A Carleson inequality for $BMOA$ functions with their derivatives on the unit ball

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Abstract. The main purpose of this note is to give a new characterization of the well-known Carleson measure in terms of the integral for $BMOA$ functions with their derivatives on the unit ball.

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1. Introduction

Let B denote the unit ball in $\mathbb{C}^n (n \geq 1)$, and v the $2n$-dimensional Lebesgue measure on B normalized so that $v(B) = 1$, while σ is the normalized surface measure on the boundary S of B.

For $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, we let $\langle z, w \rangle = z_1\overline{w}_1 + \dots + z_n\overline{w}_n$ so that $|z|^2 = \langle z, z \rangle$. For $\alpha = (\alpha_1, \dots, \alpha_n)$ with each α_i a nonnegative integer, we write $|\alpha| = \alpha_1 + \dots + \alpha_n$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\overline{w^\alpha} = \overline{w}_1{}^{\alpha_1} \cdots \overline{w}_n{}^{\alpha_n}$, and

$$\frac{\partial^{|\alpha|} f(z)}{\partial z^\alpha} = \frac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \,,$$

where $\partial^0 f(z)/\partial z^0 = f(z)$.

For $a \in B$, $a \neq 0$, let φ_a denote the automorphism of B taking 0 to a defined by

$$\varphi_a(z) = \frac{a - P_a(z) - \sqrt{(1 - |z|^2)}Q_a(z)}{1 - \langle z, a \rangle} \,,$$

where P_a is the projection of $\mathbb{C}^n$ onto the one-dimensional subspace span of a and Q_a is $I - P_a$. If $a = 0$, let $\varphi_0(z) = z$. For $0 < r < 1$ and $a \in B$, let $E(a, r) = \{z \in B : |\varphi_a(z)| < r\}$ as a pseudohyperbolic ball on B. It is easy to see that $E(a, r) = \varphi_a(rB)$ and $v(E(a, r)) \sim (1 - |a|)^{n+1}$ (see [Ru, 2.2.7]), where the symbol "$\sim$" indicates that the quantities have ratios bounded and bounded away from zero as a varies.

The Hardy space $H^p (0 < p < \infty)$ is defined as the space of holomorphic functions f on B satisfying

$$\|f\|_p = \sup_{0<r<1} \left\{ \int_S |f(r\xi)|^p \, d\sigma(\xi) \right\}^{1/p} < \infty \,. \tag{1.1}$$

The space $BMOA$ consists of the functions $f \in H^1$ for which

$$\|f\|_{BMOA} = \sup \frac{1}{\sigma(Q)} \int_Q |f - f_Q| \, d\sigma < \infty \, ,$$

where f_Q denotes the averages of f over Q and the supremum is taken over all $Q = Q_\delta(\xi) = \{\eta \in S : |1 - \langle \eta, \xi \rangle| < \delta\}$ for $\xi \in S$ and $0 < \delta \le 2$. Here we have identified f with its boundary function.

In the work on interpolation by bounded analytic functions on the unit disc $\mathcal{D}$ of $\mathbb{C}$, L. Carleson [Ca1], [Ca2] found the following well-known result:

Let μ be a positive measure on $\mathcal{D}$ and $0 < p < \infty$. Then an estimate of the form

$$\left(\int_{\mathcal{D}} |f(z)|^p \, d\mu(z) \right)^{1/p} \le C_p \|f\|_p \tag{1.2}$$

holds for all $f \in H^p$ if and only if there exists a constant $C' > 0$ such that

$$\mu(S(I)) \le C' |I|$$

for all $S(I) = \{z \in \mathcal{D} : z/|z| \in I, 1 - |I| \le |z| < 1\}$, where $|I|$ denotes the arc length of the subarc I on the unit circle and $S(I) = \mathcal{D}$ if $|I| \ge 1$. Here μ is called a Carleson measure on $\mathcal{D}$.

We say that a positive measure μ on B is a Carleson measure if there exists a constant $C > 0$ such that

$$\mu(B_\delta(\xi)) \le C \delta^n$$

for all $\xi \in S$ and all $\delta (0 < \delta \le 2)$, where $B_\delta(\xi) = \{z \in B : |1 - \langle z, \xi \rangle| < \delta\}$ is said to be a Carleson region. The definition above tells us that a Carleson measure is finite. Here and in the sequel, constants are denoted by C, they are positive and may differ from one occurrence to the other.

