Inverse distributions: the logarithmic case

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Abstract. In this paper it is proved that the distribution of the logarithmic series is not invertible while it is found to be invertible if corrected by a suitable affinity. The inverse distribution of the corrected logarithmic series is then derived.

Moreover the asymptotic behaviour of the variance function of the logarithmic distribution is determined.

It is also proved that the variance function of the inverse distribution of the corrected logarithmic distribution has a cubic asymptotic behaviour.

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1. Introduction

Let μ be a positive Radon measure on $\mathbb{R}$ such that μ is not concentrated in a single point. We denote by

$L_\mu(\theta) = \int_{\mathbb{R}} e^{\theta x}\mu(dx)$ the Laplace transform of μ,

$D_\mu = \{\theta \in \mathbb{R} : L_\mu(\theta) < \infty\}$ the domain of $L_\mu(\theta)$,

Θ_μ the interior of D_μ.

Suppose that $\Theta_\mu \neq \emptyset$; Θ_μ is an interval.

Let $\mathcal{M}$ be the set of measures described above.

We denote by $k_\mu(\theta) = \log L_\mu(\theta),\ \theta \in \Theta_\mu$ the cumulant function of μ.
$k_\mu(\theta)$ is known to be strictly convex and analytic in Θ_μ (Letac and Mora (1990)).
For all $\theta \in \Theta_\mu$ consider the probability measure $P_\mu(\theta) = \exp(\theta x - k_\mu(\theta))\mu(dx)$.
The set

$$\mathcal{P}_\mu = \{P_\mu(\theta),\ \theta \in \Theta_\mu\}$$

is called the natural exponential family (NEF) generated by μ. We also say that μ is a basis of $\mathcal{P}_\mu$.

Now we recall the concepts of inverse measure and inverse distribution (Letac (1986), Definition 1.1 and Proposition 1.2, and Letac and Mora (1990), § 5), where "reciprocal" (reciprocité in French language) is used instead of "inverse".

Definition 1.1. *Let μ and $\tilde\mu \in \mathcal{M}$. $\tilde\mu$ is the inverse measure of μ if there exists a non empty interval $\Theta^*_{\tilde\mu}$:*

$$k'_{\tilde\mu}(\theta) > 0 \quad \forall\,\theta \in \Theta^*_{\tilde\mu} \tag{1.1}$$

$$-k_\mu(-k_{\tilde\mu}(\theta)) = \theta \quad \forall\,\theta \in \Theta^*_{\tilde\mu}. \tag{1.2}$$

In this case μ is said to be invertible.

The term inverse measure is justified by the expression (1.2), that is equivalent to

$$k_{\tilde{\mu}}(t) = (-k_\mu(-t))^{-1}, \tag{1.3}$$

where f^{-1} denotes the inverse function of f, i.e. $f \circ f^{-1} = f^{-1} \circ f =$ Identity function.

Let Θ^*_μ be the image of $\Theta^*_{\tilde{\mu}}$ by the function $-k_{\tilde{\mu}}(\theta)$. Θ^*_μ is an interval and, for (1.1), by differentiating (1.2), it turns out that $k'_\mu(\theta) > 0$, $\forall\, \theta \in \Theta^*_\mu$. It follows that if $\tilde{\mu}$ is the inverse measure of μ, then μ is the inverse measure $\tilde{\mu}$.
It is remarkable that the inverse distribution of a NEF does not necessarily exist.

Example 1.

Let $\mu = \delta_1 + \delta_2$, then $k_\mu(\theta) = \log\left(e^\theta + e^{2\theta}\right)$ and from (1.3)

$$k_{\tilde{\mu}}(\theta) = \log\left(\frac{e^\theta}{2} + e^{\frac{\theta}{2}}\sqrt{\frac{e^\theta}{4} + 1}\right), \;\; \theta \in \mathbb{R}.$$

It follows that $\tilde{\mu} = \frac{1}{2}\delta_1 + \sum_{h=0}^{+\infty}\binom{1/2}{h}\frac{1}{4^h}\delta_{h+\frac{1}{2}}$, i.e. $\tilde{\mu}$ is not a positive measure.

