# On Besov spaces and absolute convergence of the Fourier transform on Heisenberg groups

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*Abstract.* In this paper the absolute convergence of the group Fourier transform for the Heisenberg group is investigated. It is proved that the Fourier transform of functions belonging to certain Besov spaces is absolutely convergent. The function spaces are defined in terms of the heat semigroup of the full Laplacian of the Heisenberg group.

*Keywords:* Besov spaces, Heisenberg groups, group Fourier transform

*Classification:* 46E35, 43A80

## 1. Introduction

Different problems in harmonic analysis on Heisenberg groups attracted attention in the last two decades. As far as the group Fourier transform is regarded a basic reference is Geller's fundamental work [5], confer also Folland's book [3]. D. Geller obtained in his paper, among other things, a characterization for the Fourier transform of rapidly decreasing Schwartz functions. On the other hand a natural analogue of the Paley-Wiener theorem for Heisenberg groups was proved by S. Thangavelu [13]. In the classical setting the third type of results that describe the connection between the smoothness of functions and the behaviour of their Fourier transform are the Bernstein type theorems. The last ones assert that the Fourier transform of a function is absolutely convergent if the function belongs to a suitable Besov space, cf. [1], [7]. In this paper we want to prove the similar results for the group Fourier transform of the Heisenberg group.

The Heisenberg group $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is a nilpotent Lie group whose group law is defined by

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + \frac{1}{2}(y_1 \cdot x_2 - x_1 \cdot y_2)). \tag{1}$$

The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}^n$ is spanned by the left invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2} y_i \frac{\partial}{\partial t},\ Y_i = \frac{\partial}{\partial y_i} - \frac{1}{2} x_i \frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t},\ i = 1, \dots n. \tag{2}$$

We have the commutation relations

$$[Y_i, X_i] = T,\ i = 1, \dots , n, \tag{3}$$

and all other commutators vanish. The Haar measure of $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$.

For each nonzero real number $\lambda$ there is a irreducible unitary representation of $\mathbb{H}^n$ on $L_2(\mathbb{R}^n)$ given by

$$\pi_\lambda(x,y,t)\phi(\xi) = e^{\pi\sqrt{-1}(2t\lambda+2y\cdot\xi+\lambda x\cdot y)}\phi(\xi+\lambda x) \tag{4}$$

where $\phi \in L_2(\mathbb{H}^n)$. There are also one dimensional unitary representation of $\mathbb{H}^n$, $\pi_{(a,b)}$, $a, b \in \mathbb{R}^n$, given by

$$\pi_{(a,b)}(x,y,t)\xi = e^{2\pi\sqrt{-1}(a\cdot x+b\cdot y)}\xi\,, \quad \xi \in \mathbb{C}. \tag{5}$$

Any irreducible unitary representation of $\mathbb{H}^n$ is unitary equivalent to one of just described representations. The representations $\pi_{(a,b)}$ are of less importance for us since they form a set of Plancherel measure zero.

For a function $f$ on the Heisenberg group, say $f \in L_1(\mathbb{H}^n)$, the Fourier transform is defined to be the operator valued function

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(x,y,t)\pi_\lambda(-x,-y-t)\,dx\,dy\,dt. \tag{6}$$

If $\phi \in L_2(\mathbb{H}^n)$, then $\hat{f}(\lambda)\phi$ is given by

$$\hat{f}(\lambda)\phi(x) = |\lambda|^n \int f(\lambda^{-1}(x-y),w,t)e^{-\pi\sqrt{-1}(y+x)\cdot w-2\pi\sqrt{-1}t\lambda}\phi(y)\,dy\,dw\,dt. \tag{7}$$

So it is an integral operators with kernel

$$K_f^\lambda(x,y) = |\lambda|^{-n}\mathcal{F}_{2,3}f(\lambda^{-1}(x-y),\frac{1}{2}(x+y),\lambda), \tag{8}$$

where $\mathcal{F}_{2,3}$ denotes Fourier transformation in the second and third variables.

