On the completeness of localic groups

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Abstract. The main purpose of this paper is to show that any localic group is complete in its two-sided uniformity, settling a problem open since work began in this area a decade ago. In addition, a number of other results are established, providing in particular a new functor from topological to localic groups and an alternative characterization of LT-groups.

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When the remarkable Closed Subgroup Theorem for localic groups, saying that any localic subgroup of a localic group is closed, made its first appearance (Isbell et al. [7]), it immediately raised the question whether a localic group must be complete in one or the other of its uniformities. That this need not be the case for the one-sided uniformities was established early on by Isbell [6] who observed that the classical example of a topological group not complete in its left uniformity, namely the automorphism group of the closed unit interval with the topology of uniform convergence (Bourbaki [3, Chapter X, § 3, Exercise 16]), was actually a localic group. This left the case of the two-sided uniformities, and the aim of this paper is to settle this by proving that any localic group is two-sidedly complete.

Specifically, we first show that the multiplication of a localic group L lifts to the completion of L relative to any of its group uniformities (Proposition 1) from which we then derive, by means of the Closed Subgroup Theorem, that L is two-sidedly complete (Proposition 2). In addition, we provide a criterion for L to be one-sidedly complete which is the localic counterpart of a familiar result on the one-sided completability of a topological group (Proposition 3). Finally, we use the techniques involved in the proof of Proposition 1 to obtain a contravariant functor from topological to localic groups which is then shown to give rise to the two-sided completion of a topological group (Proposition 4), to provide a duality between the category of two-sidedly complete topological groups and a category of certain explicitly described localic groups (Proposition 5), and to lead to a variant of the characterization of LT-groups by Kříž [9] (Proposition 6).

1. Background on uniform frames

For general facts concerning frames we refer to Johnstone [8] or Vickers [10].

Recall that a *uniformity* on a frame L is a set $\mathcal{U}$ of covers of L such that

(1) $\mathcal{U}$ is a filter relative to refinement of covers;
(2) for any $C \in \mathcal{U}$ there exist $D \in \mathcal{U}$ such that

$$\{Ds \mid s \in D\} \text{ for } Ds = \bigvee\{t \in D \mid t \wedge s \neq 0\}$$

is a refinement of C (notation: $D \leq^* C$);
(3) for any $a \in L$, $a = \bigvee\{x \in L \mid Cx \leq a$ for some $C \in \mathcal{U}\}$.

A *uniform frame* is then a frame together with a specified uniformity on it. We write L for a uniform frame, $\mathcal{U}L$ for its uniformity, $x \lhd a$ to mean that $Cx \leq a$ for some $C \in \mathcal{U}L$, and allow notational confusion between L and its underlying frame. Further, a frame homomorphism $h : M \to L$ between uniform frames is called *uniform* whenever $h[C] \in \mathcal{U}L$ for each $C \in \mathcal{U}M$, and a *surjection* provided it is onto for both, the underlying frames and the uniformities.

A uniform frame L is called *complete* if any dense surjection to L is an isomorphism, and a *completion* of L is a dense surjection $h : M \to L$ with complete M. The basic result concerning these notions is that any uniform frame L has a completion $\gamma_L : CL \to L$, unique up to isomorphism, providing a coreflection from uniform to complete uniform frames (Isbell [5]; for a recent account see Banaschewski [1]).

The specific manner in which the completion $\gamma_L : CL \to L$ may be obtained is of no concern here but we need some of the familiar properties of the right adjoint $(\gamma_L)_*$ of γ_L. To begin with, CL is generated by the image of $(\gamma_L)_*$, and $(\gamma_L)_*$ takes uniform covers to covers. Next, in order to describe the remaining property, which will be of particular importance here, we need the following notions. For a uniform frame L and an arbitrary frame T, a *T-valued Cauchy filter* on L is a $0{\wedge}e$-homomorphism $\varphi : L \to T$ (meaning φ preserves the zero 0, binary meet $\wedge$, and the unit e) which

(C) maps uniform covers of L to covers of T;

it is called *regular* whenever it satisfies the additional condition

(R) $\varphi(a) = \bigvee\{\varphi(x) \mid x \lhd a\}$ for all $a \in L$.

If T is left unspecified we refer to *general* (regular) Cauchy filters.

Regarding this terminology, we note that the **2**-valued (regular) Cauchy filters on L are exactly the characteristic functions of the usual (regular) Cauchy filters of L.

Now we have (Banaschewski [1, Proposition 4]):

For any T-valued regular Cauchy filters $\varphi : L \to T$ on a uniform frame L, there exists a unique frame homomorphism $\bar{\varphi} : CL \to T$ such that $\bar{\varphi}(\gamma_L)_* = \varphi$.

One might add that $(\gamma_L)_*$ itself is a CL-valued regular Cauchy filter on L and consequently this result identifies it as the *universal* general regular Cauchy filter on L.

We now derive a result from this which will be the major tool used in Section 3.

A frame homomorphism $h : M \to L$ between uniform frames is called a *Cauchy homomorphism* if there exists a frame homomorphism $\tilde{h} : CM \to CL$, necessarily unique, such that $h\gamma_M = \gamma_L\tilde{h}$, and we refer to $\tilde{h}$ as the *lift* of h to the completions. Note that, by the basic properties of completions mentioned earlier, any uniform homomorphism is a Cauchy homomorphism, but in our setting we are specifically concerned with Cauchy homomorphisms which are not uniform; for this, the following criterion will be useful.