Now if we consider the measure $\mu_1 = \delta_0 + \delta_1$, i.e. the image of μ by the affinity $\varphi(x) = x - 1$, it is easy to see that μ_1 is invertible and $\tilde{\mu}_1 = \sum_{n=1}^{+\infty}\delta_n$.

Regarding the probability distribution, we have the following definition:

Definition 1.2. *Let $\mu, \tilde{\mu} \in \mathcal{M}$ and let $\mathcal{P}_\mu$ and $\mathcal{P}_{\tilde{\mu}}$ be the corresponding generated NEF.*
$\mathcal{P}_{\tilde{\mu}}$ is called the inverse of $\mathcal{P}_\mu$ if $\tilde{\mu}$ is the inverse measure of μ. In this case $\mathcal{P}_\mu$ is also said to be invertible.

A sufficient condition for two NEFs to be one the inverse of the other is that their cumulant functions verify (1.1) and (1.2).

The concept of inverse distribution is due to Tweedie (1945).

The most common example is represented by the Gaussian distribution and its inverse, known as the Inverse Gaussian.
Other interesting examples are:

- the binomial distribution of parameters (p, N). Its inverse is the distribution of a random variable X/N, with X being geometrically distributed with parameter p;
- the Gamma distribution of parameters (p, N), N known. Its inverse is the distribution of a random variable X/N, where X is a Poisson of parameter p.

For this and other examples see Seshadri (1993), Cap. 5.
The problem of the invertibility of a distribution can be discussed also using the variance function, that we therefore recall.

Let: $\mu \in \mathcal{M}$, $m = m(\theta) = k'_\mu(\theta)$, $\theta \in \Theta_\mu$ and $M_\mu = k'_\mu(\Theta_\mu)$, i.e. M_μ is the image of Θ_μ by the function k'_μ.
From the strict convexity of $k_\mu(\theta)$ it follows that $k'_\mu(\theta)$ is strictly increasing; hence $m(\theta)$ is also one to one between Θ_μ and M_μ. Let $\theta(m)$ be the inverse function of $m(\theta)$; m provides $P_\mu(\theta)$ with a new parametrization, named mean-parametrization (Barndorff-Nielsen (1978), p. 121).

We have the following definition (Morris (1982)).

Definition 1.3. *The function $V_\mu(m) = k''_\mu(\theta(m))$, $m \in M_\mu$, is called the variance function of the NEF $\mathcal{P}_\mu$.*

It is remarkable that the variance function $V_\mu(m)$ and its domain M_μ characterize the natural exponential family.
Morris (1982) proved that the variance function of only six NEFs, among which the most widely used (normal, gamma, binomial, negative binomial), is a polynomial of degree less or equal to two. Later the NEFs, whose variance function is a polynomial of degree three, has been classified in six types (Mora (1986), and Letac and Mora (1990)).
The variance function has been extensively studied with the aim of characterizing those functions that can be the variance function of some NEF (Letac (1991)).

In the following theorem (Letac and Mora (1990)) the behaviour of the variance function, with respect to an affinity, is described.

Theorem 1.1. *Let $\phi(x) = ax + b$, $a \neq 0$ and $\mathcal{P}_\mu$ be the NEF generated by μ. Denote by $\mu_1 = \phi_* \mu$ the image measure of μ by ϕ; then*

(a) $k_{\mu_1}(\theta) = b\theta + k_\mu(a\theta) \;\; \forall\, \theta \in \Theta_\mu$,
(b) $M_{\mu_1} = \phi(M_\mu)$,
(c) $V_{\mu_1} = a^2 V_\mu\left(\frac{m-b}{a}\right) \;\; \forall\, m \in M_\mu$.

The following theorem analyzes the behaviour of the variance function in the context of inverse distributions (Letac and Mora (1990)).

