Comment.Math.Univ.Carolin. 39,4 (1998)749–753

Weakly uniformly rotund Banach spaces

A. Moltó, V. Montesinos, J. Orihuela, S. Troyanski

Abstract. The dual space of a WUR Banach space is weakly K-analytic.

Keywords: Banach spaces, weak uniform rotundity, K-analiticity, uniform Gâteaux differentiability
Classification: 46B22, 52A41

A Banach space is said to be *weakly uniformly rotund* (WUR for short) if given sequences (x_n) and (y_n) in the unit sphere with $\|x_n + y_n\| \to 2$ we have weak-$\lim_n(x_n - y_n) = 0$. This notion has become more important since Hájek proved that every WUR Banach space must be Asplund ([8]). To obtain this result he uses ideas of Stegall for the equivalence between being an Asplund space and having the Radon-Nikodym property on its dual. Using this result and the Fabian-Godefroy ([4]) projectional resolution of the identity in the dual of an Asplund space, Fabian, Hájek and Zizler have recently showed that for a WUR Banach space E the dual space E^* is a subspace of a WCG Banach space. Indeed they proved that for a Banach space E to have an equivalent WUR norm is equivalent to the fact that the bidual unit ball $B_{E^{**}}$, endowed with the weak-* topology, will be a uniform Eberlein compact ([5]). Consequently they obtain that E must be LUR renormable, too ([7]). The aim of this note is to provide a direct proof of the fact that every WUR Banach space E has a dual space E^* which is weakly K-analytic. This provides a topological approach to Hájek's result on the Asplundness of the space E as well as the LUR renorming consequence on E after ([6]).

In this paper, E will denote a Banach space, E^* its dual, B_E its closed unit ball, S_E its unit sphere.

Definition 1. *A Banach space* $(E, \|\cdot\|)$ *is said to be uniformly Gâteaux differentiable* (*UGD for short*) *if for every* $0 \neq x \in E$,

$$\lim_{t\to 0} \sup_{\|y\|=1} \frac{\|y+tx\| + \|y-tx\| - 2}{t} = 0.$$

The following theorem is the main result of this note:

The first named author has been supported in part by DGICYT Project PB91-0326, the second named author by DGICYT PB91-0326 and PB94-0535, the third named author by DGICYT PB95-1025 and DGICYT PB91-0326, the fourth named author by a grant from the "Conselleria de Cultura, Educació i Ciència de la Generalitat Valenciana" and by NFSR of Bulgaria Grant MM-409/94.

Theorem 1. *Let E be a Banach space such that E^* has an equivalent (not necessarily dual) UGD norm (in particular, let E be WUR Banach space). Then E^* is weakly K-analytic.*

The proof is based on the following assertions.

Fact 1 (Šmulyan, see [3, Theorem II.6.7]). *The Banach space E is WUR if and only if E^* is UGD.*

Theorem 2 (Talagrand [9]). *Let K be a compact space. The following assertions are equivalent:*

1. *$C(K)$ is weakly K-analytic;*
2. *there is an increasing mapping $\sigma \to S_\sigma$ from $\mathbb{N}^\mathbb{N}$ (endowed with the product order) in the family of compact subsets of $C(K)$ endowed with the topology of pointwise convergence, such that $\bigcup\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$ separates points of K.*

Remark 1. In [1] the validity of the previous theorem for an arbitrary topological space is studied. In particular, for every subset W of a Banach space E it follows that (W, weak) is K-analytic if and only if $W = \bigcup\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$ and every S_σ is weakly compact with $S_\sigma \subset S_\gamma$ whenever $\sigma \leq \gamma$ in the product order. This will be the only tool necessary here from the theory of K-analytic spaces.

Remark 2. From Theorem 1 and [6], see also [3, p. 296], we get that every WUR Banach space admits an equivalent LUR norm.