The Plancherel formula for $\mathbb{H}^n$ looks as follows

$$\int_{\mathbb{H}^n} |f(x,y,t)|^2\,dx\,dy\,dt = c_n\int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{HS}^2|\lambda|^n\,d\lambda. \tag{9}$$

Here $\|\cdot\|_{HS}$ stands for the Hilbert-Schmidt norm.

## 2. Besov spaces on Heisenberg groups

To describe the smoothness of functions we use the Besov spaces related to the full Laplacian on $\mathbb{H}^n$. Let $\Delta = \sum_{i=1}^n X_i^2 + Y_i^2 + T^2$ be the sum of squares of left invariant vector fields. If we equip $\mathbb{H}^n$ with the left invariant Riemannian metric $g$ such that the vector fields are the orthonormal basis in any tangent space, then the operator $\Delta$ coincides with the Laplace-Beltrami operator corresponding to this metric, and the Riemannian measure coincides with the Haar measure on $\mathbb{H}^n$.

The Bessel-potentials $(I-\Delta)^{-s/2}$ with $s>0$ can be defined in $L_2(\mathbb{H}^n)$ via the spectral theory. They can be extended afterwards from $L_2(\mathbb{H}^n)$ to $L_p(\mathbb{H}^n)$ with $1<p<\infty$, cf. [12]. For $1<p<\infty$, $s\in\mathbb{R}$ we define the Sobolev spaces $W_p^s(\mathbb{H}^n)$ in the following way:

- if $s>0$, then $W_p^s(\mathbb{H}^n)$ is the collection of all functions $f\in L_p(\mathbb{H}^n)$ such that $f=(I-\Delta)^{-s/2}h$ for some $h\in L_p(\mathbb{H}^n)$, with the norm $\|f|W_p^s(\mathbb{H}^n)\|=\|h\|_p$,
- if $s<0$, then $W_p^s(\mathbb{H}^n)$ is the collection of all distributions $f\in D'(\mathbb{H}^n)$ of the form $f=(I-\Delta)^m h$ with $h\in W_p^{2m+s}(\mathbb{H}^n)$, where $m$ is a natural number such that $2m+s>0$, and $\|f|W_p^s(\mathbb{H}^n)\|=\|h|W_p^{2m+s}(\mathbb{H}^n)\|$,
- if $s=0$, then $W_p^0(\mathbb{H}^n)=L_p(\mathbb{H}^n)$.

The spaces $W_p^s(\mathbb{H}^n)$ with $s<0$ are independent of $m$ (equivalent norms). If $s$ is a positive integer, then one can use left-invariant vector fields on $\mathbb{H}^n$ to define equivalent norms in $W_p^s(\mathbb{H}^n)$, cf. [14].

For $s\in\mathbb{R}$, $1<p<\infty$ and $1\le q\le\infty$ we define the Besov spaces $B_{p,q}^s(\mathbb{H}^n)$ via the real interpolation

$$B_{p,q}^s(\mathbb{H}^n)=(W_p^{s_0}(\mathbb{H}^n),W_p^{s_1}(\mathbb{H}^n))_{\theta,q}, \tag{10}$$

$s=(1-\theta)s_0+\theta s_1$, $0<\theta<1$.

The norm in the Besov spaces can be described by the heat semigroup $H_t=e^{t\Delta}$ ([11]). The heat semigroup $H_t$ is given by a right convolution:

$$H_tf(x)=\int_G f(y)h_t(y^{-1}x)\,dy \tag{11}$$

where $(t,x)\longrightarrow h_t(x)$ is a $C^\infty$ function on $\mathbb{R}_+\times G$ and a positive solution of $(\frac{\partial}{\partial t}+\Delta)u=0$. The semigroup is symmetric submarkovian, hence analytic in $L_p(G)$ if $1<p<\infty$. If $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$ and $m>\frac{|s|}{2}$, then the expression

$$\|f|B_{p,q}^s(\mathbb{H}^n)\|_H=\|f*h_{0,m}\|_p+\Big(\int_0^1 t^{(m-s/2)q}\|\frac{d^m}{dt^m}f*h_t\|_p^q\frac{dt}{t}\Big)^{1/q} \tag{12}$$

is an equivalent norm in $B_{p,q}^s(\mathbb{H}^n)$. Here $h_{0,m}$ is a $C^\infty$ function on $\mathbb{H}^n$ given by

$$h_{0,m}=\sum_{l=0}^{m-1}c_l\frac{\partial^l}{\partial t^l}h_t|_{t=1}. \tag{13}$$

Using the above norm one can define the Besov spaces for $p=1$ and $p=\infty$. The definition if independent of $m$ (equivalent norms). If $s>0$, then one can use $\|f\|_p$ instead of $\|f*h_{0,m}\|_p$ in (12), $1\le p\le\infty$.

