagger) < \mathfrak{J}(P_Q)$ for $Q \in \mathfrak{B}_2^\pm$ implies $\mathfrak{J}(P_Q^\ddagger) \geq \mathfrak{J}(P_Q)$. Hence, $R_Q = P_Q^\ddagger$ satisfies (2.13) and (2.14), from which the estimate (2.11) follows. The second inequality in (2.12) is derived from the relation $|P_Q^\ddagger| = |P_Q^\dagger|$ and $Q \subset P_Q^\dagger$. □

Proof of the Theorem: Suppose that ν is a constant, not less than $\big(2 \cdot (2^{m+1})^4 \cdot |\Sigma|\big)^{1/q}$, to be chosen later. Let $t \geq 1$ and $s = \nu t$.

Now we shall estimate $\sum_{Q \in \mathfrak{Q}} |Q|$ from the right-hand side of (2.2) with $s = \nu t$. This is easily done by means of (2.12):

(2.16)

$$\begin{aligned}\int_{E(g\Phi,\nu t)} (g\Phi)^q\,dz \;&\leq\; 3^{m+1}(\nu t)^q \Big(\sum_{Q \in \mathfrak{Q}_{[A]} \cup \mathfrak{B}_1} |Q| + \sum_{Q \in \mathfrak{B}_2^+} |Q| + \sum_{Q \in \mathfrak{B}_2^-} |Q| \Big)\\ &\leq\; 3^{m+1}(\nu t)^q \bigg(\Big| \bigcup_{Q \in \mathfrak{Q}_{[A]} \cup \mathfrak{B}_1} R_Q \Big| + \Big| \bigcup_{Q \in \mathfrak{B}_2^+} R_Q \Big| + \Big| \bigcup_{Q \in \mathfrak{B}_2^-} R_Q \Big| \bigg).\end{aligned}$$

On the other hand, for each R_Q, it follows from (2.11) that

$$\begin{aligned}\nu t|R_Q| &< C_* \Big\{ \int_{R_Q} g\Phi\,dz + |R_Q|^{1-1/q} \Big(\int_{R_Q} (f\Phi)^q\,dz \Big)^{1/q} \Big\}\\ &\leq\; C_* \Big\{ \int_{R_Q \cap E(g\Phi,t)} g\Phi\,dz + t|R_Q|\\ &\qquad + \frac{q-1}{q}\, t|R_Q| + \frac{1}{q}\, t^{1-q} \Big(\int_{R_Q \cap E(f\Phi,t)} (f\Phi)^q\,dz + t^q|R_Q| \Big) \Big\}.\end{aligned}$$

Thus we obtain

$$(2.17)\quad t|R_Q| \;\le\; \frac{1}{\nu/C_* - 2}\left\{\int_{R_Q\cap E(g\Phi,t)} g\Phi\,dz + t^{1-q}\int_{R_Q\cap E(f\Phi,t)} (f\Phi)^q\,dz\right\},$$

provided that

$$\nu \;>\; 2C_*(m,b,q,\theta) \equiv 2(2^{m+1})^{1+q/2}\left(2b\Big/(1-\theta)\right)^{1/q}.$$

By a well-known covering argument, the family $\{R_Q : Q \in \mathfrak{Q}_{[A]} \cup \mathfrak{B}_1\}$ has a subfamily $\mathfrak{R}_1$ of mutually disjoint cubes such that, for a positive constant $C(m)$,

$$\Big|\bigcup_{Q\in\mathfrak{Q}_{[A]}\cup\mathfrak{B}_1} R_Q\Big| \;\le\; C(m)\sum_{R_Q\in\mathfrak{R}_1} |R_Q|.$$

This and (2.17) yield the estimate

$$\Big|\bigcup_{Q\in\mathfrak{Q}_{[A]}\cup\mathfrak{B}_1} R_Q\Big| \;\le\; \frac{C(m)}{\nu/C_* - 2}\left\{t^{-1}\int_{E(g\Phi,t)} g\Phi\,dz + t^{-q}\int_{E(f\Phi,t)} (f\Phi)^q\,dz\right\}.$$

The other quantities $\Big|\bigcup_{Q\in\mathfrak{B}_2^+} R_Q\Big|$ and $\Big|\bigcup_{Q\in\mathfrak{B}_2^-} R_Q\Big|$ from the right-hand side of (2.16) are estimated in the same way. From these estimates and (2.16), we deduce (1.7).

We thus have completed the proof of the Theorem. □

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