Lemma 1. *A frame homomorphism $h : M \to L$ between uniform frames is a Cauchy homomorphism iff $(\gamma_L)_*h$ maps uniform covers to covers.*

Proof: For the non-trivial "if" part we use the fact that, for any T-valued Cauchy filter $\varphi : M \to T$ on a uniform frame M, the map of $\varphi^0 : M \to T$ such that

$$\varphi^0(a) = \bigvee\{\varphi(x) \mid x \lhd a\}$$

is a T-valued *regular* Cauchy filter on M (Banaschewski-Hong-Pultr [2]). Now, the hypothesis on h implies that $(\gamma_L)_*h$ is a CL-valued Cauchy filter: It is automatically a $0\wedge e$-homomorphism because γ_L is dense. As a result, $((\gamma_L)_*h)^0$ is a CL-valued regular Cauchy filter, as just stated, and by the properties of completions we therefore have a frame homomorphism $\tilde{h} : CM \to CL$ such that

$$\tilde{h}(\gamma_M)_*(a) = \bigvee\{(\gamma_L)_*h(x) \mid x \lhd a\}.$$

Consequently,

$$\gamma_L\tilde{h}((\gamma_M)_*(a)) = \bigvee\{h(x) \mid x \lhd a\} = h(a) = h\gamma_M((\gamma_M)_*(a))$$

for all $a \in L$, and since the $(\gamma_M)_*(a)$ generate CM this shows $\gamma_L\tilde{h} = h\gamma_M$, as desired. □

We note that this result is actually a simple extension of Remark 2 in Banaschewski-Hong-Pultr [2].

2. Background on localic groups

Here we recall some of the basic notions and results concerning localic groups. We follow the style of Isbell et al. [7], considering localic groups as cogroups in the category of frames, with Zermelo-Fraenkel set theory as foundation. We note that there are sound reasons for treating this subject constructively, in the sense of topos theory, as well as localically but for the time being we are content to deal with it in this less ambitious manner.

A *localic group*, then, is a frame L together with frame homomorphisms

$$\mu : L \to L \oplus L, \quad \iota : L \to L, \quad \varepsilon : L \to \mathbf{2},$$

its multiplication, inversion, and unit, respectively, subject to the duals of the familiar group laws by which the following diagrams commute:

$$\begin{array}{ccc} L & \xrightarrow{\mu} & L \oplus L \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu\oplus\,\mathrm{id}} \\ L \oplus L & \xrightarrow[\mathrm{id}\,\oplus\mu]{} & L \oplus L \oplus L \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\mu} & L \oplus L \\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\mathrm{id}\,\oplus(\sigma\varepsilon)} \\ L & \xrightarrow[\nabla]{} & L \oplus L \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\mu} & L \oplus L \\ {\scriptstyle\sigma\varepsilon}\downarrow & & \downarrow{\scriptstyle\mathrm{id}\,\oplus\iota} \\ L & \xrightarrow[\nabla]{} & L \oplus L \end{array}$$

(σ the initial $\mathbf{2} \to L$, $\nabla : L \oplus L \to L$ the codiagonal) together with the left-sided counterparts of the second and third diagram, with $(\sigma\varepsilon) \oplus \mathrm{id}$ and $\iota \oplus \mathrm{id}$, respectively, as the maps on the right.

In the following $L, M, \ldots$ will stand for localic groups, and their operations will be denoted $\mu_L, \iota_L, \varepsilon_L, \mu_M, \iota_M, \varepsilon_M, \ldots$, the index to be suppressed whenever convenient. Further, we permit notational confusion between a localic group and its underlying frame.

A *homomorphism* $h : M \to L$ of localic groups is a homomorphism of the underlying frames compatible with the operations of M and L, that is, such that

$$(h \oplus h)\mu_M = \mu_L h, \quad h\iota_M = \iota_L h, \quad \varepsilon_M = \varepsilon_L h,$$

and **LocG** will be the resulting category.

One of the fundamental facts concerning homomorphisms $h : M \to L$ of localic groups is that the usual dense-onto decomposition

$$M \xrightarrow{\nu} \uparrow s \xrightarrow{\bar{h}} L, \quad \uparrow s = \{a \in M \mid a \geq s\}, \quad s = \bigvee h^{-1}\{0\}$$
$$\nu(a) = a \vee s, \quad \bar{h}(a) = a$$

at the level of underlying frames is actually a decomposition in **LocG**, that is, the operations of M induce operations on $\uparrow s$ making it into a localic group such that ν and $\bar{h}$ are localic group homomorphisms. Further, the Closed Subgroup Theorem for localic groups says that $\bar{h}$ is an isomorphism whenever h is onto, or, equivalently, h is an isomorphism whenever it is dense ($s = 0$) and onto.