Hörmander [Hö] proved the higher dimensional version of Carleson's theorem above and gave a simple proof of Carleson's estimates. In this paper we shall give a new characterization of Carleson measures in terms of integrals for $BMOA$ functions with their derivatives on the unit ball. Our main result is the following:

Theorem 1. *Let μ be a positive Borel measure on B, $0 < p < \infty$ and α a multiindex. Then there exists a constant $C > 0$ such that*

$$\sup_{a \in B} \int_B \left| \frac{\partial^{|\alpha|} f(z)}{\partial z^\alpha} - \frac{\partial^{|\alpha|} f(a)}{\partial z^\alpha} \right|^p \frac{(1 - |a|^2)^{n + \frac{3|\alpha| p}{2}} (1 - |z|^2)^{\frac{|\alpha| p}{2}}}{|1 - \langle z, a \rangle|^{2n + 2|\alpha| p}} \, d\mu(z) \le C \|f\|_{BMOA}^p \tag{1.3}$$

for all $f \in BMOA$ if and only if there exists a constant $C' > 0$ such that

$$\mu(B_\delta(\xi)) \le C' \delta^{n + |\alpha| p} \tag{1.4}$$

for all $\xi \in S$ and all $\delta (0 < \delta \le 2)$.

2. Preliminary lemmas

Lemma 1. *For $0 < r < 1$ let a be a point in B with $1-|a| < 2\left(1+\sqrt{\frac{2}{1-r}}\right)^{-2}$. Then $E(a,r) \subset B_\delta(\xi)$, where $\xi = a/|a|$ and $\delta = (1-|a|)\left(1+\sqrt{\frac{2}{1-r}}\right)^{2}$.*

Proof: By the identity ([Ru])

$$1-|\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}$$

for $a \in B$ and $z \in E(a,r)$ we have (see [Je])

$$\frac{1-r}{1+r} \le \frac{1-|a|^2}{1-|z|^2} \le \frac{1+r}{1-r}\,, \tag{2.1}$$

and

$$|1-\langle z,a\rangle|^2 = \frac{(1-|a|^2)(1-|z|^2)}{1-|\varphi_a(z)|^2} \le 4\left(\frac{1-|a|}{1-r}\right)^2 . \tag{2.2}$$

Let $\xi = a/|a|$. We obtain

$$\begin{aligned} |1-\langle z,\xi\rangle|^{\frac{1}{2}} &\le |1-\langle z,a\rangle|^{\frac{1}{2}} + |1-\langle a,\xi\rangle|^{\frac{1}{2}} \\ &\le (1-|a|)^{\frac{1}{2}}\left(1+\sqrt{\frac{2}{1-r}}\right). \end{aligned}$$

Taking $\delta = (1-|a|)\left(1+\sqrt{\frac{2}{1-r}}\right)^2$ we get that $E(a,r) \subset B_\delta(\xi)$. □

Lemma 2. *Let $f \in BMOA$ and let $|\alpha|$ be a positive integer. Then there exists constant $C > 0$ such that*

$$\left|\frac{\partial^{|\alpha|} f(a)}{\partial z^\alpha}\right| \le C\|f\|_{BMOA}(1-|a|)^{-|\alpha|} \tag{2.3}$$

for all $a \in B$.

Proof: It is known that $BMOA \subset \mathcal{B}(B)$, where $\mathcal{B}(B)$ is the Bloch space of holomorphic functions f on B with $\|f\|_B = \sup\{(1-|z|^2)|\bigtriangledown f(z)| : z \in B\} < \infty$, where $\bigtriangledown f(z) = (\partial f/\partial z_1, \cdots, \partial f/\partial z_n)$ is the analytic gradient of f. From Theorem A in [Zh], for $f \in BMOA$ and positive integer $|\alpha|$, we have

$$\left|\frac{\partial^{|\alpha|} f(a)}{\partial z^\alpha}\right| \le C\|f\|_B(1-|a|)^{-|\alpha|} \le C\|f\|_{BMOA}(1-|a|)^{-|\alpha|}$$

for all $a \in B$. Here we used the estimate $\|f\|_B \le C\|f\|_{BMOA}$. □

Lemma 3 ([Wu]). *Let μ be a finite positive measure on B, $0 < r < 1$ and $\beta > 0$. Then*

$$\sup_{a\in B}\int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+\beta} d\mu(z) < \infty \tag{2.4}$$

if and only if there exists a constant $C > 0$ such that

$$\mu(E(a,r)) \leq C(1-|a|)^{n+\beta} \tag{2.5}$$

is fulfilled for all $a \in B$.