We have the following elementary topological embeddings

$$B^{s_1}_{p,q_1}(\mathbb{H}^n) \subset B^{s_2}_{p,q_2}(\mathbb{H}^n) \ \text{ if } \ s_1 > s_2, \tag{14}$$

$$B^{s}_{p,q_1}(\mathbb{H}^n) \subset B^{s}_{p,q_2}(\mathbb{H}^n) \ \text{ if } \ q_1 < q_2, \tag{15}$$

$$B^{0}_{p,p}(\mathbb{H}^n) \subset L_p(\mathbb{H}^n) \ \text{ if } \ 1 \le p \le 2. \tag{16}$$

More information about function spaces of Hardy-Sobolev-Besov type on Lie groups and more general on Riemannian manifolds with bounded geometry may be found in [15] and [10], [11].

## 3. Absolute convergence of the Fourier transform

We start with the following standard lemma.

**Lemma 1.** *Let $0 < r < c$. Then there is a positive constant $C > 0$ such that a geodesic ball $\Omega(0, r)$ is contained in the euclidean box centered at 0 with sides parallel to the coordinate axes and sizes $Cr$.*

Proof: We give a short proof of the lemma for completeness. Let $(x_1, \dots , x_{2n+1}) \in \Omega(0, r)$ be a point inside the geodesic ball. Then the is a geodesic $\gamma(t) = (\gamma_1(t), \dots , \gamma_{2n+1}(t))$ with $\gamma_i(0) = 0$, $\gamma_i(1) = x_i$ and $\|\dot\gamma(t)\| \le r$. There are functions $\beta_i(t)$ such that

$$\dot\gamma(t) = (\dot\gamma_1(t), \dots , \dot\gamma_{2n+1}(t)) = \sum_{i=1}^{n} \beta_i(t)X_i + \beta_{n+1}(t)Y_i + \beta_{2n+1}(t)T$$

and

$$\sum_{i=1}^{2n+1} \beta_i^2(t) \le r^2.$$

Thus $|\beta_i| \le r$ for each $i$. But (2) implies $\dot\gamma_i(t) = \beta_i(t)$ for $i = 1, \dots 2n$ and $\dot\gamma_{2n+1}(t) = \beta_{2n+1}(t) + 2\sum_{i=1}^{n} \gamma_{n+i}(t)\beta_i(t) - \gamma_i(t)\beta_{n+i}(t)$. Now $|\gamma_i(1)| \le \int_0^1 |\dot\gamma_i(t)|\, dt \le Cr$ since $\gamma_i(0) = 0$ and $r \le c$. □

The main result reads us follows.

**Theorem 1.** *The following inequalities*

$$\int_{\mathbb{R}} \|\hat f(\lambda)\|_{HS} |\lambda|^n \, d\lambda \le C\|f|B^{n+[\frac{n}{2}]+2}_{1,1}(\mathbb{H}^n)\| \tag{17}$$

$$\int_{\mathbb{R}} \|\hat f(\lambda)\|_{HS} |\lambda|^{\frac{n}{2}} \, d\lambda \le C\|f|B^{n+2}_{1,1}(\mathbb{H}^n)\| \tag{18}$$

*hold.*

**Remark 1.** The measure $|\lambda|^n \, d\lambda$ seems to be more natural since it is a Plancherel measure, but the measure $|\lambda|^{\frac{n}{2}} \, d\lambda$ better describes the behaviour of the Fourier transform near zero, cf. the proof of the theorem. So we decide to state the both inequalities.

The next corollary follows by real interpolation from the above theorem and Plancherel theorem since $B^0_{2,2}(\mathbb{H}^n) = L_2(\mathbb{H}^n)$.

