A remark on localized weak precompactness in Banach spaces

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Abstract. We give a characterization of K-weakly precompact sets in terms of uniform Gateaux differentiability of certain continuous convex functions.

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We begin with the requisite definition. Throughout this paper X denotes a real Banach space with topological dual X^*. If $g : X \to \mathbb{R}$ is a continuous convex function, for $x, y \in X$, we define $Dg(x, y)$ by

$$\lim_{t\to 0}\{g(x + ty) - g(x)\}/t$$

provided that this limit exists, and we also define the subdifferential of g at x ($\in X$) to be the set $\partial g(x)$ of all elements x^* of X^* satisfying that $(u, x^*) \leq g(x+u)-g(x)$ for any $u \in X$. Then $\partial g(x)$ is a non-empty weak*-compact convex subset of X^* for every $x \in X$. The triple (I, Λ, λ) refers to the Lebesgue measure space on I ($= [0, 1]$), Λ^+ to the sets in Λ with positive λ-measure. We always understand that I is endowed with Λ and λ. We denote the set $\{\chi_E/\lambda(E) : E \in \Lambda^+\}$ by $\Delta(I)$. A function $f : I \to X^*$ is said to be weak*-measurable if $(x, f(t))$ is λ-measurable for each $x \in X$. If $f : I \to X^*$ is a bounded weak*-measurable function, we obtain a bounded linear operator $T_f : X \to L_1(I, \Lambda, \lambda)$ given by $T_f(x) = x \circ f$ for every $x \in X$, where $(x \circ f)(t) = (x, f(t))$ for every $t \in I$, and the dual operator of T_f is denoted by T_f^* ($: L_\infty(I, \Lambda, \lambda) \to X^*$).

According to Bator and Lewis [1], let us define the notion of localized weak precompactness in Banach spaces as follows.

Definition 1. *Let A be a bounded subset of X and K a weak*-compact subset of X. Then we say that A is K-weakly precompact if every sequence $\{x_n\}_{n\geq 1}$ in A has a pointwise convergent subsequence $\{x_{n(k)}\}_{k\geq 1}$ on K.*

Then, in [1], they have made a systematic study of K-weakly precompact sets A in Banach spaces and obtained various characterizations of such sets.

Succeedingly, in our paper [4], we also have obtained measure theoretic characterizations of K-weakly precompact sets A by the effective use of a K-valued weak*-measurable function constructed in the case where A is non-K-weakly precompact. In this paper we wish to add a characterization of K-weakly precompact sets in terms of uniform Gateaux differentiability of certain continuous convex functions, which is our aim. This can be regarded as a slight generalization and refinement of Corollary 10 in [1]. And it should be noted that even here this K-valued function also becomes an effective means to an end. Before giving our characterization theorem, let us define some special continuous convex functions on X as follows.

Definition 2. *Let H be a non-empty bounded subset of X^*. Then the continuous convex function associated with H, which is denoted by g_H, is defined by $g_H(x) = \sup\{(x, x^*) : x^* \in H\}$ for every $x \in X$.*

In what follows, all notations and terminology used and not defined are as in [1].