Remark 3. From Theorem 1 it follows the Hájek's ([8]) result asserting that every WUR Banach space is Asplund. Indeed, if we assume that E is also separable the K-analytic structure of (E^*, weak) should imply that E^* is separable too. Let us explain here an easy argument following ideas from [2]: Assume (E^*, weak) is K-analytic. Let T be an usco mapping from $\mathbb{N}^\mathbb{N}$ into the set of subsets of (E^*, weak) with $T(\mathbb{N}^\mathbb{N}) = E^*$ (T can be assumed to be increasing by Remark 1). Let P be the natural projection from $(E^*, \text{weak}^*) \times \mathbb{N}^\mathbb{N}$ onto (E^*, weak). Consider the restriction Q of P to $\Sigma := \{(x, \alpha) : (x, \alpha) \in E^* \times \mathbb{N}^\mathbb{N},\ x \in T(\alpha)\}$. It is easy to prove that Q is continuous: let (x_i^*, α_i) be a net in Σ such that $(x_i^*, \alpha_i) \to (x, \alpha) \in \Sigma$. As $\alpha_i \to \alpha$ we can find $\beta \in \mathbb{N}^\mathbb{N}$ such that $\alpha \leq \beta$ and $\alpha_i \leq \beta$ for all $i \in \mathbb{N}$. Then $x_i \in T(\beta)$, $x \in T(\beta)$, and $x_i \xrightarrow{\text{weak}^*} x$, hence $x_i \xrightarrow{\text{weak}} x$. Therefore E^* is separable too. See also Theorem 2.4 in [9]. With more generality, any submetrizable topological space X is analytic if and only if there is a family of compact sets $\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$ in X, $S_\sigma \subset S_\gamma$ whenever $\sigma \leq \gamma$ in the product order and $X = \bigcup\{S_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\}$, [2, Theorem 7].

Proof of Theorem 1: It is well known that E admits an equivalent WUR norm. Then E^* has an equivalent dual UGD norm. Then given $x^* \in S_{E^*}$ and $\epsilon > 0$, there exists $\delta_\epsilon(x^*) > 0$ such that

$$\|y^* + tx^*\| + \|y^* - tx^*\| \leq 2 + \epsilon|t|, \text{ if } |t| < \delta_\epsilon(x^*) \text{ and } y^* \in S_{E^*}.$$

Given a positive integer p define

$$S_p(\epsilon) := \Big\{x^* \in S_{E^*} : \ \delta_\epsilon(x^*) > \frac{1}{p}\Big\}.$$

Obviously,

$$S_1(\epsilon) \subset S_2(\epsilon) \subset \ldots \subset S_p(\epsilon) \subset S_{p+1}(\epsilon) \subset \ldots$$

and $\bigcup_{p=1}^\infty S_p(\epsilon) = S_{E^*}$. Let $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$. Define

$$S_\alpha := \bigcap_{n=1}^\infty S_{a_n}\left(\frac{1}{n}\right).$$

We have

$$S_{E^*} = \bigcup \Big\{S_\alpha : \ \alpha \in \mathbb{N}^{\mathbb{N}}\Big\},$$

and

$$S_\alpha \subset S_\beta, \text{ whenever } \alpha = (a_n) \le \beta = (b_n) \text{ (i.e., } a_n \le b_n, \ \forall n).$$

This sets will give us the K-analytic structure of E^* in the weak topology. Indeed, we have the following

Claim 1. *Given $x^{**} \in B_{E^{**}}$, $\epsilon > 0$ and $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$, there is $x \in B_E$ such that*

$$|\langle x^{**} - x, x^*\rangle| < \epsilon, \ \forall\, x^* \in S_\alpha.$$

Proof of the claim: Find $n \in \mathbb{N}$ such that $\frac{3}{n} < \epsilon$. Pick $y^* \in S_{E^*}$ such that

$$\langle x^{**}, y^*\rangle > 1 - \frac{1}{na_n}.$$

Find $x \in B_E$ such that

$$\langle x, y^*\rangle > 1 - \frac{1}{na_n}.$$

Let $x^* \in S_\alpha$. Since $x^* \in S_{a_n}(\frac{1}{n})$

$$\|y^* + \frac{1}{a_n}x^*\| + \|y^* - \frac{1}{a_n}x^*\| \le 2 + \frac{1}{na_n}.$$

