Function spaces in the Stegall class

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Abstract. We prove several stability properties for the class of compact Hausdorff spaces T such that $C(T)$ with the weak or the pointwise topology is in the class of Stegall. In particular, this class is closed under arbitrary products.

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A topological space X is usually said to be of the *Stegall class* (see e.g. [Fa]) if whenever F is a minimal usco mapping from a Baire space Z to X, then there is a dense G_δ subset D of Z such that $|F(z)| = 1$ for every $z \in D$. Everywhere in this paper by a *mapping* we will mean a set-valued mapping, unless it is explicitly said to be single-valued. A mapping $F : Z \to X$ is called *usco*, if it is upper semicontinuous and $F(z)$ is a nonempty compact set for every $z \in Z$. An usco mapping is called *minimal usco* if it is minimal with respect to the graph inclusion among all usco mappings with the same domain. There are some other classes that are close to the so defined Stegall class (see [St], [KO]). For example, the condition that F be minimal usco might be substituted by the condition that F is just a minimal mapping. A mapping $F : Z \to X$ is called *minimal* (following [KO]), if whenever $U \subset Z$ and $V \subset X$ are open subsets such that $F(U) \cap V \neq \emptyset$, then there is a nonempty open subset W of U such that $F(W) \subset V$. This definition is motivated by the fact that an usco mapping is minimal usco iff it is a minimal mapping in the above sense. The requirements for the space Z also may differ. For example, Z can be completely metrizable or Čech-complete (see [KO]). The proofs of the theorems in the paper are adaptable to all these definitions, as they use only the fact that Z is Baire and that $F : Z \to X$ is a minimal mapping. Furthermore, we can weaken the condition that all the images of F are nonempty to the condition that its domain $\operatorname{dom} F := \{z \in Z : F(z) \neq \emptyset\}$ is dense in Z. For concreteness, from now on we use the following

Definition 1. The topological space X is said to be in the class $\mathcal{S}$ ($X \in \mathcal{S}$ for short) if whenever F is a minimal mapping from a Baire space Z to X with dense domain, then there is a dense G_δ subset D of Z such that $|F(z)| \leq 1$ for every $z \in D$.

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We repeat that with the requirement that F be a minimal *usco* (which is possibly more popular), all the proofs are still true, subject to some natural changes. Anyway, we prefer Definition 1 in order to show that the role of the *usco* requirement is not crucial.

In what follows, if T is a Hausdorff compact space (a compact, for short), $C(T)$ denotes the Banach space of continuous functions on it, supplied with the maximum norm, p denotes the topology of pointwise convergence on $C(T)$, and w denotes the weak topology on it. Given a set A, by $|A|$ we mean the cardinality of A. Given a real number a, by $|a|$ we mean its absolute value.

It is seen directly that, for example, every metrizable space is in the class of Stegall. A wider class of spaces appertaining to $\mathcal{S}$ is that of fragmentable spaces. The notion of fragmentability was introduced by Jayne and Rogers in [JR]. A topological space X is said to be *fragmented* by a metric ρ defined on X if for every $\varepsilon > 0$, every nonempty subset of X has a nonempty relatively open subset of ρ-diameter less than ε. X is called *fragmentable* if it is fragmented by some metric ρ on X. It is known and in fact easy to prove that every fragmentable space is in the class $\mathcal{S}$ (the inverse is false at least under some set-theoretical assumptions, by a recent result of O. Kalenda, cited in [Fa, p. 100]). Several stability properties of the class of compacts T for which $(C(T), p)$ or $(C(T), w)$ is fragmentable were proved in [K]. Some of these results are related to the papers of A. Bouziad [B2], and some unpublished results of W. Moors and N. Ribarska, concerning the notions of co-Namioka and sigma-fragmentability. Here we consider the corresponding properties for the class $\mathcal{S}$, and we add some other ones. The proofs are given only for the pointwise topology; the results regarding the weak topology are obtained by obvious changes.

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Let Z be a topological space and C be some subset of Z. We consider the Banach-Mazur game (Z, C), played by two players HP and UP, taking nonempty open subsets of Z. Put $U_0 = Z$. On the n-th move, $n \geq 1$, the player HP takes an open subset $H_n \subset U_{n-1}$ and UP answers by taking a nonempty open subset U_n of H_n. Using this way of selection, the players get a sequence $(H_n, U_n)_{n=1}^{\infty}$ which is called a *play*. The player UP is said to *have won* this play if $\bigcap_{n\geq 1} U_n \subset C$. A *partial play* is a finite (possibly empty) sequence which consists of the first several moves of a play, ending either with a move of HP or of UP. A *strategy* ζ for the player UP is a mapping which assigns to each partial play $(H_1, U_1, H_2, U_2, \ldots, H_n)$ some nonempty open subset U_n of H_n. A ζ-play is a play in which UP selects his moves according to ζ. The strategy ζ is said to be a *winning* one if every ζ-play is won by UP.

Theorem 1 (Banach-Mazur) (see [Ox]). *The player UP has a winning strategy in (Z, C) iff C is residual in Z.*

Now let $F : Z \to X$ be a minimal mapping. We denote by $BM(F)$ the Banach-

Mazur game (Z, C), in which $C = \{z \in Z : |F(z)| \leq 1\}$.

