Comment.Math.Univ.Carolin. 40,2 (1999)355–365

Condensations of Cartesian products

Oleg Pavlov

Abstract. We consider when one-to-one continuous mappings can improve normality-type and compactness-type properties of topological spaces. In particular, for any Tychonoff non-pseudocompact space X there is a μ such that X^μ can be condensed onto a normal (σ-compact) space if and only if there is no measurable cardinal. For any Tychonoff space X and any cardinal ν there is a Tychonoff space M which preserves many properties of X and such that any one-to-one continuous image of M^μ, $\mu \leq \nu$, contains a closed copy of X^μ. For any infinite compact space K there is a normal space X such that $X \times K$ cannot be mapped one-to-one onto a normal space.

Keywords: condensation, one-to-one, compact, measurable

Classification: 54C10, 54A10

0. Introduction

We consider only Tychonoff topological spaces and continuous mappings. A condensation is a one-to-one mapping onto. Throughout the paper κ denotes the first Ulam-measurable cardinal, if such a cardinal exists.

It is well-known that many key topological properties are not multiplicative. However, for many examples of a given property $\mathcal{P}$ and a space (X, τ) which has $\mathcal{P}$, but X^2 does not, there is a weaker topology τ' on X such that the square of (X, τ') does have $\mathcal{P}$. In fact, many examples are produced starting with the space (X, τ'). This observation motivated A.V. Arhangel'skii to raise the following questions. *Is it true that for any Lindelöf space X there is a condensation $f : X \to Z$ such that Z^2 is Lindelöf* (see [1])*? Is it true that the second power of any normal* (*hereditarily normal, paracompact, Lindelöf, pseudocompact, countably compact, etc.*) *space can be condensed onto a space with the same property? Can any power of a Lindelöf space be condensed onto a Lindelöf space* ([1])*? Is it true that $\mathbf{Q}^\mu$ can be condensed onto a Lindelöf (compact) space for any infinite μ?* These questions are in line with the most general problem concerning condensations: *when can a space from class $\mathcal{A}$ be condensed onto a space from $\mathcal{B}$?*, for some $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{B}$ is "better" than $\mathcal{A}$ in some sense.

R. Buzyakova answered several of these questions negatively. She constructed a normal countably compact space in [3] and a Lindelöf space in [4], whose squares cannot be condensed onto a normal space (A.N. Yakivchik constructed earlier in [10] a Hausdorff non-regular finally compact space whose square cannot be condensed onto a Hausdorff finally compact space). We generalize these results in Corollary 1: for any space X and a cardinal ν there is a larger space M which preserves many properties of X and contains many clopen copies of X in such a way, that for any $\mu \le \nu$ and for each condensation $f : M^\mu \to Z$, Z contains a closed copy of X^μ. Thus, condensations cannot improve most non-multiplicative properties of arbitrary large (but a priori fixed) powers. If also all powers of X are τ-compact for some τ, then there is an M such that for any μ, $f(M^\mu)$ contains a closed copy of X^μ.

E.G. Pytkeev proved in [9] that any separable metrizable non σ-compact Borel space can be condensed onto $\mathbf{I}^\omega$. Since $\mathbf{Q}^\omega$ is Borel (as a one-to-one continuous image of $\mathbf{N}^\omega$, see [8]) and not σ-compact ($\mathbf{N}^\omega$ is closed in $\mathbf{Q}^\omega$), $\mathbf{Q}^\omega$ can be condensed onto $\mathbf{I}^\omega$. Therefore $\mathbf{Q}^\mu$ can be condensed onto $\mathbf{I}^\mu$ for any infinite μ. This solves one of the mentioned questions. It turns out that a somewhat similar result holds for most Lindelöf spaces. We show in Theorem 1 that for any non pseudocompact X with $|X| < \kappa$, X^μ can be condensed onto a σ-compact space for many $\mu < \kappa$. On the contrary, if κ does exist, then no power of some non-pseudocompact spaces (of cardinality $\ge \kappa$) can be condensed onto a normal space (Corollary 3).