Let A be a bounded subset of X, K a weak*-compact subset of X^*, $\{x_n\}_{n\geq 1}$ a sequence in A and Y the closed linear span of $\{x_n : n \geq 1\}$ in X. In the following, we always understand that Y is a such space. Let $j : Y \to X$ be the inclusion mapping and j^* its dual mapping. For any non-empty subset H of K, the continuous convex function $g_H : Y \to \mathbb{R}$ satisfies that $\partial g_H(y) \subset \overline{\mathrm{co}}^*(j^*(K))$ for each $y \in Y$. Further let us note two preliminary facts for the proof of Theorem. One concerns separably related sets in the case where A is K-weakly precompact. Let $\{x_n\}_{n\geq 1}$ be a sequence in A and suppose that there exists a subsequence $\{x_{n(k)}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $\lim_{k\to\infty}(x_{n(k)}, x^*)$ exists for every $x^* \in K$. Then this implies that $\lim_{k\to\infty}(x_{n(k)}, y^*)$ exists for every $y^* \in \overline{\mathrm{co}}^*(j^*(K))$. Hence, by considering the mapping $L : \overline{\mathrm{co}}^*(j^*(K)) \to c$ (the Banach space of all convergent sequences of real numbers equipped with the supremum norm $\|\cdot\|_\infty$) defined by $L(y^*) = \{(x_{n(k)}, y^*)\}_{k\geq 1}$, we easily know that $\overline{\mathrm{co}}^*(j^*(K))$ is separably related to $\{x_{n(k)} : k \geq 1\}$, since c is separable. The other concerns the construction of a K-valued weak*-measurable function h and a sequence $\{x_n\}_{n\geq 1}$ in A in the case where A is non-K-weakly precompact. Then, although the construction of this function h and the sequence $\{x_n\}_{n\geq 1}$ in A is exactly the same as in § 3 of [4], for the sake of completeness, we state its outline briefly in the following. Since A is not K-weakly precompact, by the celebrated argument of Rosenthal [5], we have a sequence $\{x_n\}_{n\geq 1}$ in A and real numbers r and δ with $\delta > 0$ such that putting $A_n = \{x^* \in K : (x_n, x^*) \leq r\}$ and $B_n = \{x^* \in K : (x_n, x^*) \geq r+\delta\}$, $(A_n, B_n)_{n\geq 1}$ is an independent sequence of pairs of weak*-closed subsets of K (that is, for every $\{\varepsilon_j\}_{1\leq j\leq k}$ with $\varepsilon_j = 1$ or -1, $\bigcap\{\varepsilon_j A_j : 1 \leq j \leq k\}$ is a non-empty set, where $\varepsilon_j A_j = A_j$ if $\varepsilon_j = 1$ and $\varepsilon_j A_j = B_j$ if $\varepsilon_j = -1$). Putting $\Gamma = \bigcap_{n\geq 1}(A_n \cup B_n)$, Γ is a non-empty weak*-compact subset of K, since $(A_n, B_n)_{n\geq 1}$ is independent. Define $\varphi : \Gamma \to \mathcal{P}(N)$ (Cantor space, with its usual compact metric topology) by $\varphi(x^*) = \{p : (x_p, x^*) \leq r\}$ $(= \{p : A_p \ni x^*\}) \in \mathcal{P}(N)$. Then φ is a continuous surjection from Γ to $\mathcal{P}(N)$ (here, Γ is endowed with the weak*-topology $\sigma(X^*, X)$) and so we have a Radon probability measure γ on Γ such that $\varphi(\gamma) = \nu$ (the normalized Haar measure if we identify $\mathcal{P}(N)$ with $\{0,1\}^N$)

and $\{f \circ \varphi : f \in L_1(\mathcal{P}(N), \Sigma_\nu, \nu)\} = L_1(\Gamma, \Sigma_\gamma, \gamma)$ where Σ_ν (resp. Σ_γ) is the family of all ν (resp. γ)-measurable subsets of $\mathcal{P}(N)$ (resp. Γ). Further, consider a function $\tau : \mathcal{P}(N) \to I$ defined by $\tau(D) = \Sigma\{1/2^m : m \in D\}$ for every $D \in \mathcal{P}(N)$. Then τ is a continuous surjection such that $\tau(\nu) = \lambda$ and $\{u \circ \tau : u \in L_1(I, \Lambda, \lambda)\} = L_1(\mathcal{P}(N), \Sigma_\nu, \nu)$. Then, making use of the lifting theory, we have a weak*-measurable function $h : I \to \Gamma$ ($\subset K$) such that

$$(\alpha) \qquad \rho(x \circ h)(t) = (x, h(t)) \quad \text{for every } x \in X \text{ and every } t \in I,$$

$$(\beta) \qquad \int_E (x, h(t))\, d\lambda(t) = \int_{\varphi^{-1}(\tau^{-1}(E))} (x, x^*)\, d\gamma(x^*)$$

for every $E \in \Lambda$ and every $x \in X$. Here ρ denotes a lifting on $L_\infty(I, \Lambda, \lambda)$.

Now we are ready to state our characterization theorem (a localized version of Theorem 8 in [1]). Its main part is that (3) implies (1), whose proof is significant in the point that the characters of the K-valued function h and the sequence $\{x_n\}_{n\geq 1}$ in A obtained above are used concretely and effectively. And there, we can get a result that for every $y \in Y$ and every subsequence $\{x_{n(k)}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$, $Dg_H(y, x_{n(k)})$ does not exist uniformly in k, where $H = h(I)$ ($\subset K$).

Theorem. *Let A be a bounded subset of X and K a weak*-compact (not necessarily convex) subset of X^*. Then the following statements about A and K are equivalent.*

(1) *The set A is K-weakly precompact.*

(2) *If $\{x_n\}_{n\geq 1}$ is a sequence in A and $g : Y \to \mathbb{R}$ is a continuous convex function such that $\partial g(y) \subset \overline{\text{co}}^*(j^*(K))$ for every $y \in Y$, then there exists a dense G_δ-subset G of Y and a subsequence $\{x_{n(k)}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $Dg(y, x_{n(k)})$ exists uniformly in k for each $y \in G$.*