Corollary 1. *$X \in \mathcal{S}$ iff for any Baire space Z and every minimal mapping $F : Z \to X$ with dense domain, there exists a winning strategy for UP in $BM(F)$.*

We note that the definition of the class $\mathcal{S}$ here is somewhat more general than the definition of $\mathcal{S}$ usually adopted (in that we consider minimal set-valued mappings, that need not be *usco* and, furthermore, need not have nonempty images everywhere, but only in a dense subset of Z).

Lemma 1. *Let T be a compact and B be the closed unit ball of $C(T)$ with the pointwise topology. Then $(C(T), p) \in \mathcal{S}$ iff $B \in \mathcal{S}$.*

Proof: If $(C(T), p) \in \mathcal{S}$, then its subspace $B \in \mathcal{S}$. Inversely, if $B \in \mathcal{S}$, then $(C(T), p)$ is a countable union of the closed subspaces $nB \in \mathcal{S}$, $n \geq 1$, so $(C(T), p) \in \mathcal{S}$ (under the usually adopted definition this can be found e.g. in [Fa, Theorem 3.1.5]). Anyway, we give a proof based on Corollary 1 since our definition differs. Let Z be a Baire space and $F : Z \to (C(T), p)$ be a minimal mapping with dense domain. We will construct a winning strategy for UP in $BM(F)$.

On the n-th move, $n \geq 1$, let H_n be the last move of HP, a nonempty open subset of Z. We check whether $F(H_n) \subset nB$.

Case no. 1. If "yes", since obviously $nB \in \mathcal{S}$, there is a dense set $W_n = \bigcap_{i \geq n} W_n^i$ with W_n^i open in Z such that $|F(z)| \leq 1$ for every $z \in W_n$. Then on the i-th move for all $i \geq n$ the player UP has to play $U_i := H_i \cap W_n^i$, and this enables her to win.

Case no. 2. If "no", then $F(H_n)$ intersects the p-open set $C(T) \backslash nB$, so (by minimality) there is a nonempty open subset U_n of H_n such that $F(U_n) \subset C(T) \backslash nB$. The set U_n is the next move of UP.

As already mentioned, if for some $n \geq 1$ it is Case no. 1 that applies, then UP wins. Suppose then that for every $n \geq 1$ it is Case no. 2 that applies and take some $z \in \bigcap_{n \geq 1} U_n$. Then $F(z) \subset \bigcap_{n \geq 1} C(T) \backslash nB = \emptyset$, so UP wins again.

Remark 1. If the compact K is a continuous image of the compact T, then $(C(K), p)$ is homeomorphic to a subspace of $(C(T), p)$. Hence, if $(C(T), p) \in \mathcal{S}$, then $(C(K), p) \in \mathcal{S}$.

Remark 2. Let $S \subset T$ be two compacts, let B be the unit ball of $C(T)$ with the pointwise topology, B_S be the unit ball of $C(S)$ with the pointwise topology and Z be a Baire space. Let $F : Z \to B$ be a minimal mapping and define the mapping $F' : Z \to B_S$ by $F'(z) := \{f|S : f \in F(z)\}$. Then $F' = i \circ F$, where $i : (C(T), p) \to (C(S), p)$ is defined by $i(f) = f|S$. Thus F' is the composition of a minimal mapping and a continuous single-valued mapping, hence F' is minimal (this fact can be easily checked from the definition).

To prove the result about products of compacts we will need first the following lemma.

Lemma 2. *Let Z be a Baire space, X and Y be compacts, Ω be a nonempty open subset of Z and $\varepsilon > 0$. Let $F : Z \to (C(X \times Y), p)$ be a minimal mapping with dense domain. Assume that UP does not have a winning strategy for $BM(F_{|\Omega})$. Then there is a nonempty open subset U of Ω and a finite sequence $x_1, \dots, x_k \in X$ such that*

$$\min_{1 \le j \le k} \|f(x, \cdot) - f(x_j, \cdot)\|_{C(Y)} \le \varepsilon \quad \forall f \in F(U) \ \ \forall x \in X.$$

Proof: Assume that the conclusion of the lemma is false. For an arbitrary open set $\emptyset \ne H \subset \Omega$ and for an arbitrary finite sequence $x_1, \dots, x_k \in X$ we find an open set $\emptyset \ne \tau(H, x_1, \dots, x_k) \subset H$ and a point $\xi(H, x_1, \dots, x_k) \in X$ as follows. Find $g \in F(H)$ and $x_{k+1} \in X$ such that

$$\|g(x_{k+1}, \cdot) - g(x_j, \cdot)\|_{C(Y)} > \varepsilon \quad \forall j = 1, \dots, k.$$

Find then $y_1, \dots y_k \in Y$ so that

$$|g(x_{k+1}, y_j) - g(x_j, y_j)| > \varepsilon \quad \forall j = 1, \dots, k.$$

From the minimality of F, we find an open set $\emptyset \ne U \subset H$ so that

$$|f(x_{k+1}, y_j) - f(x_j, y_j)| > \varepsilon \quad \forall j = 1, \dots, k \quad \forall f \in F(U).$$

Put then $\tau(H, x_1, \dots, x_k) = U$ and $\xi(H, x_1, \dots, x_k) = x_{k+1}$.