1. Condensation onto a σ-compact space

Theorem 1. *Let X be a non-pseudocompact Tychonoff space and let $|X|$ be non Ulam-measurable. Let $|X| \le \mu_0 < \kappa$ and for every $k \in \omega$, $\mu_{k+1} = exp(\mu_k)$ and $\mu = sup\{\mu_k : k \in \omega\}$. Then X^μ can be condensed onto a regular σ-compact space.*

PROOF: Let $\alpha_0 = |\beta X|$ and for any $k \in \omega$, $\alpha_{k+1} = exp(\alpha_k)$. Then for $\alpha = sup\{\alpha_n : n \in \omega\}$, $\alpha = \mu$. Let $f \in C(X, [0, \infty))$ be such that for each $i \in \omega$ there is $b_i \in f^{-1}(i + 0.5)$. Let $K = \beta X$, $\tilde{K} = \{x \in K : f$ can be extended on $X \cup \{x\}\}$ and let $\tilde{f}$ be an extension of f on $\tilde{K}$. We denote $\mathcal{K} = \tilde{K} \times \prod\{K_\gamma : 1 \le \gamma < \alpha\}$ and $\mathcal{X} = \prod\{X_\gamma : \gamma < \alpha\}$, where K_γ and X_γ are copies of K and X respectively. Then $\mathcal{K}$ is a T_1 regular σ-compact space.

For any $i \in \omega$, let $A_i = \{a_{ij} \in \omega : a_{i0} = i\}$ be an increasing sequence such that for $i \ne j$, $A_i^+ \cap A_j^+ = \emptyset$ where $A_i^+ = A_i \setminus \{a_{i0}\}$. By induction, a mapping $\phi : \omega \to \omega$ can be defined such that

(1) if $i \notin \cup\{A_i^+ : i \in \omega\}$, then $\phi(i) = 0$, and
(2) if $j \ge 1$, then $\phi(a_{ij}) = \phi(i) + j + 1$.

Let $C_0 = \overline{\tilde{f}^{-1}([0; 1))}^{\tilde{K}}$ and for $i \in \omega$, $C_{i+1} = \overline{\tilde{f}^{-1}([i + \frac{1}{2}; i + 2))}^{\tilde{K}} \setminus C_i$; $\mathcal{C}_i = C_i \times \prod\{K_\gamma : 1 \le \gamma < \alpha\}$.

For $i, j \in \omega$, $j \geq 1$, let $F_{ij,0} = b_{a_{ij}} \times \prod\{K_\gamma : 1 \leq \gamma \leq \alpha_{\phi(a_{ij})}\}$, and for $1 \leq \Delta < \alpha$, $F_{ij,\Delta} = \prod\{K_\gamma : \alpha_{\phi(a_{ij})} \cdot \Delta < \gamma \leq \alpha_{\phi(a_{ij})} \cdot (\Delta + 1)\}$ (here we use a product of ordinals, see [7]), then $b_{a_{ij}} \times \prod\{K_\gamma : 1 \leq \gamma < \alpha\} = \prod\{F_{ij,\Delta} : \Delta < \alpha\}$.
For any $i, j \in \omega$, $j \geq 1$ and $\Delta \geq 1$ we denote $M_{ij,0} = b_{a_{ij}} \times \prod\{X_\gamma : 1 \leq \gamma \leq \alpha_{\phi(a_{ij})}\}$ and $M_{ij,\Delta} = \prod\{X_\gamma : \alpha_{\phi(a_{ij})} \cdot \Delta < \gamma \leq \alpha_{\phi(a_{ij})} \cdot (\Delta + 1)\}$. Then $M_{ij,0} \subset F_{ij,0}$ and $M_{ij,\Delta} \subset F_{ij,\Delta}$. Each $M_{ij,\Delta}$, $\Delta \geq 0$, contains a closed discrete subset $H_{ij,\Delta}$ of cardinality $\alpha_{\phi(a_{ij})-1}$ which is also C^*-embedded in $F_{ij,\Delta}$. Indeed, $M_{ij,0} \approx M_{ij,0} \times M_{ij,0}$. The first factor contains a closed discrete subset of cardinality $\alpha_{\phi(a_{ij})-1}$ by a theorem from [6] (since $M_{ij,0}$ is a $\alpha_{\phi(a_{ij})}$-power of a non countably compact space X). The second factor contains a C^*-embedded subset of the same cardinality. The diagonal product of these subsets is a required set $H_{ij,\Delta}$. Let us denote $\tilde{H}_{ij,\Delta} = \overline{H_{ij,\Delta}}^{F_{ij,\Delta}}$. For each τ, $\mathcal{C}_{i|\leq\tau}$ denotes projection of $\mathcal{C}$ onto ordinals not greater than τ.
If $i \in \omega$, $k \geq 1$ and $\phi(i) = 0$, let