In particular we have

$$\langle x^{**}, y^* + \frac{1}{a_n}x^*\rangle + \langle x, y^* - \frac{1}{a_n}x^*\rangle \le 2 + \frac{1}{na_n} \tag{1}$$

hence

$$\frac{1}{a_n}\langle x^{**} - x, x^*\rangle \le 2 + \frac{1}{na_n} - \langle x^{**}, y^*\rangle - \langle x, y^*\rangle < \frac{3}{na_n} < \frac{\epsilon}{a_n}$$

and so

$$\langle x^{**} - x, x^* \rangle < \epsilon, \ \forall\, x^* \in S_\alpha.$$

By interchanging x^{**} and x in (1), we get

$$|\langle x^{**} - x, x^* \rangle| < \epsilon, \ \forall\, x^* \in S_\alpha$$

and this proves the claim.

To finish the proof of the Theorem, observe that, by the claim, each S_α is weakly relatively compact since it is weak*-relatively compact. Thus, we have

$$S_{E^*} \subset \bigcup \{ \overline{S_\alpha}^{\text{weak}} : \ \alpha \in \mathbb{N}^{\mathbb{N}} \} := W$$

and W is weakly K-analytic in E^* (Theorem 2 and Remark 1).

Consider the map

$$(W, \text{weak}) \times [0, +\infty[\xrightarrow{\Psi} (E^*, \text{weak})$$

given by $\Psi(x^*, t) := t.x^*$. Ψ is continuous, $[0, +\infty[$ is a Polish space, $(W, \text{weak}) \times [0, +\infty[$ is K-analytic and $\Psi(W \times [0, +\infty[) = E^*$, so (E^*, weak) is itself K-analytic. $\square$

Acknowledgments. This paper was prepared during the visit of the fourth named author to the University of Valencia in the Spring term of the Academic Year 1995–96. He acknowledges his gratitude to the hospitality and facilities provided by the University of Valencia.

References

[1] Cascales B., *On K-analytic locally convex spaces*, Arch. Math. **49** (1987), 232–244.

[2] Cascales B., Orihuela J., *A sequential property of set-valued maps*, J. Math. Anal. Appl. **156** (1991), 86–100.

[3] Deville R., Godefroy G., Zizler V., *Smoothness and Renormings in Banach Spaces*, Longman Scientific and Technical, 1993.

[4] Fabian M., Godefroy G., *The dual of every Asplund admits a projectional resolution of the identity*, Studia Math. **91** (1988), 141–151.

[5] Fabian M., Hájek P., Zizler V., *On uniform Eberlein compacta and uniformly Gâteaux smooth norms*, Serdica Math. J. **23** (1997), 1001–1010.

[6] Fabian M., Troyanski S., *A Banach space admits a locally uniformly rotund norm if its dual is a Vašák space*, Israel J. Math. **69** (1990), 214–224.

[7] Godefroy G., Troyanski S., Whitfield J.H.M., Zizler V., *Smoothness in weakly compactly generated Banach spaces*, J. Functional Anal. **52** (1983), 344–352.

[8] Hájek P., *Dual renormings of Banach spaces*, Comment. Math. Univ. Carolinae **37** (1996), 241–253.
[9] Talagrand M., *Espaces de Banach faiblement K-analytiques*, Annals of Math. **110** (1979), 407–438.

Departament d'Anàlisi Matemàtica, Universitat de València, Dr. Moliner 50, 46100 Burjassot (València), Spain

Departamento de Matemática Aplicada, E.T.S.I. Telecomunicación, Universidad Politécnica de Valencia, C/ Vera, s/n. 46071-Valencia, Spain

Departamento de Análisis Matemático, Universidad de Murcia, Campus de Espinardo, Murcia, Spain

Faculty of Mathematics and Informatics, Sofia University, 5, James Bourchier blvd., 1126 Sofia, Bulgaria

(*Received* July 18, 1997, *revised* May 19, 1998)