Now we shall define a strategy σ for the player UP in $BM(F_{|\Omega})$. Take some $x_1 \in X$. For the partial play H_1 (where $H_1 \subset \Omega$), put $U_1 = \sigma(H_1) = \tau(H_1, x_1)$. For the partial play $H_1 \supset U_1 \supset H_2$ put $U_2 = \sigma(H_1, U_1, H_2) = \tau(H_2, x_1, x_2)$, where $x_2 = \xi(H_2, x_1)$. When the partial play $H_1 \supset U_1 \supset \dots \supset H_{k-1} \supset U_{k-1} \supset H_k$ is already defined for some $k \ge 3$, put $\sigma(H_1, U_1, \dots, H_{k-1}, U_{k-1}, H_k) = \tau(H_k, x_1, \dots x_k)$, where $x_k = \xi(H_k, x_1, \dots, x_{k-1})$.

Since we assume that the player UP does not have a winning strategy in $BM(F_{|\Omega})$, there is a play $(\Omega \supset)\ H_1 \supset U_1 \supset H_2 \supset U_2 \supset \cdots$, played by UP according to the strategy σ, such that UP does not win. This means that there is some $z \in \bigcap_{k=1}^{\infty} U_k$ so that $|Fz| > 1$. Take $f \in Fz$. Then $\|f(x_k, \cdot) - f(x_j, \cdot)\|_{C(Y)} > \varepsilon$ whenever $k \ne j$ and $k, j \in \mathbb{N}$. From the Stone-Weierstrass theorem we find $u_i \in C(X),\ v_i \in C(Y),\ i = 1, \dots, m$, such that $\left\| f - \sum_{i=1}^m u_i v_i \right\| < \varepsilon/3$. Then

$$\frac{\varepsilon}{3} < \Big\| \sum_{i=1}^m u_i(x_k) v_i - \sum_{i=1}^m u_i(x_j) v_i \Big\|_{C(Y)} \le \sum_{i=1}^m \big| u_i(x_k) - u_i(x_j) \big| . \|v_i\|_{C(Y)}$$

whenever $k, j \in \mathbb{N},\ k \ne j$. This contradicts the continuity of the functions $u_i(X),\ i = 1, \dots, m$ at the accumulation points of the sequence $(x_k)_{k \ge 1}$. □

Theorem 2. *Let X, Y be compacts such that $(C(X), p) \in \mathcal{S}$ and $(C(Y), p) \in \mathcal{S}$. Then $(C(X \times Y), p) \in \mathcal{S}$.*

PROOF: Let Z be some Baire space and consider a minimal mapping $F : Z \to \big(C(X \times Y), p\big)$, with dense domain. We shall show how UP can win in $BM(F)$ no matter how HP moves.

Let H_1 be the first move of HP. If there exists an open set $\emptyset \neq \Omega \subset H_1$ so that UP has a winning strategy in $BM(F_{|\Omega})$, then we are done. Assume that the opposite occurs, that is, UP does not have a winning strategy in $BM(F_{|\Omega})$ for any open set $\emptyset \neq \Omega \subset H_1$.

By Lemma 2, there are an open set $\emptyset \neq \Omega' \subset H_1$ and a finite set $x^1_1, \dots x^1_{k_1} \in X$ such that

$$\min_{1 \le j \le k_1} \big\| f(x, \cdot) - f(x^1_j, \cdot) \big\|_{C(Y)} < 1 \quad \forall\, f \in F(\Omega') \quad \forall\, x \in X.$$

By Lemma 2 again (if we swap the coordinates), there are open $\emptyset \neq \Omega \subset \Omega'$ and a finite set $y^1_1, \dots y^1_{k'_1} \in Y$ such that

$$\min_{1 \le j \le k'_1} \big\| f(\cdot, y) - f(\cdot, y^1_j) \big\|_{C(X)} < 1 \quad \forall\, f \in F(\Omega) \quad \forall\, y \in Y.$$

Obviously, we may arrange the things in such a way that $k'_1 = k_1$. As $(C(X), p) \in \mathcal{S}$, $(C(Y), p) \in \mathcal{S}$, there are open dense $W^1_i \subset Z$, $i \in \mathbb{N}$, such that $|F(z)(\cdot, y^1_j)| \le 1$, $|F(z)(x^1_j, \cdot)| \le 1$, $j = 1, \dots, k_n$ for every $z \in \bigcap_{i=1}^{\infty} W^1_i$. Put then $U_1 = \Omega \cap W^1_1$.

On the n-th move, $n \ge 2$, assume that for every $m = 1, 2, \dots, n-1$ UP constructed $k_m \in \mathbb{N}$, $x^m_j \in X$, $y^m_j \in Y$, $1 \le j \le k_m$, $U_m \subset H_m$, and $W^m_i \subset Z$, $i \in \mathbb{N}$. Let H_n be HP's answer to U_{n-1}. By Lemma 2, there are an open set $\emptyset \neq \Omega' \subset H_n$ and a finite set $x^n_1, \dots x^n_{k_n} \in X$ such that

$$\min_{1 \le j \le k_n} \big\| f(x, \cdot) - f(x^n_j, \cdot) \big\|_{C(Y)} < \frac{1}{n} \quad \forall\, f \in F(\Omega') \quad \forall\, x \in X.$$

Again, by Lemma 2, there are open $\emptyset \neq \Omega \subset \Omega'$ and a finite set $y^n_1, \dots y^n_{k'_n} \in Y$ such that

$$\min_{1 \le j \le k'_n} \big\| f(\cdot, y) - f(\cdot, y^n_j) \big\|_{C(X)} < \frac{1}{n} \quad \forall\, f \in F(\Omega) \quad \forall\, y \in Y.$$

Obviously, we may arrange the things in such a way that $k'_n = k_n$. As $(C(X), p) \in \mathcal{S}$, $(C(Y), p) \in \mathcal{S}$, there are open dense $W^n_i \subset Z$, $i \in \mathbb{N}$, such that $|F(z)(\cdot, y^n_j)| \le 1$, $|F(z)(x^n_j, \cdot)| \le 1$, $j = 1, \dots, k_n$ for every $z \in \bigcap_{i=1}^{\infty} W^n_i$. Put then $U_n = \Omega \cap \bigcap_{1 \le i, m \le n} W^m_i$. In this way, we described how UP should play at each step.