**Corollary 1.** *Let* $s_p = \frac{2-p}{p}(n + [\frac{n}{2}] + 2)$ *and* $\sigma_p = \frac{2-p}{p}(n+2)$ *for* $1 < p < 2$. *Then the following inequalities*

$$\left(\int_{\mathbb{R}} \|\hat{f}(\lambda)\|^p_{HS} |\lambda|^n \, d\lambda\right)^{1/p} \leq C \|f | B^{s_p}_{p,p}(\mathbb{H}^n)\| \tag{19}$$

$$\left(\int_{\mathbb{R}} \|\hat{f}(\lambda)\|^p_{HS} |\lambda|^{\frac{np}{2}} \, d\lambda\right)^{1/p} \leq C \|f | B^{\sigma_p}_{p,p}(\mathbb{H}^n)\| \tag{20}$$

*hold.*

Proof of Theorem 1: *Step* 1. To prove the theorem we use the atomic decomposition for the spaces $B^s_{p,q}$, cf. [10]. For any $j$, $j = 0, 1, \dots$, let $\{\Omega(z_{j,i}, 2^{-j})\}_{i=0}^{\infty}$ be the uniformly locally finite covering of $\mathbb{H}^n$ with the multiplicity independent of $j$. The atomic decomposition theorem asserts that $f \in B^s_{1,1}(H_n)$, $s > 0$, if and only if $f$ can be decomposed into the sum

$$f = \sum_{i,j=0}^{\infty} s_{j,i} a_{j,i}, \quad \text{convergence in} \quad S'(\mathbb{R}^{2n+1}), \tag{21}$$

where functions $a_{j,i} \in C_0^{\infty}(\mathbb{R}^{2n+1})$ are smooth atoms and the real numbers $s_{j,i}$ satisfy the condition

$$\sum_{i,j=0}^{\infty} |s_{j,i}| < \infty. \tag{22}$$

The function $a_{j,i}$ is called a smooth atom if the following two conditions are fulfilled

$$\operatorname{supp} a_{j,i} \subset \Omega(x_{j,i}, 2^{-j+1}), \tag{23}$$

$$|Z_{m_1} \dots Z_{m_k} a_{j,i}| \leq C 2^{-j(s-k-N)}, \quad k \leq L \tag{24}$$

where $Z_m$ denotes any left invariant vector field $X_i$, $Y_i$ or $T$ and $N = 2n+1$. The constant $L$ is a fixed real number $L > [s] + 1$, and $C$ is an absolute constant. Moreover, the infimum of (22) taken over all possible decompositions (21) is an equivalent norm in $B^s_{1,1}(H_n)$. The details about the atomic decomposition of Besov spaces on Riemannian manifolds can be found in [10].

*Step* 2. Let $a_{j,i}$ be the smooth atom supported in $\Omega(z_{j,i}, 2^{-j+1})$ Then $a(z) = a_{j,i}(z_{j,i} \cdot z)$ is a smooth atom supported in $\Omega(0, 2^{-j+1})$. Let $\mathcal{F}_3$ denotes Fourier transformation in the third variable. Let $z_{j,} = (x_0, y_0, t_0)$. Then

$$\begin{aligned}\|\hat{a}_{j,i}(\lambda)\|^2_{HS} &= |\lambda|^{-n} \int |\mathcal{F}_3 a_{j,i}(x, y, \lambda)|^2 \, dx \, dy \\ &= |\lambda|^{-n} \int |\mathcal{F}_3 a_{j,i}(x + x_0, y + y_0, \lambda)|^2 \, dx \, dy,\end{aligned}$$

cf. [3, p. 39]. But

$$\begin{aligned}\mathcal{F}_3 a_{j,i}(x + x_0, y + y_0, \lambda) &= C \int_{\mathbb{R}} a_{j,i}(x + x_0, y + y_0, t) e^{-\sqrt{-1}t\lambda} \, dt \\ &= C \int_{\mathbb{R}} a(x, y, t - t_0 - \frac{1}{2}(y \cdot x_0 - x \cdot y_0)) e^{-\sqrt{-1}t\lambda} \, dt \\ &= C e^{-\sqrt{-1}\lambda(t_0 - \frac{1}{2}(y \cdot x_0 - x \cdot y_0))} \int_{\mathbb{R}} a(x, y, t) e^{-\sqrt{-1}t\lambda} \, dt,\end{aligned}$$

and

$$e^{-\sqrt{-1}t\lambda} = (-\sqrt{-1}\lambda)^{-m} \frac{d^m}{dt^m} e^{-\sqrt{-1}t\lambda}.$$