We now turn to the uniformities of a localic group L. To provide a convenient and suggestive description of these we use the following approach which differs slightly from [7]. One first notes that, as a formal consequence of the group laws, the multiplication $\mu : L \to L \oplus L$ is a twisted version of the first coproduct injection and hence open so that it has a left adjoint $\mu_{\#}$ which is then used to define

$$ab = \mu_{\#}(a \oplus b)$$

for any $a, b \in L$, saying that $ab \leq c$ iff $a \oplus b \leq \mu(c)$ for all $a, b, c \in L$. Concerning this product and the operation $a^{-1} = \iota(a)$, the following rules are easy consequences of the laws concerning μ and ι:

$$(ab)c = a(bc), \quad (ab)^{-1} = b^{-1}a^{-1}, \quad (a^{-1})^{-1} = a,$$
$$\text{if } \ a \leq b \text{ then } ac \leq bc,\ ca \leq cb,\ a^{-1} \leq b^{-1}$$

for any $a, b, c \in L$.

Further, for $N = \{s \in L \mid \varepsilon(s) = 1\}$, the neighbourhood filter of the unit of L, one has:

For any $s \in N$ there exist $t \in N$ such that $t^2 \leq s$,

for any $a \in L$ and $s \in N$, $a \leq as$ and $a \leq sa$,

for any $s \in N$, $s^{-1} \in N$, and

for any $a \neq 0$ in L, aa^{-1} and $a^{-1}a$ belong to N.

Here, the first condition is proved in [7], the second is an easy consequence of the laws concerning μ and ε, and the third results from the fact that $\varepsilon\iota = \varepsilon$, a formal consequence of the group laws. Finally, since $a \oplus b \leq \mu(ab)$, we have

$$a = \nabla(a \oplus a) = \nabla(\iota \oplus \mathrm{id})(a^{-1} \oplus a) \leq \nabla(\iota \oplus \mathrm{id})\mu(a^{-1}a) = \sigma\varepsilon(a^{-1}a)$$

for any $a \in L$, showing that $\varepsilon(a^{-1}a) = 1$ whenever $a \neq 0$, and the other identity for ι implies that $\varepsilon(aa^{-1}) = 1$.

Now, one proves that the sets

$$C_s = \{a \in L \mid a^{-1}a \leq s\} \qquad (s \in N)$$

are covers of L such that $C_s \subseteq C_t$ whenever $s \leq t$, $C_t \leq^* C_s$ if $t^2 \leq s$, and, for any $a \in L$,

$$a = \bigvee\{x \in L \mid C_s x \leq a \ \text{ for some } \ s \in N\}.$$

It follows that the C_s, $s \in N$, form a basis of a uniformity on L, called the *left uniformity* of L. Analogously, one shows that the sets

$$D_s = \{a \in L \mid aa^{-1} \leq s\} \qquad (s \in N)$$

and

$$T_s = \{a \in L \mid (a^{-1}a) \vee (aa^{-1}) \leq s\}$$

form bases of uniformities, called the *right* and the *two-sided* uniformity of L, respectively.

Note that in the commutative case, where $\lambda\mu = \mu$ for the automorphism $\lambda : L \oplus L \to L \oplus L$ interchanging the two coproduct maps $L \to L \oplus L$, the three uniformities of L coincide because $ab = ba$. In general, $\iota[C_s] = D_{s^{-1}}$ and hence ι is a uniform isomorphism from L with its left uniformity to L with its right uniformity.

To put the above definitions in perspective, we note that, for a topological group G, the familiar operations on the frame $\mathcal{O}G$ of open sets of G derived from the group operations,

$$UV = \{\alpha\beta \mid \alpha \in U, \beta \in V\}, \qquad U^{-1} = \{\alpha^{-1} \mid \alpha \in U\},$$

satisfy the same conditions as the above operations for a localic group L, and the usual uniformities of G, given in terms of open covers of G, may be described in exactly the same way as the above uniformities of L.

3. Completeness

We begin with a description of the right adjoint of the coproduct of two frame homomorphisms. This may well be known but since we have no reference for it we include its short proof.

Lemma 2. *For any frame homomorphisms $f : K \to M$ and $g : L \to N$, the right adjoint of $f \oplus g : K \oplus L \to M \oplus N$ is given by*

$$(f \oplus g)_*(c) = \bigvee\{f_*(a) \oplus g_*(b) \mid a \oplus b \leq c\}.$$

Proof: For any $c \in M \oplus N$, $x \in K$, and $y \in L$, if $x \oplus y \leq (f \oplus g)_*(c)$ then $f(x) \oplus g(y) \leq c$, and since $x \oplus y \leq f_*f(x) \oplus g_*g(y)$ it follows that

$$(f \oplus g)_*(c) \leq \bigvee\{f_*(a) \oplus g_*(b) \mid a \oplus b \leq c\}.$$

The reverse inequality is obvious: act $f \oplus g$ on the join on the right. □

The next result provides the crucial step in the entire development.

Lemma 3. *For any localic group L, the multiplication $\mu : L \to L \oplus L$ is a Cauchy homomorphism for each of the group uniformities of L.*

Proof: Let $\gamma_L : CL \to L$ be any of the completions involved. Then the completion of $L \oplus L$ is $CL \oplus CL$, and hence it will be enough by Section 1 to show that

$$\varphi = (\gamma_L \oplus \gamma_L)_*\mu : L \to CL \oplus CL$$

takes uniform covers to covers for the corresponding uniformity. For this, note that

$$\varphi(c) = \bigvee\{(\gamma_L)_*(a) \oplus (\gamma_L)_*(b) \mid ab \leq c\}$$

for any $c \in L$ by Lemma 2 and the definition of ab.