It remains to show that UP wins when she uses the strategy described above. Take $z \in \bigcap_{n=1}^{\infty} U_n$, if any, and take $f, g \in Fz$, if any. Fix arbitrary $x \in X$, $y \in Y$. For every $n \in \mathbb{N}$ we find $j \le k_n$ and $i \le k_n$ so that

$$\left\|f(x,\cdot) - f(x_j^n,\cdot)\right\|_{C(X)} < \frac{1}{n}, \quad \left\|g(\cdot,y) - g(\cdot,y_i^n)\right\|_{C(Y)} < \frac{1}{n}.$$

Since $z \in \bigcap_{k\ge 1} W_k^n$, we have $f(\cdot,y_i^n) \equiv g(\cdot,y_i^n)$ and $f(x_j^n,\cdot) \equiv g(x_j^n,\cdot)$. Then

$$|f(x,y) - g(x,y)| \le |f(x,y) - f(x,y_i^n)| + |g(x,y_i^n) - g(x_j^n,y_i^n)| \\ + |f(x_j^n,y_i^n) - f(x_j^n,y)| + |g(x_j^n,y) - g(x,y)| \le \frac{4}{n},$$

so $f \equiv g$. Hence $|F(z)| \le 1$ and the strategy for UP is a winning one. This proves the theorem. □

Theorem 3. *Let $\{T_i : i \in I\}$ be a family of compacts (where I is some index set) and let $T = \prod_{i\in I} T_i$. Then $(C(T),p) \in \mathcal{S}$ iff $(C(T_i),p) \in \mathcal{S}$ for all $i \in I$.*

Proof: The "only if" part is trivial (consider the continuous projections of the product onto each factor and use Remark 1). We prove the "if" part. Fix a point $a = (a(i))_{i\in I} \in T$. If $K \subset I$ and $x = (x(i))_{i\in I} \in T$, define $p_K(x) = y = (y(i))_{i\in I}$, where $y(i) = x(i)$ for $i \in K$ and $y(i) = a(i)$ otherwise; put then $T_K = \{p_K(x) : x \in T\}$. Note that T_K (as a subspace of T) is homeomorphic to $\prod_{i\in K} T_i$. Let B be the unit ball of $C(T)$ with the topology p. For any nonempty finite $J \subset I$, let B_J be the unit ball of $C(T_J)$ with the topology p. Let $F : Z \to B$ be a minimal mapping with dense domain. By Corollary 1 and Lemma 1, it suffices to construct a winning strategy ζ for UP in $BM(F)$. Let $F_J : Z \to B_J$ be the mapping defined by $F_J(z) := \{f|T_J : f \in F(z)\}$. According to the last theorem, the space $(C(\prod_{i\in J} T_i),p) \in \mathcal{S}$, and hence $B_J \in \mathcal{S}$. We denote by π_K the canonical projection of T onto $\prod_{i\in K} T_i$, and by D the set of points in T which differ from a only in finitely many coordinates; D is dense in T.

While defining the strategy ζ we construct an increasing sequence $\{J(n) : n \ge 0\}$ of finite subsets of I. We put $J(0) = \emptyset$ and now describe the n-th move. Let the finite set $J(n-1)$ be already defined. Let the last move of the player HP be H_n (open in Z). Put

$$s_n := \sup\{|f(x) - f(y)| : \ f \in F(H_n),\ x,y \in D,\ p_{J(n-1)}(x) = p_{J(n-1)}(y)\}\ (\le 2).$$

Take $f_n \in F(H_n)$ and $x_n, y_n \in D$ such that $|f_n(x_n) - f_n(y_n)| > s_n - 1/n$ and $p_{J(n-1)}(y_n) = p_{J(n-1)}(x_n)$. Define $B_n := \{f \in B : |f(x_n) - f_n(x_n)| < 1/n, |f(y_n) - f_n(y_n)| < 1/n\}$; this is an open subset of B and $f_n \in F(H_n) \cap B_n$. Using the minimality of F, let H_n^1 be a nonempty open subset of H_n with $F(H_n^1) \subset B_n$. Also put $J(n) := J(n-1) \cup J_x^n \cup J_y^n$, where J_x^n [J_y^n] is the (finite) set of indices of the coordinates in which x_n [y_n] differs from a.

As $(C(T_{J(n)}),p) \in \mathcal{S}$, there is a dense G_δ subset W^n of Z such that $|F_{J(n)}(z)| \leq 1$ for all $z \in W^n$. Let $W^n = \bigcap_{k=1}^\infty W_k^n$, with W_k^n open in Z. Put

$$\zeta(H_1, U_1, H_2, U_2, \dots, H_n) = U_n := H_n^1 \cap \bigcap_{m,k=1}^{n} W_k^m;$$

this set is nonempty and open in H_n. The n-th move is defined.

