Initially κ-compact spaces for large κ

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Abstract. This work presents some cardinal inequalities in which appears the closed pseudo-character, ψ_c, of a space.

Using one of them — $\psi_c(X) \leq 2^{d(X)}$ for T_2 spaces — we improve, from T_3 to T_2 spaces, the well-known result that initially κ-compact T_3 spaces are λ-bounded for all cardinals λ such that $2^\lambda \leq \kappa$.

And then, using an idea of A. Dow, we prove that initially κ-compact T_2 spaces are in fact compact for $\kappa = 2^{F(X)}$, $2^{s(X)}$, $2^{t(X)}$, $2^{\chi(X)}$, $2^{\psi_c(X)}$ or $\kappa = \max\{\tau^+, \tau^{<\tau}\}$, where $\tau > t(p, X)$ for all $p \in X$.

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1. Introduction

If X is an initially κ-compact space and κ is sufficiently large with respect to other cardinal numbers associated with X (e.g. $\kappa = |X|$, $\omega(X)$ or $L(X)$), then X is in fact compact.

A. Dow [D85] asked if there is some T_2 first-countable, initially ω_1-compact, non-compact space; and showed that under CH the answer is no. Moreover the same holds in Cohen Models ([D89]) and under PFA ([BDFN]).

But, P. Koszmider [Koz] showed that it is consistent with any cardinal arithmetic consistent with $\neg$ CH, that there is a normal, first-countable, initially ω_1-compact, non-compact space.

We do not know if a similar result may hold for larger cardinals. But, from Koszmider's example X and Corollary 3.3 it follows that it is also consistent that there is a T_2, first countable, separable, initially ω_1-compact, non-compact space Y: since, from Corollary 3.3, X cannot be ω-bounded, just take $Y = \overline{A}$, where $A \subseteq X$ is such that $|A| = \omega$ and $\overline{A}$ is not compact.

Here, following the ideas of [D85], it is shown that initially κ-compact T_2 spaces are compact for $\kappa = 2^{s(X)}$, $2^{F(X)}$, $2^{t(X)}$, $2^{\chi(X)}$, $2^{\psi_c(X)}$ or $\kappa = \max\{\tau^+, \tau^{<\tau}\}$, where $\tau > t(p, X)$ for all $p \in X$.

For this purpose we improved, from T_3 to T_2 spaces, the result that initially κ-compact T_3 spaces are λ-bounded for all cardinals λ such that $2^\lambda \leq \kappa$. From

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this result it also follows that every subspace of density λ such that $2^\lambda \leq \kappa$ of an initially κ-compact T_2 space is completely regular.

The improvement was obtained using a new bound for the closed pseudocharacter ψ_c of a T_2 space X: $\psi_c(X) \leq 2^{d(X)}$.

2. Some cardinal inequalities

Here we present two bounds for the closed pseudocharacter $\psi_c(X)$ of a space X and two bounds for $|X|$ using $\psi_c(X)$.

We recall the definition of ψ_c, as in [Ju80]: for a T_2 space $\langle X, \tau_X \rangle$ we define, for each $p \in X$,

$$\psi_c(p, X) = \min\{|\mathcal{V}| : \mathcal{V} \subseteq \tau_X,\ p \in \bigcap \mathcal{V},\ \bigcap\{\overline{V} : V \in \mathcal{V}\} = \{p\}\},$$

and $\psi_c(X) = \sup\{\psi_c(p, X) : p \in X\} + \omega$.

In [St], the closed pseudocharacter ψ_c is called the H-pseudocharacter and it is proved there (Theorem 3.4) that every initially κ-compact T_2 space of H-pseudocharacter κ is a regular space of character κ (this result will be used in Proposition 3.1 and Lemma 3.1).

The other cardinal functions are of more common usage and are as in [Ju80] or [Ho].

Proposition 2.1. (i) *For a T_2 space X, $\psi_c(X) \leq 2^{d(X)}$.*
(ii) *For a Urysohn space X, $\psi_c(X) \leq 2^{s(X)}$.*

Proof: (i) For each $p \in X$, let $\mathcal{V}_p$ be the family of all open neighborhoods of p and let $D \subseteq X$ be a dense subspace of X such that $|D| \leq d(X)$.