We first deal with the left uniformity, showing that $\varphi[C_s]$ is a cover for each $s \in N$. For this, take $u \in N$ such that $u^3 \leq s$ and consider

$$(*) \qquad \{(\gamma_L)_*(a) \oplus (\gamma_L)_*(b) \mid a^{-1}a \leq bub^{-1},\ b^{-1}b \leq u\}$$

which is a cover of $CL \oplus CL$ since $bub^{-1} \geq bb^{-1}$ belongs to N and

$$\{(\gamma_L)_*(a) \mid a^{-1}a \leq bub^{-1}\} \text{ (any fixed } b\text{) and } \{(\gamma_L)_*(b) \mid b^{-1}b \leq u\}$$

are covers of CL by the properties of completions. Further, for any a and b involved here,

$$(ab)^{-1}ab = b^{-1}a^{-1}ab \leq b^{-1}bub^{1-}b \leq u^3 \leq s$$

showing that $ab \in C_s$. Finally $(\gamma_L)_*(a) \oplus (\gamma_L)_*(b) \leq \varphi(ab)$ trivially, hence the cover $(*)$ is a refinement of $\varphi[C_s]$, and consequently the latter is a cover.

Of course, the case of the right uniformity is perfectly analogous, with the obvious left-right interchange. For the slightly more subtle two-sided case, we proceed as follows. Consider the two sets

$$\{(\gamma_L)_*(a) \oplus (\gamma_L)_*(b) \mid (a^{-1}a) \vee (aa^{-1}) \leq bub^{-1},\ (b^{-1}b) \vee (bb^{-1}) \leq u\}$$

and

$$\{(\gamma_L)_*(a) \oplus (\gamma_L)_*(b) \mid (b^{-1}b) \vee (bb^{-1}) \leq a^{-1}ua,\ (a^{-1}a) \vee (aa^{-1}) \leq u\}$$

where $u \in N$ such that $u^3 \leq s$ as before. Now, the same argument as above shows that either of these is a cover of $CL \oplus CL$, and for the a and b involved we have, respectively,

$$(ab)^{-1}ab = b^{-1}a^{-1}ab \leq b^{-1}bub^{-1}b \leq u^3 \leq s$$

and

$$ab(ab)^{-1} = abb^{-1}a^{-1} \leq aa^{-1}uaa^{-1} \leq u^3 \leq s.$$

Then, taking the elementwise meet of these two covers we obtain a cover by elements $(\gamma_L)_*(a) \oplus (\gamma_L)_*(b)$ where $((ab)^{-1}ab) \vee (ab(ab)^{-1}) \leq s$ showing that $ab \in T_s$. On the other hand $(\gamma_L)_*(a) \oplus (\gamma_L)_*(b) \leq \varphi(ab)$ as before, and hence $\varphi[T_s]$ is a cover. □

Remark. Given that $(\gamma_L)_* : L \to CL$ is the universal regular Cauchy filter on L for the uniformity in question (Section 1), the corresponding $(\gamma_L \oplus \gamma_L)_* : L \oplus L \to CL \oplus CL$ describes the *universal pair* of general regular Cauchy filters on L, and hence the composite $(\gamma_L \oplus \gamma_L)_* \mu$ considered above represents the product of the universal pair of general regular Cauchy filters on L. Hence the above proof that this is a general Cauchy filter amounts to showing, at the appropriate level of generality, that *any product of regular Cauchy filters is a Cauchy filter*. This is what Raikov proved for the classical Cauchy filters of a topological group (see Bourbaki [3, Chapter III, § 3.4]), and the arguments given above are really nothing else but an adaptation of Raikov's to the pointfree case. It should be added that the appropriate localic proof of this lemma makes this point particularly evident in that it is literally a direct translation of the classical proof.

Our first result now is

Proposition 1. *For any of the uniform completions $\gamma_L : CL \to L$ of a localic group L, CL is a localic monoid with multiplication $\tilde{\mu} : CL \to CL \oplus CL$ and unit $\tilde{\varepsilon} = \varepsilon\gamma_L$ such that γ_L is a monoid homomorphism.*

Proof: The monoid identities for $\tilde{\mu}$ and $\tilde{\varepsilon}$ follow immediately from the corresponding identities for μ and ε and the fact that, in the relevant diagrams such as

$$\begin{array}{ccccccc}
 & & & \tilde{\mu} & & & \\
CL & \xrightarrow{\gamma L} & L & \xrightarrow{\mu} & L \oplus L & \xleftarrow{\gamma L \oplus \gamma L} & CL \oplus CL \\
\tilde{\mu} \downarrow & & \mu \downarrow & & \downarrow \mu \oplus \mathrm{id} & & \downarrow \tilde{\mu} \oplus \mathrm{id} \\
CL \oplus CL & \xrightarrow{\gamma L \oplus \gamma L} & L \oplus L & \xrightarrow{\mathrm{id} \oplus \mu} & L \oplus L \oplus L & \xleftarrow{\gamma L \oplus \gamma L \oplus \gamma L} & CL \oplus CL \oplus CL \\
 & & & \mathrm{id} \oplus \tilde{\mu} & & &
\end{array},$$

all squares commute while $\gamma_L \oplus \gamma_L \oplus \gamma_L$ is dense and hence monic because all frames involved are regular. □

It might be worth adding that this result is considerably more obvious in the commutative case because there $\mu : L \to L \oplus L$ is actually uniform and hence trivially a Cauchy homomorphism: the uniformity of $L \oplus L$ is generated by the covers

$$W_s = \{a \oplus b \mid a^{-1}a, b^{-1}b \le s\}$$

and by commutativity $(ab)^{-1}ab \le s^2$ whenever $a^{-1}a, b^{-1}b \le s$; consequently, W_s is a refinement of $\mu[C_t]$ if $s^2 \le t$ since $a \oplus b \le \mu(ab)$.