Theorem 1.2. *Let $\mathcal{P}_\mu$ be the NEF generated by μ and $\mathcal{P}_{\tilde\mu}$ its inverse; define $M^+_\mu = M_\mu \cap (0, +\infty)$ and $M^+_{\tilde\mu} = M_{\tilde\mu} \cap (0, +\infty)$. Then*

(a) *$M^+_\mu \neq \emptyset$ and $M^+_{\tilde\mu} \neq \emptyset$ and $\frac{1}{m}$ is a one to one mapping between M^+_μ and $M^+_{\tilde\mu}$,*
(b) *$V_{\tilde\mu}(m) = m^3 V_\mu\left(\frac{1}{m}\right) \;\; \forall\, m \in M^+_\mu$.*

We observe that point (b) of Theorem 1.2 shows that the set of cubic variance is closed under invertibility.

Sometimes this theorem allows to face and solve the inverting problem in a different way, because, computing first the variance function of the distribution to be inverted and then deriving, by means of Theorem 1.2, the variance function of the inverse distribution, the corresponding distribution is identified.

As an example, consider the measure $\mu = \delta_1 + \delta_2$ of Example 1. $V_\mu(m) = (m-1)(2-m)$ and $M_\mu = (1,2)$, then from Theorem 1.2 $V_{\tilde{\mu}}(m) = m(1-m)(2m-1)$ and $M_{\tilde{\mu}} = (1/2, 1)$ should hold, but a NEF with cubic variance and limited domain does not exist (Seshadri (1993)).
Moreover there are measures such that the variance function of the NEF they generate is very difficult to be computed. An example of this kind of measures is the logarithmic measure.

In this paper starting from a result given in Sacchetti (1992), first we prove in Theorem 2.1 that the logarithmic series distribution (L.S.D.) is not invertible, then in Theorem 2.3 we show that a measure $\mu \in \mathcal{M}$ defined on $1, 0, -1, -2, -3, -4, \dots$ is invertible and we derive its inverse measure.

It is worth observing that the invertibility of this kind of measure μ is well-known and has the following probabilistic interpretation: consider the random walk in $\mathbb{Z}$ ruled by an element of the exponential family concentrated on $1, 0, -1, -2, \dots$, then the first passage time of 1 gives a base for the inverse exponential family (Letac and Mora (1990), Theorem 5.6, p. 27). Anyway the proof of Theorem 2.3 follows from the Lagrange's formula (Theorem 2.2) and it does not rely on the martingale theory as Theorem 5.6, quoted above, does; moreover the explicit computation of the inverse measure is provided by this theorem.

In Corollary 2.1 the results of Theorem 2.3 are applied to the logarithmic measure corrected with a suitable affinity: the family generated by the inverse measure of the corrected logarithmic measure is called Inverse Logarithmic Series distribution (I.L.S.D.).

In Section 3, Theorems 3.1 and 3.2, we prove that the variance functions of L.S.D. and I.L.S.D. are infinity, as $m \to +\infty$, of the same order as $m^2 \log m$ and αm^3, $\alpha > 0$ respectively.

2. Logarithmic measure

Let
$$\mu = \sum_{n=1}^{+\infty} \frac{1}{n} \delta_n$$
where δ_n is the Dirac function in $n \in \mathbb{N}$.

The logarithmic series distribution (L.S.D.) is the NEF generated by μ, i.e. it is defined as follows (Johnson and Kotz (1969)):
$$\mathcal{P}_\mu(\theta) = \sum_{n=1}^{+\infty} -\frac{1}{\log(1-\theta)} \frac{\theta^n}{n} \delta_n.$$

Theorem 2.1. *If $\mu = \sum_{n=1}^{+\infty} \frac{1}{n}\delta_n$, then μ is not invertible.*

Proof: Let
$$\tilde{\mu} = \delta_1 + \frac{1}{2}\delta_0 + \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{B_n}{(2n)!} \delta_{-(2n-1)}$$

where the B_n are known as Bernoulli numbers and $B_n > 0$, $\forall\, n \in \mathbb{N}$; $\tilde{\mu}$ is a nonpositive measure. We will show that $\tilde{\mu}$ verifies (1.1) and (1.2).
From Fichtenholz (1970) it is known that the series $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{B_n}{(2n)!}x^{2n-1}$ has convergence radius $2\pi > 0$ and that

$$(2.1)\qquad 1-\frac{1}{2}x+\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{B_n}{(2n)!}x^{2n}=\frac{x}{e^x-1}\quad if\ |x|<2\pi.$$