Thus Lemma 1 implies

$$\begin{aligned}\|\hat{a}_{j,i}(\lambda)\|^2_{HS} &\le C|\lambda|^{-n-m} \int \left| \int_{\mathbb{R}} e^{-\sqrt{-1}t\lambda} T^m a(x, y, t) \, dt \right|^2 dx \, dy \\ &\le |\lambda|^{-n-2m} \int \Big( \int_{-C2^{-j}}^{C2^{-j}} |T^m a(x, y, t)| \, dt \Big)^2 dx \, dy \\ &\le C|\lambda|^{-n-2m} 2^{-j} \int \int \int |T^m a(x, y, t)|^2 \, dx \, dy \, dt.\end{aligned}$$

Now from the definition of an atom we get

$$\|\hat{a}_{j,i}(\lambda)\|_{HS} \le C|\lambda|^{-m-\frac{n}{2}} 2^{-j(s-m-\frac{N-1}{2})} = C|\lambda|^{-m-\frac{n}{2}} 2^{-j(s-m-n)}. \tag{25}$$

Using these estimates with $m = 0$ for small values of $|\lambda|$ and $m = 2$ for $|\lambda|$ big we get

$$\int_{\mathbb{R}\setminus\{0\}} \|\hat{a}_{j,i}(\lambda)\|_{HS} |\lambda|^{\frac{n}{2}} \, d\lambda \le C 2^{-j(s-2-n)}. \tag{26}$$

Similarly, taking $m = [\frac{n+4}{2}]$ for big values of $|\lambda|$ we get

$$\int_{\mathbb{R}\setminus\{0\}} \|\hat{a}_{j,i}(\lambda)\|_{HS} |\lambda|^n \, d\lambda \le C2^{-j(s-[\frac{n}{2}]-n-2)}. \tag{27}$$

*Step* 3. If $f \in L_1(\mathbb{H}^n)$, then the operator $\hat{f}(\lambda)$ is an integral operator with kernel

$$K_f^\lambda(x,y) = |\lambda|^{-n} \mathcal{F}_{2,3} f(\lambda^{-1}(x-y), \frac{1}{2}(x+y), \lambda).$$

If $a_{j,i}$ is an atom, then the corresponding kernels $K_{j,i}^\lambda$ are smooth functions.

If $f \in B_{1,1}^{n+2}(H_n)$ and $f = \sum_{j,i=0}^\infty s_{j,i} a_{j,i}$, then $f \in L_1(\mathbb{H}^n)$ and

$$\int f(x,y,t)\psi(x,y,t)\, dx\, dy\, dt = \sum_{j,i=0}^\infty s_{j,i} \int\int\int a_{j,i}(x,y,t)\psi(x,y,t)\, dx\, dy\, dt,$$

$\psi \in S(\mathbb{R}^N)$. Thus

$$|K_f^\lambda(x,y)| \le \sum_{j,i=0}^\infty |s_{j,i}|\, |\lambda|^{-n} |\, |\mathcal{F}_{2,3} a_{j,i}(\lambda^{-1}(x-y), \frac{1}{2}(x+y), \lambda)| = \sum_{j,i=0}^\infty |s_{j,i}| K_{j,i}^\lambda$$

where $K_{j,i}^\lambda$ is a kernel of $\hat{a}_{j,i}(\lambda)$. Thus

$$\begin{aligned}\|\hat{f}(\lambda)\|_{HS} &= (\int |K_f^\lambda(x,y)|^2\, dx\, dy)^{1/2} \\ &\le \sum_{j,i=0}^\infty |s_{j,i}| (\int |K_{j,i}^\lambda(x,y)|^2\, dx\, dy)^{1/2} = \sum_{j,i=0}^\infty |s_{j,i}| \|\hat{a}_{j,i}\|_{HS}.\end{aligned}$$