We now prove that ζ is a winning strategy for UP in $BM(F)$. Note that s_n is a non-increasing (in the definition of s_{n+1}, the supremum is taken over smaller sets of functions f and points x, y than that from the definition of s_n). Let $\lim_{n\to\infty} s_n = s_\infty$ and $\bigcup_{n\geq 1} J(n) = J_\infty$.

Case (a). Assume $s_\infty > 0$. Take some $z \in \bigcap_{i\geq 1} U_i$ (if any). Suppose there is some $f \in F(z)$. Let (x_∞, y_∞) be an accumulation point of $\{(x_n, y_n)\}_{n\geq 1}$. For each $q \in I \backslash J_\infty$ and all natural n, we have $x_n(q) = y_n(q) = a(q)$, so $x_\infty(q) = y_\infty(q)$. If $q \in J_\infty$, then there is some n_0 with $q \in J(n)$ for all $n \geq n_0$. But $x_n(q) = y_n(q)$ for $n \geq n_o$, so $x_\infty(q) = y_\infty(q)$. Hence $x_\infty = y_\infty$. Then

$$|f(x_n) - f(y_n)| \geq |f_n(x_n) - f_n(y_n)| - |f(x_n) - f_n(x_n)| - |f(y_n) - f_n(y_n)| > s_n - 3/n,$$

which contradicts the continuity of f at x_∞. This means that $F(z) = \emptyset$ and so ζ is a winning strategy.

Case (b). If $s_\infty = 0$, fix an arbitrary $\varepsilon > 0$. Take some n such that $s_n < \varepsilon/2$. Let $z \in \bigcap_{n\geq 1} U_n$ and $f, g \in F(z)$. Then $f|T_{J(n)}, g|T_{J(n)} \in F_{J(n)}(z)$, $z \in \bigcap_{k\geq 1} W_k^n = W^n$ and by the definition of W^n we have $f|T_{J(n)} = g|T_{J(n)}$. Now if $t \in D$, we have $p_{J(n)}(t) \in D$ too, so

$$|f(t) - g(t)| \leq |f(t) - f(p_{J(n)}(t))| + |g(p_{J(n)}(t)) - g(t)| \leq 2s_n < \varepsilon.$$

D being dense in T, we get $\|f - g\| \leq \varepsilon$. But $\varepsilon > 0$ is arbitrary, hence $f \equiv g$ and so $|F(z)| \leq 1$. Thus ζ is again a winning strategy. This concludes the proof. □

Theorem 4. *Let $(T_\gamma)_{\gamma\in\Gamma}$ be an infinite family of compacts and let T be the Alexandroff compactification of the free sum $\bigoplus_{\gamma\in\Gamma} T_\gamma$. Then $(C(T),p) \in \mathcal{S}$ if and only if $(C(T_\gamma),p) \in \mathcal{S}$ for every $\gamma \in \Gamma$.*

Remark 3. If T is the free sum of a finite family $(T_\gamma)_{\gamma\in\Gamma}$ of compacts, then it is easy to see that $(C(T),p) \in \mathcal{S}$ iff each $(C(T_\gamma),p) \in \mathcal{S}$. This can also be proved by a simplified variant of what follows.

Proof of Theorem 4: By Remark 1, just the "if" direction is to be proved. Let a be the "infinite" element in T, so $T = (\bigoplus_{\gamma\in\Gamma} T_\gamma) \cup \{a\}$. Let B be the unit ball of $C(T)$ with the p-topology. Let B_γ be the unit ball of $C(T_\gamma)$ with the p-topology for any $\gamma \in \Gamma$. Let $F : Z \to B$ be a minimal mapping with dense domain. By Corollary 1 and Lemma 1, it suffices to construct a winning strategy ζ for UP in

$BM(F)$. Let $F_\gamma : Z \to B_\gamma$ be the mapping defined by $F_\gamma(z) := \{f|T_\gamma : f \in F(z)\}$. By Remark 2, F_γ is minimal. When defining the strategy ζ we will inductively construct a sequence of subsets I_n of Γ, with $|I_n| = n$. We put $I_0 = \emptyset$.

We now describe the n-th move of UP in ζ. We assume that the set I_{n-1} and the partial ζ-play $p_{n-1} := H_1, U_1, H_2, U_2, \ldots, H_{n-1}, U_{n-1}$ are already constructed. Let the n-th move of HP be H_n. We put

$$s_n := \sup\{|f(t) - f(a)| : \ f \in F(H_n), \ t \in \bigoplus_{\gamma \in \Gamma \setminus I_{n-1}} T_\gamma\} \ (\leq 2).$$

Let $f_n \in F(H_n)$ and $t_n \in \bigoplus_{\gamma \in \Gamma \setminus I_{n-1}} T_\gamma$ be such that $|f_n(t_n) - f_n(a)| > s_n - 1/n$ and let $\gamma_n \in \Gamma \setminus I_{n-1}$ be such that $t_n \in T_{\gamma_n}$. We put $I_n := I_{n-1} \cup \{\gamma_n\}$. Now we define

$$A_n := \{f \in B : |f(t_n) - f_n(t_n)| < 1/n, |f(a) - f_n(a)| < 1/n\}.$$

The mapping F_{γ_n} is minimal and $(C(T_{\gamma_n}), p) \in \mathcal{S}$, so there is a dense G_δ subset W^n of Z such that $|F_{\gamma_n}(z)| \leq 1$ for all $z \in W^n$. Let $W^n = \bigcap_{k=1}^\infty W_k^n$ with W_k^n open in Z. A_n is (p-)open in B and $f_n \in F(H_n) \cap A_n$. By minimality of F, let U'_n be nonempty open in H_n with $F(U'_n) \subset A_n$. Then put $\zeta(p_{n-1}, H_n) = U_n := U'_n \cap \bigcap_{m,k=1}^n W_k^m$. This finishes the description of the n-th move of UP for ζ.