Let $\mathcal{C} = \{V \cap D : V \in \mathcal{V}_p\}$, and for each $C \in \mathcal{C}$ let $V_c \in \mathcal{V}_p$ be such that $C = V_c \cap D$.

Let $\mathcal{V} = \{V_c : C \in \mathcal{C}\}$. Then $\mathcal{V} \subseteq \tau_X$, $p \in \bigcap \mathcal{V}$ and $\bigcap\{\overline{V} : V \in \mathcal{V}\} = \bigcap\{\overline{V}_c : C \in \mathcal{C}\} = \bigcap\{\overline{V_c \cap D} : C \in \mathcal{C}\} = \bigcap\{\overline{C} : C \in \mathcal{C}\} = \bigcap\{\overline{V \cap D} : V \in \mathcal{V}_p\} = \bigcap\{\overline{V} : V \in \mathcal{V}_p\} = \{p\}$.

Hence, $\psi_c(p, X) \leq |\mathcal{V}| \leq |\mathcal{C}| \leq |\mathcal{P}(D)| \leq 2^{d(X)}$.

(ii) Let $p \in X$. For each $q \in X \setminus \{p\} = Y$ let U_q and V_q be open neighborhoods of p and q respectively such that $\overline{U}_q \cap \overline{V}_q = \emptyset$.

$\mathcal{V} = \{V_q : q \in Y\}$ is an open cover of Y. Applying to the space Y, with the open cover $\mathcal{V}$, Šapirovskiĭ's result (Proposition 4.8 of [Ho]), we get $A, B \subseteq Y$ such that $|A| \leq s(X)$, $|B| \leq s(X)$ and $Y = \overline{A} \cup \cup\{V_q : q \in B\}$.

Let $\mathcal{C} = \{C \subseteq A : \emptyset \neq C = V_q \cap A \text{ for some } q \in Y\}$; and for each $C \in \mathcal{C}$ let $q_c \in Y$ be such that $C = V_{q_c} \cap A$.

Let $\mathcal{U} = \{U_{q_c} : C \in \mathcal{C}\} \cup \{U_q : q \in B\}$. Then $\mathcal{U} \subseteq \tau_X$, $p \in \bigcap \mathcal{U}$ and for $y \in X \setminus \{p\} = Y$ we have:

- if $y \in \overline{A}$, then $C = V_y \cap A \neq \emptyset$ and hence $C \in \mathcal{C}$. So, $y \in \overline{V_y \cap A} = \overline{C} = \overline{V_{q_c} \cap A} \subseteq \overline{V}_{q_c}$ and therefore $y \notin \overline{U}_{q_c}$;
- if $y \in \cup\{V_q : q \in B\}$, then $y \in V_q$ for some $q \in B$; and hence $y \notin \overline{U}_q$.

In either case, $y \notin \bigcap\{\overline{U} : U \in \mathcal{U}\}$ and hence $\{p\} \subseteq \bigcap\{\overline{U} : U \in \mathcal{U}\} \subseteq \{p\}$; i.e. $\bigcap\{\overline{U} : U \in \mathcal{U}\} = \{p\}$. Therefore, $\psi_c(p, X) \leq |\mathcal{U}| \leq |\mathcal{C}| + |B| \leq 2^{s(X)} + s(X) = 2^{s(X)}$. □

Proposition 2.2. *For a T_2 space X,*

(i) $|X| \leq 2^{d(X)\psi_c(X)}$;
(ii) $|X| \leq d(X)^{t(X)\psi_c(X)}$.

Proof: Both results follow immediately form Lemma 4.3 of [Ho], which may be stated as: let κ be an infinite cardinal and let X be a T_2 space such that $\psi_c(X) \leq \kappa$ and there is a subset S of X such that $X = \cup\{\overline{A} : A \subseteq S, |A| \leq \kappa\}$. Then $|X| \leq |S|^\kappa$.