$$C_{i0} = \mathcal{C}_{i|\leq\alpha_0} \setminus \prod\{X_\gamma : \gamma \leq \alpha_0\},$$

and

$$C_{ik} = \{x \in (\mathcal{C}_{i|\leq\alpha_k} \setminus \prod\{X_\gamma : \gamma \leq \alpha_k\}) : x_{|\leq\alpha_{k-1}} \in \prod\{X_\gamma : \gamma \leq \alpha_{k-1}\}\}.$$

If $n, k \geq 1$ and $i = a_{jn}$, let

$$C_{i0} = \mathcal{C}_{i|\leq\alpha_{\phi(i)}} \setminus (\prod\{X_\gamma : \gamma \leq \alpha_{\phi(i)}\} \cup \tilde{H}_{jn,0}),$$

and

$$C_{ik} = \{x \in (\mathcal{C}_{i|\leq\alpha_{\phi(i)+k}} \setminus (\prod\{\tilde{H}_{jn,\Delta} : \Delta < \alpha\})_{|\leq\alpha_{\phi(i)+k}}) : \\ x \notin \prod\{X_\gamma : \gamma \leq \alpha_{\phi(i)+k}\}, \text{ and } x_{\phi(i)+k-1} \in \prod\{X_\gamma : \gamma \leq \alpha_{\phi(i)+k-1}\}.$$

Then for every $i, j \in \omega$, $|C_{ij}| = exp(\alpha_{\phi(i)+j}) = \alpha_{\phi(i)+j+1}$. Let also $\mathcal{C}_{ik} = C_{ik} \times \prod\{K_\gamma : \alpha_{\phi(i)+k} < \gamma < \alpha\}$. Therefore, if $\phi(i) = 0$, then $\{\mathcal{C}_{ik} : k \in \omega\}$ is a partition of $\mathcal{C}_i \setminus \mathcal{X}$. If $\phi(i) \neq 0$ and $i = a_{jn}$, then $\{\mathcal{C}_{ik} : k \in \omega\}$ is a partition of $\mathcal{C}_i \setminus (\mathcal{X} \cup \prod\{\tilde{H}_{jn,\Delta} : \Delta < \alpha\})$.
For $i, j \in \omega$, $j \geq 1$, let $\psi_{ij,0}$ be a one-to-one mapping of $H_{ij,0}$ onto $C_{i(j-1)}$. Such a mapping exists since $|H_{ij,0}| = \alpha_{\phi(a_{ij})-1} = \alpha_{(\phi(i)+j+1)-1} = \alpha_{\phi(i)+j} = |C_{i(j-1)}|$. This mapping can be extended to a continuous mapping $\tilde{\psi}_{ij,0} : \tilde{H}_{ij,0} \to \overline{C_{i(j-1)}}^{\mathcal{K}_{|\leq\alpha_{\phi(i)+j-1}}} = \overline{C_i} \times \prod\{K_\gamma : 1 \leq \gamma \leq \alpha_{\phi(i)+j-1}\}$. In the same way for $i, j \in \omega$, $j \geq 1$ and $1 \leq \Delta < \alpha$ there is a one-to-one continuous mapping $\psi_{ij,\Delta}$ of $H_{ij,\Delta}$ onto $F_{i(j-1),\Delta}$. This mapping can be extended to a continuous mapping $\tilde{\psi}_{ij,\Delta} : \tilde{H}_{ij,\Delta} \to F_{i(j-1),\Delta}$. For any $i, j \in \omega$, $j \geq 1$, let $\tilde{\psi}_{ij} = \prod\{\tilde{\psi}_{ij,\Delta} : \Delta < \alpha\} : \prod\{\tilde{H}_{ij,\Delta} : \Delta < \alpha\} \to \overline{\mathcal{C}}_i$ and $\psi_{ij} = \tilde{\psi}_{ij|\mathcal{X}}$. It then follows that $\tilde{\psi}_{ij}$ is a mapping "onto" and that ψ_{ij} is a condensation of $\prod\{H_{ij,\Delta} : \Delta < \alpha\}$ onto $\mathcal{C}_{i(j-1)}$.