We now prove that ζ is a winning strategy. Take some $z \in \bigcap_{n \geq 1} U_n$ (if any). We have to prove that $|F(z)| \leq 1$. The sequence $(s_n)_{n \geq 1}$ is non-increasing, so let s_∞ be its limit.

Case (a). Assume $s_\infty > 0$. Suppose that there is some $f \in F(z)$. For every positive integer n we have $f \in F(U'_n) \subset A_n$, so

$$|f(t_n) - f(a)| \geq |f_n(t_n) - f_n(a)| - |f_n(t_n) - f(t_n)| - |f_n(a) - f(a)| > s_n - 3/n.$$

Recall that $t_n \in T_{\gamma_n}$ for every n and $\{\gamma_n\}$ is an injective sequence. Hence a is an accumulation point of $(t_n)_{n \geq 1}$, and f is not continuous at a. This contradiction means that $F(z) = \emptyset$ and so ζ is winning.

Case (b). If $s_\infty = 0$, fix some $\varepsilon > 0$. Take some $n > 6/\varepsilon$ such that $s_n < \varepsilon/3$. Let $f, g \in F(z)$. Then $|f(a) - g(a)| \leq |f(a) - f_n(a)| + |f_n(a) - g(a)| < 2/n < \varepsilon/3$. If $t \in T_\gamma$ for some $\gamma \in \Gamma \setminus I_{n-1}$, then

$$|f(t) - g(t)| \leq |f(t) - f(a)| + |f(a) - g(a)| + |g(a) - g(t)| < s_n + \frac{\varepsilon}{3} + s_n < \varepsilon.$$

If $t \in T_{\gamma_m}$ with $\gamma_m \in I_{n-1}$, then as $z \in \bigcap_{k \geq 1} W_k^m = W^m$, we get $f|T_{\gamma_m} = g|T_{\gamma_m}$ and $f(t) = g(t)$. We have thus proved that $|f(t) - g(t)| < \varepsilon$ for every $t \in T$. As $\varepsilon > 0$ was arbitrary we get that $f \equiv g$. Therefore $|F(z)| \leq 1$ and so ζ is a winning strategy. This concludes the proof. □

Theorem 5. *Let T be a compact and $(T_i)_{i\in\mathbb{N}}$ a sequence of compact subspaces of T such that $\bigcup_{i\in\mathbb{N}} T_i$ is dense in T. If $(C(T_i),p)\in\mathcal{S}$ for all i, then $(C(T),p)\in\mathcal{S}$.*

PROOF: Let Z be some Baire space and $F : Z \to (C(T),p)$ be a minimal mapping with dense domain. Let $F_i : Z \to (C(T_i),p)$ be defined by $F_i(z) := \{f|T_i : f \in F(z)\}$; it is minimal by Remark 2. Let W^i be a residual subset of Z such that $|F_i(z)| \le 1$ for all $z \in W^i$. Then $W := \bigcap_{i\ge 1} W^i$ is also residual and if $z\in W$ and $f,g\in F(z)$, then $f|T_i \equiv g|T_i$ for all $i\ge 1$, so $f\equiv g$ (as $\bigcup_{i\ge1} T_i$ is dense in T). □

Definition 2 (see [AP, III.81], or [B1])**.** Let T be a compact and let T' be another copy of T having discrete topology. Let $q : T \to T'$ be the corresponding bijection. Let $eT = T\cup T'$ be supplied with the following topology (called, after [B1], "porc-épic", that is, "porcupine"): every point of T' is isolated in eT and if $\mathcal{U}$ is a local base at t in T, then the local base of t in eT is

$$\{U\cup q(U)\backslash\{q(t)\} : U\in\mathcal{U}\}.$$

It is easy to check that eT is also a Hausdorff compact space. For example, if S is a circle with the natural topology, then eS is the space "Two circles of Alexandroff".

Theorem 6. *Let T be a compact. Then $(C(T),p)\in\mathcal{S}$ iff $(C(eT),p)\in\mathcal{S}$.*

PROOF: By Remark 1, just the "only if" direction is to be proved, as eT maps onto T in a natural way. Let Z be a Baire space, B be the unit ball of $C(eT)$ supplied with the pointwise topology, B_T be the unit ball of $(C(T),p)$ and $F : Z\to B$ be a minimal mapping with dense domain. By Corollary 1 and Lemma 1, it suffices to construct a winning strategy ζ for UP in $BM(F)$. Let $F' : Z\to B_T$ be the (minimal) mapping defined by $F'(z) := \{f|T : f\in F(z)\}$. As $B_T\in\mathcal{S}$, let W be a dense G_δ in Z such that $|F'(z)|\le 1$ for all $z\in W$. Let $W=\bigcap_{k=1}^\infty W_k$ with W_k open in Z. While defining the strategy ζ, we will inductively construct a sequence $(J(n))_{n=1}^\infty$ of subsets of T, $|J(n)| = n$. Put $J(0) := \emptyset$.