Hence we have (Guest (1991), Proposition 45.2) that

$$\tilde{\mu}=\delta_1+\frac{1}{2}\delta_0+\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{B_n}{(2n)!}\delta_{-(2n-1)}$$

is term by term Laplace transformable and

$$L_{\tilde{\mu}}=e^{\theta}+\frac{1}{2}+\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{B_n}{(2n)!}\,\mathrm{e}^{-(2n-1)\theta}\quad \text{if}\ \ |\,\mathrm{e}^{-\theta}\,|<2\pi.$$

Then, substituting x with $-\,\mathrm{e}^{-\theta}$ in (2.1) and multiplying for e^{θ} we have

$$L_{\tilde{\mu}}(\theta)=\frac{1}{1-\mathrm{e}^{-\,\mathrm{e}^{-\theta}}}\quad \text{if}\ \ |\,\mathrm{e}^{-\theta}\,|<2\pi.$$

Then $\Theta_{\tilde{\mu}}=(-\log 2\pi,+\infty)$ and

$$k_{\tilde{\mu}}(\theta)=-\log(1-\mathrm{e}^{-\,\mathrm{e}^{-\theta}})\ \ \text{if}\ \ \theta\in(-\log 2\pi,+\infty).$$

We have that $k'_{\tilde{\mu}}>0$, $\forall\,\theta\in(-\log 2\pi,+\infty)$, i.e. that (1.1) is satisfied.
Since $k_{\mu}(\theta)=-\log\left[-\log(1-\mathrm{e}^{\theta})\right]$, $\Theta_{\mu}=(-\infty,0)$ and

$$-k_{\mu}\left(-k_{\tilde{\mu}}(\theta)\right)=\theta\quad\forall\,\theta\in(-\log 2\pi,+\infty),$$

that is expression (1.2), the theorem is proved. □

Before showing the main result of this section, we recall the following theorem (Dieudonné (1971)).

Theorem 2.2 (Lagrange's formula)**.** *Let g be an analytic function in $(-r,r)$, $r>0$ and $g(0)\neq 0$. Then there exist an $R>0$ and an analytic function $t=t(w)$ in $(-R,R)$ such that*

$$t=wg(t)\quad\forall\,w\in(-R,R).$$

Furthermore, if F is analytic on $(-R,R)$, then $\forall\,w\in(-R,R)$ we have that

$$F(t)=F(0)+\sum_{n=1}^{+\infty}\frac{w^n}{n!}\left[\left(\frac{d}{dz}\right)^{n-1}\{F'(z)(g(z))^n\}\right]_{z=0}.$$

In the following remark we provide a more suitable definition of invertibility.

Remark 2.1. Let $\mu \in \mathcal{M}$ and $f_\mu(t) = L_\mu(\log t)$; f_μ is called the generating function of μ. The domain of f_μ is $I_\mu = \{t \in \mathbb{R}^+ : \log t \in \Theta_\mu\}$. We observe that I_μ is an interval and that $f_\mu(t) > 0$ in I_μ.

From Definition 1.1 it follows that $\tilde{\mu} \in \mathcal{M}$ is the inverse measure of μ if:

$$\text{there exists a non empty interval } (a,b) \in \mathbb{R}^+ \text{ such that } f'_{\tilde{\mu}}(t) > 0 \; \forall\, t \in (a,b), \tag{2.2}$$

$$f_{\tilde{\mu}}(t) = \left(\frac{1}{f_\mu\left(\frac{1}{t}\right)}\right)^{-1}. \tag{2.3}$$

We just observe that the condition (2.3) easily follows from (1.3).