(3) *If $\{x_n\}_{n\geq 1}$ is a sequence in A and H is a non-empty subset of K, then there exists an element y of Y and a subsequence $\{x_{n(k)}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $Dg_H(y, x_{n(k)})$ exists uniformly in k.*

Proof: (1) $\Rightarrow$ (2). The proof is analogous to that of the corresponding part of Theorem 8 in [1]. Suppose that (1) holds. Take any sequence $\{x_n\}_{n\geq 1}$ in A and any continuous convex function $g : Y \to \mathbb{R}$ such that $\partial g(y) \subset \overline{\text{co}}^*(j^*(K))$ for every $y \in Y$. As A is K-weakly precompact, we have a subsequence $\{x_{n(k)}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $\lim_{k\to\infty}(x_{n(k)}, x^*)$ exists for every $x^* \in K$. Therefore, by the first preliminary fact preceding Theorem, $\overline{\text{co}}^*(j^*(K))$ is separably related to B ($= \{x_{n(k)} : k \geq 1\}$). So it is separably related to $\text{aco}(B)$ (: the absolutely convex hull of B). Since $\partial g(y) \subset \overline{\text{co}}^*(j^*(K))$ for every $y \in Y$, by the same argument as in Theorem 3.14 and Proposition 3.15 of [2], we have a dense G_δ-subset G of Y such that g is $\text{aco}(B)$-differentiable (cf. [2]) at every $y \in G$, whence (2) holds.

(2) $\Rightarrow$ (3). This follows immediately from the fact that $\partial g_H(y) \subset \overline{\text{co}}^*(j^*(K))$ for every non-empty subset H of K and every $y \in Y$.

(3) $\Rightarrow$ (1). The proof of this part is crucial. Suppose that (1) fails. By the second preliminary fact preceding Theorem, we have a function $h : I \to K$ and a sequence $\{x_n\}_{n\geq 1}$ in A as stated above. Take $H = h(I)$, and let $\{U(n,k) : n = 0,1,\ldots; k = 0,\ldots,2^n - 1\}$ be a system of open intervals in I given by $U(n,k) = (k/2^n, (k+1)/2^n)$ if $n \geq 0$, $0 \leq k \leq 2^n - 1$. Then we get that $\varphi^{-1}(\tau^{-1}(U(n,2k))) \subset B_n$ and $\varphi^{-1}(\tau^{-1}(U(n,2k+1))) \subset A_n$ for $n = 1,2,\ldots$ and $k = 0,\ldots,2^{n-1} - 1$. Further we note a following elementary fact: Let $E \in \Lambda^+$ and $\{n(i)\}_{i\geq 1}$ be a strictly increasing sequence of natural numbers. Then there exists a natural number i and a non-negative number q with $0 \leq 2q < 2^{n(i)} - 1$ such that both $E \cap U(n(i),2q)$ and $E \cap U(n(i),2q+1)$ are in Λ^+, which can be easily shown by an argument used in Lemma 2 of [3].

Now, let us show that for every subsequence $\{x_{n(k)}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ and every $y \in Y$, $Dg_H(y, x_{n(k)})$ does not exist uniformly in k. To this end, take any point y in Y and any subsequence $\{x_{n(k)}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$, and set $y_k = x_{n(k)}$ for every k. Consider a family of weak*-open slices of $\overline{\text{co}}^*(j^*(T_h^*(\Delta(I))))$ $(= M)$: $\{S(y, \delta/3i, M) : i \geq 1\}$. Then we have that for every i

$$\begin{aligned} S(y,\delta/3i,M) &= \Big\{y^* \in M : (y,y^*) > \sup_{z^*\in M} (y,z^*) - \delta/3i\Big\} \\ &= \Big\{y^* \in M : (y,y^*) > \operatorname*{ess\text{-}sup}_{t\in I} (j(y),h(t)) - \delta/3i\Big\} \\ &= \Big\{y^* \in M : (y,y^*) > g_H(y) - \delta/3i\Big\}, \end{aligned}$$

since $g_H(y) = \sup_{t\in I}(j(y),h(t)) = \text{ess-sup}_{t\in I}(j(y),h(t))$ by virtue of (α) above. So, letting $E_i = \{t \in I : (j(y),h(t)) > g_H(y) - \delta/3i\}$, we easily get that $E_i \in \Lambda^+$ and $j^*(h(E_i)) \subset S(y,\delta/3i,M)$ for every i. Hence, by the elementary fact stated above, there exists a natural number $k(i)$ and a non-negative number $q(i)$ with $0 \leq 2q(i) < 2^{n(k(i))} - 1$ such that both $E_i \cap U(n(k(i)),2q(i))$ and $E_i \cap U(n(k(i)),2q(i)+1)$ are in Λ^+. For every i, let $F_i = E_i \cap U(n(k(i)),2q(i))$ and $G_i = E_i \cap U(n(k(i)),2q(i)+1)$, and let $u_i^* = j^*(T_h^*(\chi_{F_i}/\lambda(F_i)))$ and $v_i^* = j^*(T_h^*(\chi_{G_i}/\lambda(G_i)))$. Then we have that for every i