3. Proof of Theorem 1

We first consider the case $|\alpha| = 0$. Suppose that (1.4) holds for all $\xi \in S$ and all $\delta (0 < \delta \leq 2)$, that is, μ is a Carleson measure on B. For a holomorphic function f on B and $0 < p < \infty$, from [Ch] we have that $\|f\|_{BMOA} < \infty$ implies

$$\sup_{a\in B}\left\{\int_S |f(\xi)-f(a)|^p \frac{(1-|a|^2)^n}{|1-\langle \xi,a\rangle|^{2n}}\, d\sigma(\xi)\right\}^{1/p} < \infty\,. \tag{3.1}$$

Thus for each $a \in B$ and each $f \in BMOA$ we have

$$F_a(z) = (f(z)-f(a))\left(\frac{(1-|a|^2)^n}{(1-\langle z,a\rangle)^{2n}}\right)^{1/p} \in H^p,\ 0<p<\infty\,.$$

By Hörmander's result we have

$$\int_B |F_a(z)|^p d\mu \leq C_p \int_S |F_a(\xi)|^p\, d\sigma(\xi),$$

it follows that

$$\begin{aligned}
&\sup_{a\in B}\left\{\int_B |f(z)-f(a)|^p \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}}\, d\mu(z)\right\}^{1/p}\\
&\leq C' \sup_{a\in B}\left\{\int_S |f(\xi)-f(a)|^p \frac{(1-|a|^2)^n}{|1-\langle \xi,a\rangle|^{2n}}\, d\sigma(\xi)\right\}^{1/p}\\
&\leq C'\|f\|_{BMOA}.
\end{aligned}$$

To prove that (1.4) follows from (1.3) we only need to prove that (1.4) is valid for all $\xi \in S$ and all $\delta (0 < \delta \leq \frac{1}{4})$ since μ is finite. For each $\xi \in S$ and each $\delta (0 < \delta \leq \frac{1}{4})$ we take $a' = (1-2\delta)\xi \in B$. For $z \in B_\delta(\xi)$ we have

$$\begin{aligned}
2\delta &\leq |1-\langle z,a'\rangle| \leq (|1-\langle z,\xi\rangle|^{\frac{1}{2}} + |1-\langle \xi,a'\rangle|^{\frac{1}{2}})^2\\
&\leq \sqrt{5}\delta < 3\delta < 4\delta(1-\delta) = 1-|a'|^2.
\end{aligned} \tag{3.2}$$

For fixed $a' \in B$ above we choose a function $f(z) = (1 - \langle z, a' \rangle)^{-n} \in BMOA$. From (3.2) we have

$$
\begin{aligned}
&\sup_{a\in B} \int_B |f(z) - f(a)|^p \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}}\, d\mu(z) \\
&\geq \int_B |f(z) - f(a')|^p \frac{(1-|a'|^2)^n}{|1-\langle z,a'\rangle|^{2n}}\, d\mu(z) \\
&\geq \int_{B_\delta(\xi)} \Big(|1-\langle z,a'\rangle|^{-n} - (1-|a'|^2)^{-n}\Big)^p \frac{(1-|a'|^2)^n}{|1-\langle z,a'\rangle|^{2n}}\, d\mu(z) \\
&\geq (\frac{1}{\sqrt{5}} - \frac{1}{3})^{np} \delta^{-np} \int_{B_\delta(\xi)} \frac{(1-|a'|^2)^n}{|1-\langle z,a'\rangle|^{2n}}\, d\mu(z) \\
&\geq C\delta^{-np-n} \mu(B_\delta(\xi)).
\end{aligned} \tag{3.3}
$$

On the other hand,

$$
\|f\|^p_{BMOA} \leq C(1-|a'|)^{-np} \leq C\delta^{-np}. \tag{3.4}
$$

Therefore, from (1.3), (3.3) and (3.4) there exists a constant $C' = C(n,p)$ such that

$$
\mu(B_\delta(\xi)) \leq C'\delta^n,
$$

this shows that μ is a Carleson measure on B since μ is finite.