Theorem 2.3. *Let $\mu \in \mathcal{M}$; $\mu = \sum_{n=-1}^{+\infty} a_{-n}\delta_{-n}$ and $a_1 > 0$. Then μ is invertible and*

$$\tilde{\mu} = \sum_{n=1}^{+\infty} \frac{b_n}{n!}\delta_n \tag{2.4}$$

where

$$b_n = \left\{ D^{n-1}\left[\sum_{n=-1}^{+\infty} a_{-n}t^{n+1}\right]^n \right\}_{t=0}. \tag{2.5}$$

Proof: $\mu \in \mathcal{M}$ then: $a_n \geq 0$, $n = -1, 0, 1, 2, \ldots$, the integer series $\sum_{n=0}^{+\infty} a_{-n}z^n$ has convergence radius $r > 0$, $L_\mu(\theta) = \sum_{n=-1}^{+\infty} a_{-n}\,\mathrm{e}^{-n\theta}$, $\Theta_\mu = (-\log r, +\infty)$ and the generating function of μ is $f_\mu(t) = \sum_{n=-1}^{+\infty} a_{-n}t^{-n}$ with $t > \frac{1}{r}$.
Let

$$g(t) = \sum_{n=-1}^{+\infty} a_{-n}t^{n+1}; \tag{2.6}$$

we observe that the convergence radius of series (2.6) is r and that, by hypothesis, $g(0) = a_1 \neq 0$, then for Theorem 2.2 with F being the identity function, there exists $R > 0$ and an analytic function $t = t(w)$ in $(-R, R)$ such that

$$t - wg(t) = 0 \quad \forall\, w \in (-R, R). \tag{2.7}$$

Furthermore we have

$$t = t(w) = \sum_{n=1}^{+\infty} \frac{w^n}{n!} b_n, \; w \in (-R, R) \tag{2.8}$$

where

$$b_n = \left\{D^{n-1}\,(g(t))^n\right\}_{t=0} = \left\{D^{n-1}\left(\sum_{n=1}^{+\infty} a_{-n}t^{n+1}\right)^n\right\}_{t=0}, \quad \text{that is (2.5).}$$

We notice that $b_n \geq 0\ \forall\, n \in \mathbb{N}$ because $a_{-n} \geq 0$, $n = -1, 0, 1, \dots$.
On the other hand $g(t) = tf_\mu\left(\frac{1}{t}\right)$, $\forall\, t \in (0, r)$ and from (2.8) it follows that $t = t(w) > 0\ \forall\, w \in (0, R)$. Hence from (2.7) and (2.d6) we have that:

$$\frac{1}{f_\mu\left(\frac{1}{t}\right)} = w \quad \forall\, w \in (0, R), \tag{2.9}$$

that is the function $t = t(w) = \sum_{n=1}^{+\infty} b_n \frac{w^n}{n!}$ is the inverse function of $1/\left[f_\mu\left(\frac{1}{t}\right)\right]$. It can be easily seen that $t'(w) > 0\ \forall\, w \in (0, R)$ and that $t = t(w)$ is the generating function of the measure $\tilde{\mu}$ where $\tilde{\mu} = \sum_{n=1}^{+\infty} \frac{b_n}{n!}\delta_n$.
$\tilde{\mu}$ belongs to $\mathcal{M}$ because $b_n \geq 0\ \forall\, n \in \mathbb{N}$ and the series $\sum_{n=1}^{+\infty} \frac{b_n}{n!} w^n$ has convergence radius $R > 0$; furthermore $\tilde{\mu}$ satisfies the expressions (2.2) and (2.3), that is $\tilde{\mu}$ is the inverse measure of μ. □

Corollary 2.1. *Let $\mu = \sum_{n=1}^{+\infty} \frac{1}{n}\delta_n$ be the logarithmic measure, $\phi(x) = -x + 2$ and let $\mu_1 = \phi_*\mu$ be the image measure of μ by ϕ, i.e. $\mu_1 = \sum_{n=1}^{+\infty} \frac{1}{n}\delta_{-n+2}$; then μ_1 is invertible and its inverse measure is*

$$\tilde{\mu}_1 = \sum_{n=1}^{+\infty} \frac{a_n}{n}\delta_n \tag{2.10}$$

where a_n is defined as follows

$$a_n = \sum_{\substack{k_i \in \mathbb{N} \\ k_1+\ldots+k_n = n-1}} \prod_{i=1}^{n} \frac{1}{k_i + 1}\,. \tag{2.11}$$