For the first inequality, let $\kappa = d(X)\psi_c(X)$ and $S \subseteq X$ a dense subspace with $|S| \leq d(X)$. Then,

$$|X| \leq |S|^\kappa \leq [d(X)]^{d(X)\psi_c(X)} = 2^{d(X)\psi_c(X)}.$$

For the second, let $\kappa = t(X)\psi_c(X)$ and $S \subseteq X$ dense with $|S| \leq d(X)$. Then $|X| \leq |S|^\kappa \leq d(X)^{t(X)\psi_c(X)}$. □

Remarks.

1. From these results some well-known inequalities follow:
   (i) For T_2 spaces X,
      – from 2.1(i) and 2.2(i), follows $|X| \leq 2^{2^{d(X)}}$;
      – since $t(X)\psi_c(X) \leq \chi(X)$, from 2.2(ii), follows $|X| \leq d(X)^{\chi(X)}$;
      – since $\psi_c(X) \leq L(X)\psi(X)$ (2.8(c) of [Ju80]), from 2.2(ii) follows $|X| \leq d(X)^{L(X)t(X)\psi(X)}$.
   (ii) For T_3 spaces X, since $\psi_c(X) = \psi(X)$, from 2.2(i) and 2.2(ii), it follows that $|X| \leq 2^{d(X)\psi(X)}$ and $|X| \leq d(X)^{t(X)\psi(X)}$.
2. In 1.0 of [Ju84] a T_3 space X is given such that $d(X)^{\psi(X)} < |X|$ and, consequently, $d(X)^{\psi_c(X)} < |X|$. This shows that 2(i) and 2(ii) cannot be strengthened to $|X| \leq d(X)^{\psi_c(X)}$.
3. In Example 7.1 of [Ju84], for each cardinal κ a T_2 space X is given such that $d(X) = \kappa$, $|X| = s(X) = \exp_2(\kappa)$ and $\chi(X) = w(X) = \exp_3(\kappa)$; where $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

   Then $\psi_c(X) \leq 2^{d(X)} \leq 2^\kappa$; $\exp_2(\kappa) = |X| \leq 2^{d(X)\psi_c(X)} \leq 2^{\kappa.2^\kappa} = \exp_2(\kappa)$ and hence $|X| = 2^{d(X)\psi_c(X)}$. But $2^{s(X)\psi(X)} = \exp_3(\kappa) > |X|$ and $2^{c(X)\chi(X)} = 2^{L(X)\chi(X)} = \exp_4(\kappa) > |X|$.

   This shows that 2.2(i) might give a more accurate bound for $|X|$, than the three traditional inequalities above.

3. Initially κ-compact spaces

We recall some definitions. Let $\kappa \geq \omega$ be a cardinal; a space X is called:

- *initially κ-compact* iff every open cover of X of size $\leq \kappa$ has a finite subcover;
- *κ-bounded* iff for every $A \subseteq X$ with $|A| \leq \kappa$ there is $Y \subseteq X$, Y compact such that $A \subseteq Y$ (for X T_2, this is equivalent to $\overline{A}$ being compact);
- *$< \kappa$-bounded* iff X is λ-bounded for every cardinal $\lambda < \kappa$.

Proposition 3.1. *Let $\kappa \geq \omega$ be a cardinal and let X be an initially κ-compact T_2 space. Then X is λ-bounded for every cardinal λ such that $2^\lambda \leq \kappa$.*

Proof: Let λ be a cardinal such that $2^\lambda \leq \kappa$, let $A \subseteq X$ with $|A| = \lambda$ and let $Y = \overline{A}$. Then Y, being closed in X, is also initially κ-compact ([St, Theorem 3.1]). $A \subseteq Y$ being dense in Y, gives $d(Y) \leq |A| = \lambda$; hence, from Proposition 2.1(i), $\psi_c(Y) \leq 2^{d(Y)} \leq 2^\lambda \leq \kappa$. Now, from Theorem 3.4 of [St], it follows that Y is a regular space of character $\psi_c(Y) \leq \kappa$. Y being regular, we may use the well-known inequality (3.3(b) of [Ho]) $w(Y) \leq 2^{d(Y)} \leq \kappa$.

Given an open cover of Y, there is a subcover of it of size $\leq w(Y) \leq \kappa$; and (since Y is initially κ-compact) there is a finite subcover of it. Hence Y is compact and X is λ-bounded. □

The next result uses an idea from Theorem 2 of [D85].