Next we have our principal result:

Proposition 2. *Any localic group L is complete in its two-sided uniformity.*

Proof: Obviously $\iota[T_s] = T_{s^{-1}}$ for the inversion of ι of L, showing it is uniform and hence a Cauchy homomorphism in the present setting, so that we also have its lift $\tilde{\iota} : CL \to CL$. Now, CL with the operations $\tilde{\mu}$, $\tilde{\iota}$, $\tilde{\varepsilon}$ is a localic group: it is a localic monoid with the operations $\tilde{\mu}$ and $\tilde{\varepsilon}$ by Proposition 1, and $\tilde{\iota}$ satisfies the corresponding inversion laws because in the diagram

$$\begin{array}{ccccccc} & & & \overset{\tilde{\mu}}{\overbrace{\hspace{8cm}}} & & & \\ CL & \underset{\gamma L}{\longrightarrow} & L & \underset{\mu}{\rightarrow} & L \oplus L & \underset{\gamma L \oplus \gamma L}{\longleftarrow} & CL \oplus CL \\ \sigma\tilde{\varepsilon} \downarrow & & \sigma\varepsilon \downarrow & & \downarrow \mathrm{id} \oplus \iota & & \downarrow \mathrm{id} \oplus \tilde{\iota} \\ CL & \underset{\gamma L}{\longrightarrow} & L & \underset{\nabla}{\longrightarrow} & L \oplus L & \underset{\gamma L \oplus \gamma L}{\longleftarrow} & CL \oplus CL \\ & & & \underset{\nabla}{\underbrace{\hspace{8cm}}} & & & \end{array}$$

and its left-sided counterpart all squares commute and $\gamma_L \oplus \gamma_L$ is dense and hence monic. It follows that $\gamma_L : CL \to L$ is a dense onto homomorphism of localic groups and therefore an isomorphism. □

Remark. The proofs of the above propositions could alternatively be based on the general principle, for finitary algebras in an arbitrary category with products, that an algebra B will satisfy all identities satisfied by an algebra A whenever there exists a homomorphism $A \to B$ inducing epimorphisms for all finite powers of the corresponding underlying objects. However, in view of a later argument in connection with Proposition 4, the ad hoc proofs given here are preferable.

Finally, we have the following characterization of one-sided completeness:

Proposition 3. *The following are equivalent for a localic group L:*

(1) *L is complete in its left uniformity;*
(2) *for the left completion $\gamma_L : CL \to L$, $(\gamma_L)_* \iota_L$ maps left uniform covers to covers;*
(3) *for the left completion $\gamma_L : CL \to L$, $(\gamma_L)_*$ maps right uniform covers to covers.*

Proof: Since $\iota_L[C_s] = D_{s^{-1}}$ for any $s \in N$ it is clear that (2) $\Leftrightarrow$ (3), and (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Here ι_L is a Cauchy homomorphism for the left uniformity and the resulting lift $\tilde{\iota}_L : CL \to CL$ provides the inversion for the localic monoid structure of CL given by Proposition 1, making it into a localic group and hence γ_L an isomorphism by the Closed Subgroup Theorem. □

We note that Proposition 3 is the exact counterpart of the familiar result that a topological group has a left completion iff its inversion takes left Cauchy filters to left Cauchy filters (Bourbaki [3, III, § 3.4, Theorem 1]), or, equivalently, every left Cauchy filter is a right Cauchy filter. Again, we observe that the localic version of the above proof amounts to a direct translation of the classical argument for topological groups.

4. The functor from topological to localic groups

There is an obvious contravariant functor Σ from localic to topological groups, the enriched version of the usual spectrum functor from frames to spaces, for which ΣL is the topological group with underlying space the frame spectrum of L and with multiplication $\Sigma\mu$, inversion $\Sigma\iota$, and unit ε. We now use some of the ideas of the last section to obtain a functor in the opposite direction.

For any topological group G (always assumed to be *Hausdorff*) we consider $\mathcal{O}G$ as a uniform frame with the two-sided uniformity, that it, with basic uniform covers

$$\mathcal{T}_S = \{U \in \mathcal{O}G \mid U^{-1}U \cup UU^{-1} \subseteq S\}, \quad S \in \mathcal{N},$$

where $\mathcal{N}$ is the filter of open neighbourhoods of the unit of G, and let $\tau_G : C\mathcal{O}G \to \mathcal{O}G$ be the corresponding completion. Then, the composite $\delta_G : C\mathcal{O}G \oplus C\mathcal{O}G \to \mathcal{O}(G \times G)$ of $\tau_G \oplus \tau_G : C\mathcal{O}G \oplus C\mathcal{O}G \to \mathcal{O}G \oplus \mathcal{O}G$ with the standard homomorphism $\pi_G : \mathcal{O}G \oplus \mathcal{O}G \to \mathcal{O}(G \times G)$ taking $U \oplus V$ to $U \times V$ is clearly a dense surjection for the coproduct uniformity on $C\mathcal{O}G \oplus C\mathcal{O}G$ and the uniformity of $\mathcal{O}(G \times G)$ given by taking $G \times G$ as product of uniform spaces, making it the completion of $\mathcal{O}(G \times G)$.