Proof: From Theorem 2.3 it follows that μ_1 is invertible and its inverse is

$$\tilde{\mu}_1 = \sum_{n=1}^{+\infty} \frac{b_n}{n!}\delta_n$$

where

$$b_n = \left\{D^{n-1}\left(tf_\mu\left(\frac{1}{t}\right)\right)^n\right\}_{t=0}.$$

Since $tf_\mu\left(\frac{1}{t}\right) = \sum_{n=1}^{+\infty}\frac{1}{n}t^{n-1} = \sum_{n=0}^{+\infty}\frac{1}{n+1}t^n$, it turns out that $\left(tf_\mu\left(\frac{1}{t}\right)\right)^n = \sum_{n=0}^{+\infty} c_n t^n$ where

$$c_n = \sum_{\substack{k_i\in\mathbb{N}\\ k_1+\ldots+k_n=n}} \prod_{i=1}^{n} \frac{1}{k_i+1}\,.$$

Then we have

$$b_n = (n-1)!c_{n-1} = (n-1)! \sum_{\substack{k_i\in\mathbb{N}\\ k_1+\ldots+k_n=n-1}} \prod_{i=1}^{n} \frac{1}{k_i+1}$$

and the theorem is proved. □

Corollary 2.2. *Let $\mu = \sum_{n=1}^{+\infty}\frac{1}{n}\delta_n$ be the base of the logarithmic NEF, and let $\mathcal{P}_{\mu_1}$ be the NEF generated by $\mu_1 = \phi_*\mu$ where $\phi(x) = -x+2$.*
Then $\mathcal{P}_{\mu_1}$ is invertible and its inverse is $\mathcal{P}_{\tilde{\mu}_1}$, with $\tilde{\mu}_1$ defined by (2.10) *and* (2.11)*.*

For a weaker notation, we denote the inverse logarithmic series distribution, $\mathcal{P}_{\tilde{\mu}_1}$, by I.L.S.D.

3. Asymptotic behaviour of the variance function

We recall some notation:

$\mu = \sum_{n=1}^{+\infty}\frac{1}{n}\delta_n$,
$\mu_1 = \phi_*\mu$ where $\phi(x) = -x+2$, i.e. $\mu_1 = \sum_{n=1}^{+\infty}\frac{1}{n}\delta_{-n+2}$,
$\tilde{\mu}_1$ defined in Corollary 2.1 is the inverse measure of μ_1.

The following two theorems describe the asymptotic behaviour of the variance functions V_μ and $V_{\tilde{\mu}_1}$.

Theorem 3.1. *The following results hold:*

(a) $M_\mu = (1,+\infty)$;
(b) $V_\mu(m) = m^2(h(m)-1)$ *where the function $h(m)$ is such that:*
 (b$_1$) $h(m) = \log(m\log m) + \frac{\log\log m}{\log m} + o\left(\frac{\log\log m}{\log m}\right)$ *as $m\to+\infty$,*
 (b$_2$) $h(m) - 1 = m - 1 + o(m-1)$ *as $m \to 1^+$.*

Proof: (a) Let $\mu = \sum_{n=1}^{+\infty}\frac{1}{n}\delta_n$; we have

$$k_\mu(\theta) = \log\left[-\log(1-e^\theta)\right],\ \Theta_\mu = (-\infty,0),\ m(\theta) = k'_\mu(\theta) =$$

$$= -\frac{e^\theta}{(1-e^\theta)\log(1-e^\theta)},\ \forall\,\theta\in\Theta_\mu.$$

M_μ is the image of $k'_\mu(\theta)$, thus

$$M_\mu = (1,+\infty).$$

(b) From

$$k''_\mu(\theta) = -e^\theta \frac{\log(1-e^\theta)+e^\theta}{(1-e^\theta)^2\log^2(1-e^\theta)}$$

it follows that

$$k''_\mu(\theta) = (k'_\mu(\theta))^2(\varphi(\theta)-1)$$

where $\varphi(\theta) = -(\log(1-e^\theta))/\theta$.