The strategy ζ is the following (we define the n-th move). Suppose that the partial ζ-play $p_{n-1} := (H_1,U_1,\dots,H_{n-1},U_{n-1})$ and the set $J(n-1)\subset T$ are already constructed. Let the n-th move of HP be H_n. Put $U'_n := H_n\cap\bigcap_{k=1}^n W_k$ and

$$s_n := \sup\{|f(t)-f(q(t))| :\ t\in T\backslash J(n-1),\ f\in F(U'_n)\}\ (\le 2).$$

Let $f_n\in F(U'_n)$ and $t_n\in T\backslash J(n-1)$ satisfy $|f_n(t_n)-f_n(q(t_n))| > s_n - 1/n$. Put $J(n) := J(n-1)\cup\{t_n\}$ and

$$A_n := \{f\in B : |f(t)-f_n(t)| < 1/n,\ |f(q(t))-f_n(q(t))| < 1/n,\ t\in J(n)\}.$$

A_n is open in B and $f_n\in F(U'_n)\cap A_n$. By minimality of F, let U_n be nonempty open in U'_n with $F(U_n)\subset A_n$. Put $\zeta(p_{n-1},H_n) := U_n$. The n-th move of UP for ζ is described.

We now prove that ζ is a winning strategy for UP in $BM(F)$. The sequence $(s_n)_{n\ge1}$ is non-increasing. Let s_∞ be its limit.

Case (1). If $s_\infty > 0$, take some $z \in \bigcap_{n\geq 1} U_n$ (if any). Suppose there is some $f \in F(z)$. Then $f \in \bigcap_{n\geq 1} A_n$ and so for every n,

$$f(t_n) - f(q(t_n))| \geq |f_n(t_n) - f_n(q(t_n))| \\ - |f(t_n) - f_n(t_n)| - |f(q(t_n)) - f_n(q(t_n))| > s_n - 3/n.$$

Let t_∞ be an accumulation point of $(t_n)_{n\geq 1}$. By definition the points $q(t_n)$ are different for the different n, so t_∞ is an accumulation point of $(q(t_n))_{n\geq 1}$, too. But then f is not continuous at t_∞. This contradiction shows that $F(z) = \emptyset$ and so ζ is a winning strategy.

Case (2). If $s_\infty = 0$, then take an arbitrary $\varepsilon > 0$. Choose $n > 2/\varepsilon$ such that $s_n \leq \varepsilon/2$. Take some $z \in \bigcap_{n\geq 1} U_n$ and some $f, g \in F(z)$. We have $z \in \bigcap_{i\geq 1} W_i = W$, $f|T \in F'(z)$ and $g|T \in F'(z)$, so $f|T \equiv g|T$. Then (as $f, g \in A_n$), $|f(q(t)) - g(q(t))| < 2n^{-1} < \varepsilon$ for $t \in J(n)$, and for $t \in T\backslash J(n)$ one has

$$|f(q(t)) - g(q(t))| < |f(t) - f(q(t))| + |g(t) - g(q(t))| \leq 2s_n < \varepsilon.$$

$\varepsilon > 0$ being arbitrary, we get $f|_{q(T)} \equiv g|_{q(T)}$, so $f \equiv g$ and $|F(z)| \leq 1$. Thus ζ is a winning strategy. This finishes the proof. □

Theorem 7. *Let h be a continuous mapping of the compact T onto the compact S. Let $D := \{x \in S : |h^{-1}(x)| > 1\}$ contain only isolated points. Then the following are equivalent:*

(a) $(C(T), p) \in \mathcal{S}$;
(b) $(C(h^{-1}(x)), p) \in \mathcal{S}$ *for all* $x \in S$ *and* $(C(S), p) \in \mathcal{S}$;
(c) $(C(h^{-1}(x)), p) \in \mathcal{S}$ *for all* $x \in D$ *and* $(C(S), p) \in \mathcal{S}$.

Proof: For (a) ⇒ (b), $h : T \to S$ being continuous, by Remark 1, $(C(T), p) \in \mathcal{S}$ implies $(C(S), p) \in \mathcal{S}$. For any $x \in S$, put $T_x := h^{-1}(x)$, fix some $t_x \in T_x$ and define the mapping $p_x : T \to T_x$ to be the identity on T_x and to send $T\backslash T_x$ onto $\{t_x\}$. The mapping p_x is continuous: it is constant for $x \in S\backslash D$, and for $x \in D$ the continuity follows from the fact that x is isolated in S (hence $h^{-1}(x)$ is clopen in T). Thus by Remark 1 we have $(C(h^{-1}(x)), p) \in \mathcal{S}$.

(b) ⇒ (c) is obvious. Let us prove (c) ⇒ (a). The subspace $K := \{t_x : x \in S\}$ of T is closed in T (its complement in T is $\bigcup_{d\in D}(T_d\backslash\{t_d\})$, which is open, as D consists of isolated points of S). Then K is a compact and $h|K$ is a homeomorphism of K onto S. Let Z be some Baire space, B be the unit ball of $C(T)$ supplied with the pointwise topology, and $F : Z \to B$ be a minimal mapping with dense domain. By Corollary 1 and Lemma 1, it suffices to construct a winning strategy ζ for UP in $BM(F)$.