For $i, j \in \omega$, $j \geq 1$, let $D_{ij} = Dom(\tilde{\psi}_{ij})$, then $\tilde{\psi}_{ij}$ induces an upper semicontinuous decomposition E_{ij} of D_{ij} since D_{ij} is compact. We define a decomposition E of $\mathcal{K}$ as follows:

(1) if $x \notin \cup\{D_{ij} : i, j \in \omega, j \geq 1\}$, then $xEy \leftrightarrow x = y$;
(2) if $j_0 \geq 1$ and $x \in D_{i_0 j_0}$, then xEy if and only if $y \in D_{i_0 j_0}$ and $xE_{i_0 j_0}y$.

This decomposition is well defined and it is upper semicontinuous since $\{D_{ij} \subset \mathcal{K} : i, j \in \omega, j \geq 1\}$ is a locally finite family of disjoint closed subsets of $\mathcal{K}$. Then the quotient mapping $q : \mathcal{K} \to \mathcal{K}' = \mathcal{K}_{/E}$ is closed, therefore $\mathcal{K}'$ is a T_1 regular σ-compact space. For $i \in \omega$, let $D_{i0} = \overline{\mathcal{C}}_i$, $D_i = \cup\{D_{ij} : j \in \omega\}$, $\mathcal{K}_i = \cup\{D_j : j \leq i\}$ and $G_i = \cup\{\overline{\mathcal{C}}_j : j \leq i\}$. By a theorem from [2] the space $\mathcal{K}$ is an inductive limit of its closed subsets $\mathcal{K}_i$ and also of the compacta G_i. The same is true for the space $\mathcal{K}'$ and sets $\mathcal{K}'_i = q(\mathcal{K}_i)$ and $G'_i = q(G_i)$ since q is a quotient mapping. Let $D'_i = q(D_i)$, $D'_{ij} = q(D_{ij})$ and $\mathcal{X}' = q(\mathcal{X})$.

We claim that $q_{|\mathcal{X}}$ is a condensation. To see this, note that from the definition of the decomposition E it is sufficient to prove that $q_{|D_{ij} \cap \mathcal{X}}$ is a condensation. But this is obvious since E_{ij} is generated by a mapping $\tilde{\psi}_{ij}$ whose restriction ψ_{ij} is a condensation. In general, $\mathcal{X}'$ is not a σ-compact space. The desired condensation of $\mathcal{X}'$ onto a σ-compact space will be a restriction $g_{|\mathcal{X}'}$ of a quotient map $g : \mathcal{K}' \to g(\mathcal{K}')$ which we define at the end of the proof. g will be the limit of maps g_i, $i \in \omega$, which are defined below, in the sense of Lemma 1. It will be constructed in such a way that $g(\mathcal{X}') = g(\mathcal{K}')$ which ensures that $g(\mathcal{X}')$ is σ-compact. In the next paragraph we introduce an auxiliary notation which will be used in the definition of maps g_i.