Let $\theta(m)$ be the inverse function of $k'(\theta) = m(\theta)$; we have

$$V(m) = k''(\theta(m)) = m^2(\varphi(\theta(m))-1).$$

Denoting $\varphi(\theta(m)) = h(m)$, it follows $V(m) = m^2(h(m)-1)$, that is (b).

(b$_1$) This point can be proved equivalently by showing that

$$\lim_{m\to+\infty} \frac{h(m)-\log(m\log m)}{\frac{\log\log m}{\log m}} = 1$$

that is

$$\lim_{m\to+\infty} \frac{\mathrm{e}^{h(m)-\log(m\log m)}-1}{\frac{\log\log m}{\log m}} = 1$$

or equivalently

$$\frac{\mathrm{e}^{h(m)}}{m\log m} - 1 \sim \frac{\log\log m}{\log m} \quad \text{as} \quad m\to+\infty.$$

Since $m\to+\infty \Leftrightarrow \theta\to 0$, changing variable, we find that

$$\begin{aligned}\frac{\mathrm{e}^{h(m)}}{m\log m} - 1 &= (1-\mathrm{e}^\theta)^{-1/\mathrm{e}^\theta}\left[-\frac{(1-\mathrm{e}^\theta)\log(1-\mathrm{e}^\theta)}{\mathrm{e}^\theta}\right]\frac{1}{\log\left[-\frac{\mathrm{e}^\theta}{(1-\mathrm{e}^\theta)\log(1-\mathrm{e}^\theta)}\right]} - 1\\ &= \frac{\left[\log(1-\mathrm{e}^\theta)\right]\left(1-(1-\mathrm{e}^\theta)^{1-1/\mathrm{e}^\theta}\right) - \theta + \log\left[-\log(1-\mathrm{e}^\theta)\right]}{\mathrm{e}^\theta\left\{\theta - \log(1-\mathrm{e}^\theta) - \log\left[-\log(1-\mathrm{e}^\theta)\right]\right\}};\end{aligned}$$

furthermore it is easy to show that

$$\lim_{\theta\to 0}\left[\log(1-\mathrm{e}^\theta)\right]\left(1-(1-\mathrm{e}^\theta)^{1-1/\mathrm{e}^\theta}\right) = 0.$$

Hence

$$\frac{\mathrm{e}^{h(m)}}{m\log m} - 1 \sim \frac{-\log\left[-\log(1-\mathrm{e}^\theta)\right]}{\log(1-\mathrm{e}^\theta)} \quad \text{as} \quad m\to+\infty \quad (\theta\to 0).$$

We have also that

$$\frac{-\log\left[-\log(1-\mathrm{e}^\theta)\right]}{\log(1-\mathrm{e}^\theta)} \sim \frac{\log\log m}{\log m} \quad \text{as} \quad m \to +\infty.$$

Hence, as $m \to +\infty$

$$\frac{\mathrm{e}^{h(m)}}{m \log m} - 1 \sim \frac{\log\log m}{\log m}\,.$$

(b_2) First, we observe that $m \to 1 \Leftrightarrow \theta \to -\infty$.
Then, we treat separately $m-1$ and $h(m)-1$ and express them in terms of θ. We find respectively that, when $\theta \to -\infty$

$$\begin{aligned} m-1 &= -\frac{\mathrm{e}^\theta}{(1-\mathrm{e}^\theta)\log(1-\mathrm{e}^\theta)} - 1 = \frac{-\mathrm{e}^\theta - (1-\mathrm{e}^\theta)\log(1-\mathrm{e}^\theta)}{(1-\mathrm{e}^\theta)\log(1-\mathrm{e}^\theta)} \\ &\sim \frac{-\mathrm{e}^\theta + \mathrm{e}^\theta - \frac{1}{2}\mathrm{e}^{2\theta}}{-\mathrm{e}^\theta} = \frac{1}{2}\mathrm{e}^\theta \end{aligned}$$

and

$$h(m) - 1 = \frac{-\log(1-\mathrm{e}^\theta) - \mathrm{e}^\theta}{\mathrm{e}^\theta} \sim \frac{1}{2}\mathrm{e}^\theta\,.$$