Now the theorem follows from inequalities (26)–(27). □

**Theorem 2.** *Let* $1 \le q \le 2$ *and* $s_q = \frac{2-q}{q}(n + \frac{1}{2})$. *Then the following inequality*

$$\left(\int_{\mathbb{R}} \|\hat{f}(\lambda)\|_{HS}^q |\lambda|^n \, d\lambda\right)^{1/q} \le C\|f|B_{2,q}^{s_q}(\mathbb{H}^n)\| \tag{28}$$

*holds.*

Proof: For $q = 2$ the theorem is obvious since $\mathcal{B}_{2,2}^0(\mathbb{H}^n) = L_2(\mathbb{H}^n)$, cf. [14]. We prove the theorem for $q = 1$. The rest follows by interpolation. Any function $f \in L_2(\mathbb{H}^n)$ can be decomposed in the following way

$$f(x) = C\left(f * h_{0,m} + \int_0^2 t^m \frac{d^m}{dt^m} f * h_t \frac{dt}{t}\right) \tag{29}$$

where $h_{0,m} \in S(\mathbb{R}^{2n+1})$ is given by (13), cf. [11]. Thus

$$\|\hat{f}(\lambda)\|_{HS} \leq \|\hat{f}(\lambda)\hat{h}_{0,m}(\lambda)\|_{HS} + \int_0^1 t^m \|\hat{f}(\lambda)\hat{h}^m_{t/2}(\lambda)\hat{h}_{t/2}(\lambda)\|_{HS}\frac{dt}{t}$$

$$\leq \|\hat{f}(\lambda)\|_{HS}\|\hat{h}_{0,m}(\lambda)\|_{HS} + \int_0^1 t^m \|\widehat{f * h^m_{t/2}}\|_{HS}\|\hat{h}_{t/2}(\lambda)\|_{HS}\frac{dt}{t}$$

$$\int_{\mathbb{R}} \|\hat{f}(\lambda)\|_{HS}|\lambda|^n\,d\lambda \leq \left(\int_{\mathbb{R}} \|\hat{f}(\lambda)\|^2_{HS}|\lambda|^n\,d\lambda\right)^{1/2}\left(\int_{\mathbb{R}} \|\hat{h}_{0,m}(\lambda)\|^2_{HS}|\lambda|^n\,d\lambda\right)^{1/2}$$

$$+ \int_0^2 t^m \left(\int_{\mathbb{R}} \|\widehat{f * h^m_{t/2}}\|^2_{HS}|\lambda|^n\,d\lambda\right)^{1/2}\left(\int_{\mathbb{R}} \|\hat{h}_{t/2}(\lambda)\|^2_{HS}|\lambda|^n\,d\lambda\right)^{1/2}\frac{dt}{t}$$

$$= \|f\|_2\|h_{0,m}\|_2 + \int_0^1 t^m\|f * h^m_{t/2}\|_2\|h_{t/2}\|_2\frac{dt}{t}$$

$$\leq \|f\|_2\|h_{m,0}\|_2 + \int_0^2 t^{(m-\frac{N}{4})}\|f * h^m_{t/2}\|_2\frac{dt}{t} \leq C\|f|B^{n+\frac{1}{2}}_{2,1}(H_n)\|.$$

$\square$

**Corollary 2.** *Let $1 \leq q \leq p \leq 2$ and $s_{p,q} = n + \frac{2-p}{p}[\frac{n+2}{2}] + \frac{1}{p} + \frac{1}{q} - 1$. Then the following inequality*

$$\left(\int_{\mathbb{R}} \|\hat{f}(\lambda)\|^q_{HS}|\lambda|^n\,d\lambda\right)^{1/q} \leq C\|f|B^{s_{pq}}_{p,q}(\mathbb{H}^n)\| \tag{30}$$

*holds.*

**Remark 2.** There is also a heat semigroup version of Bernstein theorem on a unimodular Lie group, cf. [8] and [4].

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(*Received* February 23, 1998)