Now we have, for the multiplication map $m : G \times G \to G$:

Lemma 4. *The frame homomorphism $\mathcal{O}m : \mathcal{O}G \to \mathcal{O}(G \times G)$ taking U to $m^{-1}[U]$ is a Cauchy homomorphism.*

Proof: We show that $\varphi = (\delta_G)_* \mathcal{O}m$ is a general Cauchy filter. Now, for any $U \in \mathcal{O}G$,

$$\begin{aligned}
\varphi(U) &= (\tau_G \oplus \tau_G)_*(\pi_G)_*(m^{-1}[U]) \\
&= \bigvee\{(\tau_G)_*(V) \oplus (\tau_G)_*(W) \mid V \oplus W \subseteq (\pi_G)_*(m^{-1}[U])\} \\
&= \bigvee\{(\tau_G)_*(V) \oplus (\tau_G)_*(W) \mid V \times W \subseteq m^{-1}[U]\} \\
&= \bigvee\{(\tau_G)(V) \oplus (\tau_G)(W) \mid VW \subseteq U\}
\end{aligned}$$

and hence in particular $(\tau_G)_*(V) \oplus (\tau_G)_*(W) \leq \varphi(VW)$ for any $V, W \in \mathcal{O}G$. It follows from this that we can use formally the same argument as in the proof of Lemma 3 to obtain, for any $S \in \mathcal{N}$, a cover of $C\mathcal{O}G \oplus C\mathcal{O}G$ consisting of elements $(\tau_G)_*(V) \oplus (\tau_G)_*(W)$ where $(VW)^{-1}VW \cup VW(VW)^{-1} \subseteq S$, showing that $VW \in \mathcal{F}_S$ and hence that $\varphi[\mathcal{T}_S]$ is a cover. □

It follows that $C\mathcal{O}G$ can be equipped with localic operations such that μ is the lift of $\mathcal{O}m$ by Lemma 4, ι that of $\mathcal{O}i$ for the uniformly continuous inversion i of G, and $\varepsilon(a) = 1$ iff $\tau_G(a) \in \mathcal{N}$. Furthermore, acting $\mathcal{O}$ on the group identities of G one obtains corresponding commuting diagrams for $\mathcal{O}m$, $\mathcal{O}i$, and the map $\mathcal{O}G \to \mathbf{2}$ given by the unit of G, and from these, in turn, one derives that μ, ι, and ε are localic group operations by the same kind of argument used in the proofs of Propositions 1 and 2.

Next, for any homomorphism $\varphi : G \to H$ of topological groups, the commuting diagrams expressing the compatibility of φ with the operations of G and H yield diagrams, again by acting $\mathcal{O}$, from which one readily derives that the lift $\tilde{\varphi} : C\mathcal{O}H \to C\mathcal{O}G$ of $\mathcal{O}\varphi$, given by the fact that $\mathcal{O}\varphi$ is uniform, is a localic group homomorphism. Finally, the correspondence $\varphi \mapsto \tilde{\varphi}$ clearly preserves composition and units. In all, this has shown the first part of

Proposition 4. *Completion of the frame $\mathcal{O}G$ of open sets of a topological group G with respect to the two-sided uniformity determines a faithful contravariant functor $C\mathcal{O}$ from* **TopG** *to* **LocG** *such that $\Sigma C\mathcal{O}$ is the reflection to the category* **2CTopG** *of two-sidedly complete topological groups, with reflection maps $\rho_G : G \to \Sigma C\mathcal{O}G$ taking $\alpha \in G$ to $\hat{\alpha}\tau_G$ where $\hat{\alpha}(U) = 1$ iff $\alpha \in U$.*

Proof: By basic facts concerning uniform spaces and the completion of their uniform frames of open sets, ρ_G is the embedding of G into its completion for the two-sided uniformity on G and the uniformity on $\Sigma C\mathcal{O}G$ which is induced by the uniformity of $C\mathcal{O}G$ as the completion of $\mathcal{O}G$. Further, ρ_G is natural in G and hence $C\mathcal{O}$ is faithful: if $C\mathcal{O}\varphi = C\mathcal{O}\psi$ for any $\varphi, \psi : G \to H$ then also $\rho_H\varphi = \rho_H\psi$ and consequently $\varphi = \psi$.