(a) $(y,u_i^*) > g_H(y) - \delta/3i$ and $(y,v_i^*) > g_H(y) - \delta/3i$,

(b) $(y_{k(i)}, u_i^* - v_i^*) \geq \delta$,

(c) $g_H(y + y_{k(i)}/i) \geq (y + y_{k(i)}/i, u_i^*)$ and $g_H(y - y_{k(i)}/i) \geq (y - y_{k(i)}/i, v_i^*)$.

Indeed, we have that

$$\begin{aligned} (y,u_i^*) &= (j(y), T_h^*(\chi_{F_i}/\lambda(F_i))) \\ &= \Big\{\int_{F_i} (j(y),h(t))\,d\lambda(t)\Big\}/\lambda(F_i) > g_H(y) - \delta/3i, \end{aligned}$$

since $j^*(h(F_i)) \subset S(y,\delta/3i,M)$. Similarly, $(y,v_i^*) > g_H(y) - \delta/3i$. Thus we have (a). And we can prove (b) as follows. In virtue of (β), we have that for every i

$$\begin{aligned}
&(y_{k(i)}, u_i^* - v_i^*) \\
&= (j(y_{k(i)}), T_h^*(\chi_{F_i}/\lambda(F_i))) - (j(y_{k(i)}), T_h^*(\chi_{G_i}/\lambda(G_i))) \\
&= (j(x_{n(k(i))}), T_h^*(\chi_{F_i}/\lambda(F_i))) - (j(x_{n(k(i))}), T_h^*(\chi_{G_i}/\lambda(G_i))) \\
&= \Big\{ \int_{F_i} (j(x_{n(k(i))}), h(t))\, d\lambda(t) \Big\} / \lambda(F_i) \\
&\qquad - \Big\{ \int_{G_i} (j(x_{n(k(i))}), h(t))\, d\lambda(t) \Big\} / \lambda(G_i) \\
&= \Big\{ \int_{\varphi^{-1}(\tau^{-1}(F_i))} (j(x_{n(k(i))}), x^*)\, d\gamma(x^*) \Big\} / \lambda(F_i) \\
&\qquad - \Big\{ \int_{\varphi^{-1}(\tau^{-1}(G_i))} (j(x_{n(k(i))}), x^*)\, d\gamma(x^*) \Big\} / \lambda(G_i) \\
&\geq (r+\delta) - r = \delta,
\end{aligned}$$

since $\varphi^{-1}(\tau^{-1}(F_i))$ $(\subset \varphi^{-1}(\tau^{-1}(U(n(k(i)), 2q(i))))) \subset B_{n(k(i))}$, $\varphi^{-1}(\tau^{-1}(G_i))$ $(\subset \varphi^{-1}(\tau^{-1}(U(n(k(i)), 2q(i)+1)))) \subset A_{n(k(i))}$ and $\tau(\varphi(\gamma)) = \lambda$. As to (c), we have that for every i

$$\begin{aligned}
g_H(y + y_{k(i)}/i) &= \sup_{t\in I} (j(y + y_{k(i)}/i), h(t)) \\
&\geq \Big\{ \int_{F_i} (j(y + y_{k(i)}/i), h(t))\, d\lambda(t) \Big\} / \lambda(F_i) = (y + y_{k(i)}/i, u_i^*).
\end{aligned}$$

Similarly, $g_H(y - y_{k(i)}/i) \geq (y - y_{k(i)}/i, v_i^*)$. Then, making use of (a), (b) and (c), we have that for every i

$$\begin{aligned}
&g_H(y + y_{k(i)}/i) + g_H(y - y_{k(i)}/i) - 2\cdot g_H(y) \\
&> (y + y_{k(i)}/i, u_i^*) + (y - y_{k(i)}/i, v_i^*) - \{(y, u_i^* + v_i^*) + 2\delta/3i\} \\
&= (y_{k(i)}, u_i^* - v_i^*)/i - 2\delta/3i \geq \delta/3i.
\end{aligned}$$

Consequently, we have that for every i

$$\{g_H(y + y_{k(i)}/i) + g_H(y - y_{k(i)}/i) - 2\cdot g_H(y)\}/(1/i) > \delta/3,$$

which implies that $Dg_H(y, x_{n(k)})$ does not exist uniformly in k. Thus the proof is complete. □

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