Let B_K be the unit ball of $(C(K), p)$, and for any $d \in D$, let B_d be the unit ball of $(C(T_d), p)$. Let $F_0 : Z \to B_K$ be defined by $F_0(z) := \{f|K : f \in F(z)\}$. For any $d \in D$, let $F_d : Z \to B_d$ be defined by $F_d(z) := \{f|T_d : f \in F(z)\}$. By

Remark 2, F_0 and all F_d are minimal; of course their domains are dense in Z. We have $B_K \in \mathcal{S}$, so let W^0 be a dense G_δ in Z such that $|F_0(z)| \le 1$ for all $z \in W^0$. Let $W^0 = \bigcap_{k=1}^\infty W_k^0$ with W_k^0 open in Z. For any $d \in D$, as $B_d \in \mathcal{S}$, let W^d be a dense G_δ in Z such that $|F_d(z)| \le 1$ for all $z \in W^d$. Let $W^d = \bigcap_{k=1}^\infty W_k^d$ with W_k^d open in Z. We now define ζ. While constructing the strategy, we define a sequence $\{d_n\}_n \subset D$ and sequences $\{t_n\}_n, \{t^n\}_n \subset T$. By $J(n)$ we will denote the set $\{d_i : i = 1, \ldots, n\}$.

Move $n \ge 1$**.** Assume that we have constructed the points $\{d_j\}_{j=1}^{n-1}$ and the partial ζ-play $p_{n-1} := (H_i, U_i)_{i=1}^{n-1}$. Let the next move of HP be H_n. Let

$$s_n := \sup\{|f(t) - f(t_d)| : \ f \in F(H_n),\ t \in T_d,\ d \in D\backslash J(n-1)\}\ (\le 2).$$

Take some $f_n \in F(H_n)$, $d_n \in D\backslash J(n-1)$ and $t_n \in T_{d_n}$ in such a way that $|f_n(t_n) - f_n(t_{d_n})| > s_n - 1/n$; denote $t^n := t_{d_n}$. Now put

$$A_n := \{f \in B : |f(t_n) - f_n(t_n)| < 1/n,\ |f(t^n) - f_n(t^n)| < 1/n\},$$

which is a (p-)open subset of B. Now $f_n \in F(H_n) \cap A_n$, so let H_n' be a nonempty open subset of H_n such that $F(H_n') \subset A_n$. Now put

$$U_n = \zeta(p_{n-1}, H_n) := H_n' \cap \bigcap_{k=1}^{n} (W_k^0 \cap \bigcap_{j=1}^{n} W_k^{d_j}).$$

The n-th move of UP for ζ is defined.

Let us prove that ζ is a winning strategy. The sequence $(s_n)_n$ is nonincreasing, so let $s_n \to s_\infty$.

Case (a). Let $s_\infty > 0$. Let t_∞, t^∞ be any accumulation points of $(t_n)_n$, $(t^n)_n$, respectively. As $h(t_n) = h(t^n) = d_n$ and $(d_n)_n$ is injective, one has $h(t_\infty) = h(t^\infty) \in S\backslash D$. Therefore $t_\infty = t^\infty$. Take some $z \in \bigcap_{n\ge 1} U_n$ (if no such z exists, we are done). Suppose there is some $f \in F(z)$. Then for every $n \ge 1$ we have $F(z) \subset F(U_n) \subset F(H_n') \subset A_n$, so

$$|f(t_n) - f(t^n)| \ge |f_n(t_n) - f_n(t^n)| - |f(t_n) - f_n(t_n)| - |f(t^n) - f_n(t^n)| > s_n - 3/n,$$

that contradicts the continuity of f at t_∞. Therefore $F(z) = \emptyset$ and so ζ is a winning strategy.

Case (b). If $s_\infty = 0$, fix some $\varepsilon > 0$. Let n be such that $s_n < \varepsilon/2$. Let $z \in \bigcap_{i\ge 1} U_i$ and take some $f, g \in F(z)$. We have to prove that $f \equiv g$.

(1) As $z \in \bigcap_{k=1}^\infty W_k^0 = W^0$ and $f|K, g|K \in F_0(z)$, we have $f|K \equiv g|K$.

(2) For any $j \ge 1$, as $z \in \bigcap_{k=1}^\infty W_k^{d_j} = W^{d_j}$ and $f|T_{d_j}, g|T_{d_j} \in F_{d_j}(z)$, we have $f|T_{d_j} \equiv g|T_{d_j}$.

(3) If $t \in T_d$ for some $d \in D\backslash\{d_j : j < n\}$, then by $s_n < \varepsilon/2$ and by (1) we have

$$|f(t) - g(t)| \le |f(t) - f(t_d)| + |g(t_d) - g(t)| \le 2s_n < \varepsilon.$$

By (1), (2) and (3) we have $\|f - g\| \le \varepsilon$. But $\varepsilon > 0$ was arbitrary, so $f \equiv g$, and $|F(z)| \le 1$. Thus ζ is a winning strategy. This finishes the proof. □

Remark 4. All the theorems proven in this paper hold true when everywhere the pointwise convergence topology is substituted by the weak one. The changes in the proofs are natural.

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