Now we consider the case $|\alpha| > 0$. Assume that μ satisfies (1.4) and let $f \in BMOA$. By Lemma 2 and the elementary inequality

$$
(a+b)^p \leq 2^p(a^p + b^p), \quad 0 < p < \infty,\ a > 0,\ b > 0,
$$

we have

$$
\begin{aligned}
&\int_B \left| \frac{\partial^{|\alpha|} f(z)}{\partial z^\alpha} - \frac{\partial^{|\alpha|} f(a)}{\partial z^\alpha} \right|^p \frac{(1-|a|^2)^{n+\frac{3|\alpha|p}{2}}(1-|z|^2)^{\frac{|\alpha|p}{2}}}{|1-\langle z,a\rangle|^{2n+2|\alpha|p}}\, d\mu(z) \\
&\leq C\|f\|^p_{BMOA} \int_B \frac{(1-|a|^2)^{n+\frac{3|\alpha|p}{2}}(1-|z|^2)^{-\frac{|\alpha|p}{2}}}{|1-\langle z,a\rangle|^{2n+2|\alpha|p}}\, d\mu(z) + \\
&+ C\|f\|^p_{BMOA} \int_B \frac{(1-|a|^2)^{n+\frac{|\alpha|p}{2}}(1-|z|^2)^{\frac{|\alpha|p}{2}}}{|1-\langle z,a\rangle|^{2n+2|\alpha|p}}\, d\mu(z) \\
&\leq C\|f\|^p_{BMOA} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+\frac{|\alpha|p}{2}} d\nu(z),
\end{aligned} \tag{3.5}
$$

where $d\nu(z) = (1-|z|^2)^{-\frac{|\alpha|p}{2}}\, d\mu(z)$. For a fixed $r(0<r<1)$ and $a \in B$ with $1-|a| < 2\left(1+\sqrt{\frac{2}{1-r}}\right)^{-2}$, we set $\xi = a/|a|$ and $\delta = (1-|a|)\left(1+\sqrt{\frac{2}{1-r}}\right)^2$. By (1.4), (2.1) and Lemma 1 we have

$$\nu(E(a,r)) = \int_{E(a,r)} (1-|z|^2)^{-\frac{|\alpha|p}{2}}\, d\mu(z) \le C(1-|a|)^{n+\frac{|\alpha|p}{2}}\,. \tag{3.6}$$

Since μ is finite, we see that (3.6) holds for all $a \in B$. Using Lemma 3 for the case $|\alpha| > 0$ and $0 < p < \infty$ we obtain

$$\sup_{a\in B} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+\frac{|\alpha|p}{2}} d\nu(z) < \infty\,. \tag{3.7}$$

Therefore, from the estimates (3.5) and (3.7) we get (1.3).

Conversely, suppose that (1.3) holds for all $f \in BMOA$. Let $a \in B$ with $|a| > 191/192$ and take $a' = (32|a|-31)a/|a|$, then $Q_a(a') = 0$ and $|\varphi_a(a')| > 176/197$. Given $r, 0 < r < 1/33$, then $a' \notin E(a,r)$. By Lemma 1 and (2.2) for $z \in E(a,r)$ we have

$$|1-\langle z,a'\rangle| \le (|1-\langle z,a\rangle|^{\frac{1}{2}} + |1-\langle a,a'\rangle|^{\frac{1}{2}})^2 \le \frac{825}{16}(1-|a|), \tag{3.8}$$

and

$$\frac{176}{3}(1-|a|) \le 1-|a'|^2 \le 64(1-|a|). \tag{3.9}$$

Combining (3.8) with (3.9) we have

$$|1-\langle z,a'\rangle| \le \left(\frac{15}{16}\right)^2 (1-|a'|^2), \quad z \in E(a,r). \tag{3.10}$$

For fixed a' above we take $f(z) = (1-\langle z,a'\rangle)^{-n}$. It is easy to see that $f \in BMOA$ and for any positive integer $|\alpha|$ we have

$$\frac{\partial^{|\alpha|} f(z)}{\partial z^\alpha} = n(n+1)\cdots(n+|\alpha|-1)\overline{a'}^{|\alpha|}(1-\langle z,a'\rangle)^{-n-|\alpha|}\,. \tag{3.11}$$