On the other hand, the way the operations of $C\mathcal{O}G$ are related to those of G makes it immediately obvious that ρ_G is also a group homomorphism and hence a dense embedding of topological groups. Furthermore, $\Sigma C\mathcal{O}G$ is two-sidedly complete. To see this we first show that, for any $S \in \mathcal{N}$,

$$(*) \qquad \left\{\rho_G^{-1}[\Sigma_a] \mid \Sigma_a^{-1}\Sigma_a \cup \Sigma_a\Sigma_a^{-1} \subseteq \Sigma_{(\tau_G)_*(S)}\right\} \subseteq \mathcal{T}_S,$$

where Σ_a is the open subset of $\Sigma C\mathcal{O}G$ consisting of all ξ such that $\xi(a) = 1$. For any $a \in C\mathcal{O}G$ as in $(*)$; if $\alpha, \beta \in \rho_G^{-1}[\Sigma_a]$ then

$$\rho_G(\alpha^{-1}\beta) = \rho_G(\alpha)^{-1}\rho_G(\beta) \in \Sigma_a^{-1}\Sigma_a \subseteq \Sigma_{(\tau_G)_*(S)}$$

so that $\alpha^{-1}\beta \in \rho_G^{-1}[\Sigma_{(\tau_G)_*(S)}]$, and the same for $\alpha\beta^{-1}$. Furthermore,

$$\rho_G^{-1}[\Sigma_{(\tau_G)_*(S)}] = \{\gamma \in G \mid \hat{\gamma}\tau_G(\tau_G)_*(S) = 1\} = \{\gamma \in G \mid \hat{\gamma}(S) = 1\} = S$$

and therefore $\rho_G^{-1}[\Sigma_a] \in \mathcal{T}_S$, as claimed. Next, $(*)$ implies that the cover

$$\left\{\Sigma_a \mid \Sigma_a^{-1}\Sigma_a \cup \Sigma_a\Sigma_a^{-1} \subseteq \Sigma_{(\tau_G)_*(S)}\right\}$$

itself is a refinement of $\{U^\# \mid U \in \mathcal{F}_S\}$ where $U^\#$ is the largest open $W \subseteq \Sigma C\mathcal{O}G$ such that $\rho_G^{-1}[W] = U$. Now, the latter covers generate the uniformity of the two-sided completion of G, and hence $\Sigma C\mathcal{O}G$ is also complete in its finer two-sided uniformity.

In all, this shows that the functor $\Sigma C\mathcal{O}$ takes topological groups to two-sidedly complete topological groups such that the dense embedding $\rho_G : G \to \Sigma C\mathcal{O}G$, natural in G, is an isomorphism iff G is two-sidedly complete. □

Remark. In the commutative case, the crucial Lemma 4 is trivial because then the multiplication is uniformly continuous and $\mathcal{O}m$ obviously lifts to the completions. This case was already considered in [7].

Proposition 5. *$C\mathcal{O}$ is a full dual embedding of* **2CTopG** *into* **LocG**, *providing a dual equivalence with the category of localic groups L for which the spatial reflection $\eta_L : L \to \mathcal{O}\Sigma L$ is the completion relative to the two-sided uniformity of $\mathcal{O}\Sigma L$.*

Proof: We first show that $C\mathcal{O}$ is full. Given any localic group homomorphism $h : C\mathcal{O}G \to C\mathcal{O}H$ for two-sidedly complete topological groups G and H, we have to find a homomorphism $\varphi : H \to G$ such that $C\mathcal{O}\varphi = h$. Now, since ρ_G is an isomorphism by hypothesis, an obvious candidate is $\varphi = \rho_G^{-1}\Sigma h\rho_H$. In order to see this is indeed the right choice, we first establish a couple of auxiliary results.

(i) For any topological group G, $\tau_G = (\mathcal{O}\rho_G)\eta_{C\mathcal{O}G}$: for any $U \in \mathcal{O}G$,

$$\begin{aligned}(\mathcal{O}\rho_G)\eta_{C\mathcal{O}G}((\tau_G)_*(U)) = \mathcal{O}\rho_G(\Sigma_{(\tau_G)_*(U)}) = \rho_G^{-1}[\Sigma_{(\tau_G)_*(U)}] = U \\ = \tau_G((\tau_G)_*(U)),\end{aligned}$$

the third step by the proof of Proposition 4, and since the $(\tau_G)_*(U)$ generate $C\mathcal{O}G$ this proves the claim.

(ii) For any two-sidedly complete topological group G, $\eta_{C\mathcal{O}G}C\mathcal{O}\rho_G = \tau_{\Sigma C\mathcal{O}G}$: by (i) and the definition of τ,

$$(\mathcal{O}\rho_G)\eta_{C\mathcal{O}G}\, C\mathcal{O}\rho_G = \tau_G\, C\mathcal{O}\rho_G = (\mathcal{O}\rho_G)\tau_{\Sigma C\mathcal{O}G},$$

and $\mathcal{O}\rho_G$ can be cancelled since it is an isomorphism.

Now, by the definition of η and τ and by (ii),

$$\begin{aligned}\eta_{C\mathcal{O}H}\, h\, C\mathcal{O}\rho_G = \mathcal{O}\Sigma h\, \eta_{C\mathcal{O}G}\, C\mathcal{O}\rho_G = \mathcal{O}\Sigma h\, \tau_{\Sigma C\mathcal{O}G} \\ = \tau_{\Sigma C\mathcal{O}H}\, C\mathcal{O}\Sigma h = \eta_{C\mathcal{O}H}\, C\mathcal{O}\rho_H\, C\mathcal{O}\Sigma h,\end{aligned}$$

and $\eta_{\mathcal{C}\mathcal{O}H}$ can be cancelled since it is dense by (i). Finally, for the $\varphi : H \to G$ suggested above,

$$C\mathcal{O}\varphi = C\mathcal{O}\rho_H C\,\mathcal{O}\Sigma h\, C\mathcal{O}\rho_G^{-1} = h,$$

as desired.