Proposition 3.2. *Let $\kappa > \omega$ be a cardinal and let X be a T_2 $< \kappa$-bounded, non-compact space. Then X has a free sequence of length κ (i.e. $F(X) \geq \kappa$).*

Proof: Let $\mathcal{U}$ be an open cover of X which does not have a finite subcover. Let $\lambda < \kappa$ be a cardinal. Since X is λ-bounded, it follows that X is initially λ-compact and hence $\mathcal{U}$ does not have a subcover of size λ; i.e. there is no $\mathcal{V} \subseteq \mathcal{U}$ with $|\mathcal{V}| < \kappa$ covering X.

We define by transfinite recursion on $\alpha < \kappa$, a sequence $\langle x_\alpha : \alpha < \kappa \rangle$ of points of X and an increasing sequence $\langle U_\alpha : \alpha < \kappa \rangle$ of open subsets of X such that for every $\alpha < \kappa$,

(i) U_α is a union of $\leq \max\{|\alpha|, \omega\}$ elements of $\mathcal{U}$;
(ii) $x_\alpha \notin U_\alpha$;
(iii) $\overline{\{x_\gamma : \gamma < \alpha\}} \subseteq U_\alpha$.

We start with any $U_0 \in \mathcal{U}$ and, since $U_0 \neq X$, we may choose some $x_0 \in X \setminus U_0$.

Let $0 < \alpha < \kappa$ and suppose that x_γ, U_γ have been already chosen for every $\gamma < \alpha$ satisfying the three conditions above. Let $A_\alpha = \{x_\gamma : \gamma < \alpha\}$. $|A_\alpha| \leq |\alpha| < \kappa$, and so $\overline{A}_\alpha$ is compact. Hence there is $\mathcal{V}_\alpha \subseteq \mathcal{U}$ finite such that $\overline{A}_\alpha \subseteq \bigcup \mathcal{V}_\alpha$. Let $U_\alpha = \cup\{U_\gamma : \gamma < \alpha\} \cup (\cup \mathcal{V}_\alpha)$. Since each U_γ is a union of $\leq \max\{|\gamma|, \omega\} \leq \max\{|\alpha|, \omega\}$ elements of $\mathcal{U}$, it follows that U_α is a union of $\leq \max\{|\alpha|, \omega\}.|\alpha| + \omega = \max\{|\alpha|, \omega\}$ elements of $\mathcal{U}$. (iii) also holds since $\overline{\{x_\gamma : \gamma < \alpha\}} = \overline{A}_\alpha \subseteq \cup \mathcal{V}_\alpha \subseteq U_\alpha$. And finally, since $\max\{|\alpha|, \omega\} < \kappa$, $U_\alpha \neq X$ and we may choose some $x_\alpha \in X \setminus U_\alpha$.

We claim that the sequence $\langle x_\alpha : \alpha < \kappa\rangle$ is a free sequence of X: Let $\alpha < \kappa$, then $\overline{\{x_\gamma : \gamma < \alpha\}} \subseteq U_\alpha$ and $\{x_\gamma : \alpha \leq \gamma < \kappa\} \subseteq X \setminus U_\alpha$. Hence $\overline{\{x_\gamma : \alpha \leq \gamma < \kappa\}} \subseteq \overline{X \setminus U_\alpha} = X \setminus U_\alpha$ and therefore

$$\overline{\{x_\gamma : \gamma < \kappa\}} \cap \overline{\{x_\gamma : \alpha \leq \gamma < \kappa\}} = \emptyset.$$

□

In the special case, where $\kappa = \theta^+$, Proposition 3.2 implies that if X is a T_2 θ-bounded, non-compact space, then $F(X) \geq \theta^+$; so we have the immediate:

Corollary 3.1. *Let $\theta \geq \omega$ be a cardinal and let X be a T_2 θ-bounded space with $F(X) \leq \theta$ or $s(X) \leq \theta$. Then X is compact.* □

The space $X = \kappa^+$ for $\kappa \geq \omega$, with the order topology shows that a space X may be κ-bounded with $t(X) = \kappa$ and $\chi(X) = \kappa$ and non-compact. But:

Corollary 3.2. *Let $\kappa \geq \omega$ be a cardinal and let X be a T_2 κ-bounded, initially κ^+-compact space, with $t(X) \leq \kappa$. Then X is compact.*

Proof: Let X be κ-bounded, initially κ^+-compact, with $t(X) \leq \kappa$ and suppose that X is non-compact. Then X has some free sequence of length κ^+. Since X is initially κ^+-compact, this free sequence has some complete accumulation point $p \in X$; which satisfies $t(p, X) \geq \kappa^+$, against $t(X) \leq \kappa$. □

Lemma 3.1. *Let $\kappa \geq \omega$ be a cardinal and let X be a T_2 initially κ-compact space with $\psi_c(X) \leq \kappa$. Then $t(X) \leq \kappa$.*

Proof: From Theorem 3.4 of [St], $\chi(X) = \psi_c(X) \leq \kappa$; and $t(X) \leq \chi(X)$. □

Corollary 3.3. *Let $\kappa \geq \omega$ be a cardinal and let X be a T_2 κ-bounded, initially κ^+-compact space with $\chi(X) \leq \kappa$ or $\psi_c(X) \leq \kappa$. Then X is compact.* □

Combining these results with Proposition 3.1, we get:

Corollary 3.4. *Let X be an initially κ-compact T_2 space with $\kappa = 2^{F(X)}$, or $\kappa = 2^{s(X)}$, or $\kappa = 2^{t(X)}$, or $\kappa = 2^{\chi(X)}$ or $\kappa = 2^{\psi_c(X)}$. Then X is compact.*

Proof: From Proposition 3.1 it follows that X is λ-bounded for $\lambda = F(X)$, or $\lambda = s(X)$, or ... or $\lambda = \psi_c(X)$; and, since $\lambda^+ \leq 2^\lambda \leq \kappa$ for all these λ's, X is also initially λ^+-compact. Hence, from Corollaries 3.1 to 3.3, it follows that X is compact. □

Remarks.

1. Koszmider's example of a normal, non-compact, initially ω_1-compact, first-countable space, shows that it is not possible to improve in ZFC Corollary 3.4 to $\kappa = t(X)^+$, or $\kappa = \chi(X)^+$ or $\kappa = \psi_c(X)^+$.
2. Also this result does not hold for $\kappa = 2^{c(X)}$, as the following example shows: Let $\kappa = 2^\omega$ and let $X = \{f \in {}^{\kappa^+}2 : |f^{-1}(\{1\})| \leq \kappa\}$. Then X is a

T_2, initially κ-compact, non-initially κ^+-compact space (cf. Example 4.2 of [St]). Also X is dense in $Y = {}^{\kappa^+}2$, hence $c(X) \leq c(Y) = \omega$. Therefore X is a T_2 initially $2^{c(X)}$-compact, non-compact space.

3. For T_1 spaces the conclusion of Corollary 3.4 does not hold for $\kappa = 2^{F(X)}$, $2^{s(X)}$, $2^{t(X)}$, $2^{d(X)}$, as the following example shows: Let $\theta > \omega$ be a regular cardinal and let $X = \theta$ with the cofinite topology refined by the initial segments — i.e. $\emptyset \neq U \subseteq X$ is open iff there exist $\alpha \leq \theta$ and $F \subseteq \alpha$ finite such that $U = \{\xi < \alpha : \xi \notin F\} = \alpha \setminus F$.

   It is easy to see that X is a non-compact T_1 (not T_2) space. Also X is initially κ-compact for every $\kappa < \theta$: given an open cover $\mathcal{U}$ of X with $|\mathcal{U}| = \kappa < \theta$, let, for each $U \in \mathcal{U}$, $\beta_U = \sup U$. If $\beta_U < \theta$ for every $U \in \mathcal{U}$, then $\sup\{\beta_U : U \in \mathcal{U}\} < \theta$ and hence $\cup\mathcal{U} \neq X$. Therefore $\beta_{U_0} = \theta$ for some $U_0 \in \mathcal{U}$. From this it follows that $U_0 = \theta \setminus F = X \setminus F$ for some $F \subseteq \theta$ finite and consequently $\mathcal{U}$ admits some finite subcover.