From (2.1), (3.9), (3.10) and (3.11) we get

$$
\begin{aligned}
&\sup_{a\in B}\int_B \left|\frac{\partial^{|\alpha|} f(z)}{\partial z^\alpha}-\frac{\partial^{|\alpha|} f(a)}{\partial z^\alpha}\right|^p \frac{(1-|a|^2)^{n+\frac{3|\alpha|p}{2}}(1-|z|^2)^{\frac{|\alpha|p}{2}}}{|1-\langle z,a\rangle|^{2n+2|\alpha|p}}\, d\mu(z)\\
&\quad\geq \int_B \left|\frac{\partial^{|\alpha|} f(z)}{\partial z^\alpha}-\frac{\partial^{|\alpha|} f(a')}{\partial z^\alpha}\right|^p \frac{(1-|a'|^2)^{n+\frac{3|\alpha|p}{2}}(1-|z|^2)^{\frac{|\alpha|p}{2}}}{|1-\langle z,a'\rangle|^{2n+2|\alpha|p}}\, d\mu(z)\\
&\quad\geq C(n,|\alpha|,p)\int_{E(a,r)} \left(|1-\langle z,a'\rangle|^{-n-|\alpha|}-(1-|a'|^2)^{-n-|\alpha|}\right)^p \times\\
&\quad\times \frac{(1-|a'|^2)^{n+\frac{3|\alpha|p}{2}}(1-|z|^2)^{\frac{|\alpha|p}{2}}}{|1-\langle z,a'\rangle|^{2n+2|\alpha|p}}\, d\mu(z)\\
&\quad\geq C(1-|a'|^2)^{-np-|\alpha|p}\int_{E(a,r)} \frac{(1-|a'|^2)^{n+\frac{3|\alpha|p}{2}}(1-|z|^2)^{\frac{|\alpha|p}{2}}}{|1-\langle z,a'\rangle|^{2n+2|\alpha|p}}\, d\mu(z)\\
&\quad\geq C(1-|a|)^{-n-np-|\alpha|p}\mu(E(a,r)).
\end{aligned}
\tag{3.12}
$$

Also, we have

$$
\|f\|^p_{BMOA}\leq C(1-|a|)^{-np}. \tag{3.13}
$$

Hence, from the estimates (3.12), (3.13) above and (1.3), we obtain

$$
\mu(E(a,r))\leq C'(1-|a|)^{n+|\alpha|p} \tag{3.14}
$$

for $\frac{191}{192}<|a|<1$. In fact, we can get that (3.14) is fulfilled for all $a\in B$ since μ is finite. Since $|\alpha|>0$ and $0<p<\infty$, then by Lemma 3 we know that (3.14) implies

$$
K=\sup_{a\in B}\int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+|\alpha|p} d\mu(z)<\infty\,. \tag{3.15}
$$

For each $\xi\in S$ and each $\delta(0<\delta\leq 2)$, we set $a=(1-\frac{\delta}{2})\xi$. For $z\in B_\delta(\xi)$, we have

$$
|1-\langle z,a\rangle|\leq (|1-\langle z,\xi\rangle|^{\frac12}+|1-\langle \xi,a\rangle|^{\frac12})^2\leq 4\delta.
$$

This implies that

$$
\begin{aligned}
K&\geq \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+|\alpha|p} d\mu(z)\\
&\geq \int_{B_\delta(\xi)} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+|\alpha|p} d\mu(z)\\
&\geq C(p,n,|\alpha|)\delta^{-n-|\alpha|p}\mu(B_\delta(\xi)).
\end{aligned}
\tag{3.16}
$$

Therefore, from (3.15) and (3.16), we have

$$\mu(B_\delta(\xi)) \leq C\delta^{n+|\alpha|p}$$

for all $\xi \in S$ and all $0 < \delta \leq 2$. Thus the proof of Theorem 1 is complete. □

From the second part of the proof of Theorem 1, we can get the following result:

Theorem 2. *Let μ be a finite positive measure on B, $0 < r < 1$ and $\alpha > n$. Then the following statements are equivalent:*

(i) $\mu(B_\delta(\xi)) \leq C\delta^\alpha$ *for all* $\xi \in S$ *and all* $0 < \delta \leq 2$;
(ii) $\mu(E(a,r)) \leq C(1-|a|)^\alpha$ *for all* $a \in B$.

Notice that (i) implies (ii), but the converse fails if $\alpha = n$ (see [Lu] for case $n = 1$), that is, the Carleson region $B_\delta(\xi)$ cannot be replaced by the pseudohyperbolic ball $E(a,r)$ for case $n = \alpha$.

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