Regarding the image of $C\mathcal{O}$, it is clear that $L \simeq C\mathcal{O}\Sigma L$ for any L of the type described, and the converse follows from (i). □

It should be pointed out that there is an open question concerning the above characterization of the image of the functor $C\mathcal{O}$. We do not know whether the weaker assumption that $\eta_L : L \to \mathcal{O}\Sigma L$ is *merely dense* is sufficient to ensure its equivalence with the two-sided completion $\tau_{\Sigma L} : C\mathcal{O}\Sigma L \to \mathcal{O}\Sigma L$ of $\mathcal{O}\Sigma L$: if dense η_L is trivially the completion relative to the uniformity it induces on $\mathcal{O}\Sigma L$ from the two-sided uniformity of L, but how that relates to the two-sided uniformity of $\mathcal{O}\Sigma L$ itself is another question. The latter is determined by the covers

$$(1) \qquad \{\Sigma_a \mid \Sigma_a^{-1}\Sigma_a \cup \Sigma_a\Sigma_a^{-1} \subseteq \Sigma_s\} \qquad (s \in N)$$

whereas the covers generating the former are

$$(2) \qquad \{\Sigma_a \mid a^{-1}a \vee aa^{-1} \leq s\} \qquad (s \in N).$$

It is easy to see that (2) is contained in, and hence a refinement of, (1) for each $s \in N$, and consequently there exists a homomorphism $h : C\mathcal{O}\Sigma L \to L$ such that $\eta_L h = \tau_{\Sigma L}$, but it is by no means clear why this should be an isomorphism. We note that in [7] it is argued, in the commutative case considered there, that this is indeed so, based on the implicit claim that the two uniformities of $\mathcal{O}\Sigma L$ in question are the same, but no explanation is given, and we have so far been unable to verify this. On the other hand, equality of these uniformities is not really required here: for h to be an isomorphism it is sufficient that $(\tau_{\Sigma L})_*\eta_L$ is a general Cauchy filter — but that seems just as elusive.

Another open question in connection with Propositions 4 and 5 concerns the precise relation between the functors Σ and $C\mathcal{O}$. Given that, for frames, Σ and $\mathcal{O}$ are adjoint on the right to each other, one might wonder whether this carries over to the present situation for Σ and $C\mathcal{O}$, but the following considerations show this is not the case — at least not if one expects the $\rho_G : G \to \Sigma C\mathcal{O}G$ to provide one of the adjunctions. Suppose this to be so and let $\lambda_L : L \to C\mathcal{O}\Sigma L$ be the other adjunction maps. Then we have the identity $C\mathcal{O}\rho_G\,\lambda_{C\mathcal{O}G} = \mathrm{id}_{C\mathcal{O}G}$, and as $C\mathcal{O}$ is full by Proposition 5 there exists a topological group homomorphism $h : \Sigma C\mathcal{O}G \to G$ such that $\lambda_{C\mathcal{O}G} = C\mathcal{O}h$. It follows from this that $C\mathcal{O}(h\rho_G) = \mathrm{id}_{C\mathcal{O}G}$, implying that $h\rho_G = \mathrm{id}_G$ because $C\mathcal{O}$ is faithful, and this makes ρ_G an isomorphism since it is dense, contradicting the existence of topological groups which are not two-sidedly complete.

In conclusion, we show how the present setting furnishes a short proof of a variant of a result of Kříž [9]. For this, recall that a topological group G is called an LT-group whenever $\mathcal{O}m : \mathcal{O}G \to \mathcal{O}(G \times G)$ factors through $\pi_G : \mathcal{O}G \oplus \mathcal{O}G \to \mathcal{O}(G \times G)$, thereby making $\mathcal{O}G$ into a localic group with the resulting map $\mu_0 : \mathcal{O}G \to \mathcal{O}G \oplus \mathcal{O}G$ as multiplication and the obvious inversion and unit. Then we have the following characterization.

Proposition 6. *A topological group G is an LT-group iff the frame $\mathcal{O}G$ is complete in its two-sided uniformity.*

Proof: ($\Rightarrow$) Since the outer square and the right hand triangle commute in the diagram

$$\begin{array}{ccccc} C\mathcal{O}G & \xrightarrow{\tau_G} & \mathcal{O}G & \xrightarrow{\quad \mathcal{O}m \quad} & \searrow \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu_0} & & \\ C\mathcal{O}G \oplus C\mathcal{O}G & \xrightarrow[\tau_G \oplus \tau_G]{} & \mathcal{O}G \oplus \mathcal{O}G & \xrightarrow[\pi_G]{} & \mathcal{O}(G \times G) \end{array}$$

and π_G is dense, τ_G is a homomorphism of localic groups and hence an isomorphism by the Closed Subgroup Theorem.

($\Leftarrow$) Trivially, $\pi_G\,(\tau_G \oplus \tau_G)\mu\tau_G^{-1} = \mathcal{O}m$. □

We note that, in general, for any uniform space X, the corresponding uniform frame $\mathcal{O}X$ is complete iff X is supercomplete in the terminology of Isbell [4], and consequently *a topological group is an LT-group iff it is supercomplete in its two-sided uniformity.*

It should be added that the corresponding result of Kříž [9, Theorem 4.3.3], is formally weaker but a closer analysis of the relation between our $C\mathcal{O}G$ and the object considered there shows it is actually equivalent to the above proposition. We omit the details.

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