   And finally $hd(X) = \omega$, from which it follows that $d(X) = s(X) = F(X) = t(X) = \omega$. To see that $hd(X) = \omega$, let $Y \subseteq X$ be infinite. $Y = \{\alpha_\xi : \xi < o.t.(Y)\}$ — with $\omega \leq o.t.(Y)$ and $\alpha_\xi < \alpha_\eta$ for $\xi < \eta < o.t.(Y)$. It is immediate to see that $A = \{\alpha_\xi : \xi < \omega\}$ is a countable dense subspace of Y.

   Hence, for $\theta = (2^\omega)^+$, X is a non-compact, initially 2^ω-compact, T_1 space, with $d(X) = s(X) = F(X) = t(X) = \omega$ (and more generally, for every $\kappa > \omega$, $X = \kappa^+$ is a non-compact, initially κ-compact, T_1 space with $d(X) = s(X) = F(X) = t(X) = \omega$).

   In these spaces $\chi(X) = \psi(X) = \kappa$ if $\theta = \kappa^+$ (and $\chi(X) = \psi(X) = \theta$ if θ is a limit cardinal). Hence X is not initially $2^{\chi(X)}$-compact; and in fact we do not know if Corollary 3.4 holds for T_1 spaces with $\kappa = 2^{\chi(X)}$, or if it holds (for T_1 or T_2 spaces) with $\kappa = 2^{\psi(X)}$ (for T_3 spaces, $\psi(X) = \psi_c(X)$, so it holds).

4. Let $\kappa_0 = \omega$, $\kappa_{n+1} = 2^{\kappa_n}$ for $n < \omega$ and $\kappa = \sup\{\kappa_n : n < \omega\}$; κ is a strong limit cardinal with $cf(\kappa) = \omega$. Let $X = \kappa^+$ with the order topology. Then X is an initially κ-compact, non-compact, T_2 (in fact T_4) space with $\kappa > 2^{t(p,X)}$ for all $p \in X$ (since for $p = \alpha < \kappa^+$, $t(p, X) = cf(\alpha) < \kappa$).

   This example shows that the conclusion of Corollary 3.4 does not hold with $\kappa > 2^{t(p,X)}$ for all $p \in X$ (instead of $\kappa = 2^{t(X)}$). But, we may prove the following:

Proposition 3.3. *Let X be an initially κ-compact T_2 space, with $\kappa = \max\{\tau^+, \tau^{<\tau}\}$, where $\tau > t(p, X)$ for all $p \in X$. Then X is compact.*

PROOF: Since $\kappa \geq \tau^+$, it suffices (from Corollary 3.3) to show that X is τ-bounded.

Let then $S \subseteq X$, $|S| = \tau$ and $Y = \overline{S}$. We have that $Y = \cup\{\overline{A} : A \in [S]^{<\tau}\}$, since, given $p \in Y = \overline{S}$, there exists some $A \subseteq S$ with $|A| \leq t(p, X) < \tau$ such that $p \in \overline{A}$.

Given a cardinal $\lambda < \tau$ we have that $2^\lambda \leq 2^{<\tau} \leq \tau^{<\tau} \leq \kappa$. Hence, from Proposition 3.1, X is λ-bounded and thus $\overline{A}$ is compact for every $A \in [S]^{<\tau}$. Therefore Y is a union of $\leq |[S]^{<\tau}| = \tau^{<\tau} \leq \kappa$ compact subsets of X.

Let now $\mathcal{U}$ be an open cover of Y. For each $A \in [S]^{<\tau}$ let $\mathcal{U}_A \in [\mathcal{U}]^{<\omega}$ be a finite subcover of $\overline{A}$ and let $\mathcal{V} = \cup\{\mathcal{U}_A : A \in [S]^{<\tau}\} \subseteq \mathcal{U}$.

$\mathcal{V}$ is an open cover of Y of cardinality $\leq \omega.\tau^{<\tau} \leq \kappa$. Y is initially κ-compact (since it is a closed subspace of X). Therefore there exists some finite $\mathcal{V}_0 \subseteq \mathcal{V} \subseteq \mathcal{U}$ which covers Y; and Y is compact. □

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