Hence we have proved that

$$\lim_{m\to 1} \frac{h(m)-1}{m-1} = 1,$$

that is the thesis. □

Now we state and prove the theorem describing the asymptotic behaviour of the variance function of $\tilde{\mu}_1$, where $\tilde{\mu}_1$ is the inverse measure of $\mu_1 = \phi_*\mu$.

Theorem 3.2. *The following results hold:*

(i) $M_{\tilde{\mu}_1} = (1, +\infty)$,
(ii) *as* $m \to +\infty$, $V_{\tilde{\mu}_1}(m) \sim \alpha m^3$, *where* $\alpha = V_\mu(2)$.

Proof: Recall that $V_\mu(m)$ is the variance function of the NEF generated by $\mu = \sum_{n=1}^{+\infty} \frac{1}{n}\delta_n$ and that M_μ is its domain. From (a) of Theorem 3.1 we know that $M_\mu = (1, +\infty)$, then $M_\mu^+ = (1, +\infty)$. If $\phi(x) = -x + 2$ and $\mu_1 = \phi_*\mu$, from Theorem 1.1 we derive that $M_{\mu_1} = (-\infty, 1)$ and $V_{\mu_1}(m) = V(-m+2)$ implying $M_{\mu_1}^+ = M_{\mu_1} \cap (0, +\infty) = (0, 1)$.

From Theorem 1.2 we conclude that

(i) $M_{\tilde{\mu}_1} = (1, +\infty)$ and
(ii) $V_{\tilde{\mu}_1}(m) = m^3 V_{\mu_1}\left(\frac{1}{m}\right) = m^3 V_\mu\left(-\frac{1}{m} + 2\right)$ from which it follows that

$$\lim_{m\to+\infty} \frac{V_{\tilde{\mu}_1}(m)}{m^3} = V_\mu(2) > 0$$

and the theorem is proved. □

References

Barndorff-Nielsen O., *Information and exponential families in statistical inference*, Wiley, New York, 1978.

Dieudonné, J., *Infinitesimal Calculus*, Houghton Mifflin, Boston, 1971.

Fichtenholz G.M., *Functional Series*, Gordon and Breach, Science Publishers, 1970.

Guest G., *Laplace Transform and an Introduction to Distributions*, Ellis Horwood, 1991.

Johnson N.L., Kotz S., *Discrete Distributions*, Houghton Mifflin, Boston, 1969.

Jorgensen B., Martinez J.R., Tsao M., *Asymptotic behaviour of the variance function*, Scand. J. Statist. **21** (1994), 223–243.

Letac G., *La reciprocité des familles exponentielles naturelles sur* $\mathbb{R}$, C.R. Acad. Sci. Paris **303** Ser. I 2 (1986), 61–64.

Letac G., *Lectures on natural exponential families and their variance functions*, I.M.P.A., Rio de Janeiro, 1991.

Letac G., Mora M., *Natural real exponential families with cubic variance functions*, Ann. Statist. **18** (1990), 1–37.

Mora M., *Classification de fonctions variance cubiques des familles exponentielles sur* $\mathbb{R}$, C.R. Acad. Sci. Paris Sér I Math. **302** (1986), 587–590.

Morris C.N., *Natural exponential families with quadratic variance functions*, Ann. Statist. **10** (1982), 65–80.

Sacchetti D., *Inverse distribution: an example of non existence*, Atti dell' Accademia delle Scienze Lettere ed Arti di Palermo, 1993.

Seshadri V., *The Inverse Gaussian distribution*, Oxford University Press, Oxford, 1993.

Tweedie M.C.K., *Inverse statistical variates*, Nature **155** (1945), 453.

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