Let H be a closed subset of some topological space M, and let h be a quotient mapping of H. Then h induces a decomposition E_H of H and an associate decomposition E_M of M by the rules: if $x \notin H$, then $xE_My \Leftrightarrow x = y$; if $x \in H$, then $xE_My \Leftrightarrow y \in H$ and xE_Hy. The decomposition E_M defines a quotient mapping of M, which we will denote by $h_{H,M}$. It is clear that if h is closed then so is $h_{H,M}$, that $h_{H,M|M \setminus H}$ is a homeomorphism, and that $h_{H,M}(M \setminus H) \cap h_{H,M}(H) = \emptyset$.

Let us define quotient mappings g_{-1}, $g_{-1,0}$ and g_i, $g_{i,i+1}$ as follows:

(1) $g_{-1} \equiv id_{\mathcal{K}'}$;
(2) if g_{i-1} is already defined, then $g_{i-1,i} = g_{i-1,i_{g_{i-1}(D'_i), g_{i-1}(\mathcal{K}')}}$ and $g_i = g_{i-1,i} \circ g_{i-1}$;
(3) let $g_{i-1,i|g_{i-1}(D')}$ be a quotient mapping corresponding to decomposition E'_i of the space $g_{i-1}(D'_i)$, where for $y \in \overline{\mathcal{C}}_i$, $E'_i(g_{i-1}q(y)) = \{g_{i-1}(q(y))\} \cup \{g_{i-1}(q(X)) : \text{there is } j \geq 1, x \in D_{i,j} \text{ and } \tilde{\psi}_{i,j}(x) = y\}$.

The following are the properties of the mappings g_{i-1}, $g_{i-1,i}$ for $i \in \omega$:

(a) $g_{i-1}(\mathcal{K})$ is a T_1 normal space;
(b) every compact $g_{i-1}(D'_{in})$ $(n \in \omega)$ has a neighborhood $U_{i,n}$ in $g_{i-1}(\mathcal{K}')$ such that $\{U_{i,n} : n \in \omega\}$ is a discrete family in $g_{i-1}(\mathcal{K})$;

(c) $g_{i-1}(D')$ is closed in $g_{i-1}(\mathcal{K}')$;
(d) for any $i, j \in \omega$, $g_{i-1|D'_{j,n}}$ is a homeomorphism;
(e) $g_{i-1|D'_i}$ is a homeomorphism in a closed subset of $g_{i-1}(\mathcal{K}')$;
(f) $B_{i-1} = g_{i-1}(\mathcal{K}')$ is compact for $i > 0$;
(g) $g_{i-1,i|B_{i-1}}$ is a homeomorphism for $i > 0$.

First, let us check properties (a)–(g) for $i = 0$. (a) holds trivially. The family $\{U_{0n} \subset \mathcal{K}' : n \in \omega\}$, where $U_{00} = q(\tilde{f}^{-1}[0; \frac{4}{3}))$ and $U_{0i} = q(\tilde{f}^{-1}(b_{a_{0j}} - \frac{1}{3}; b_{a_{0j}} + \frac{1}{3}))$ for $i \geq 1$ satisfies (b). (c) follows from (b) and the fact that $D'_0 = \bigoplus\{D'_{0,n} : n \in \omega\}$ and each $D'_{0,n}$ is compact. (d) holds trivially, (e) follows directly from (b)–(d). Now let mappings g_k, $g_{k-1,k}$ be constructed for all $k \leq i-1$ and satisfy properties (a)–(e).

Lemma 1. *Let a T_1 normal space M be an inductive limit of an increasing sequence of its closed subsets M_n, where $n \in \omega$. Let $\{h_{n,n+1} : n \in \omega\}$ be a family of quotient mappings such that $Dom(h_{0,1}) = M$, $Dom(h_{n+1,n+2}) = Ran(h_{n,n+1})$ and $h_{n+1} = h_{n,n+1} \circ \ldots \circ h_{0,1}$.. Let $\mathcal{M}$ be an equivalence relation on M such that $x\mathcal{M}y \Leftrightarrow h_k(x) = h_k(y)$ for some $n \in \omega$. Let also for $n \in \omega$ sets $B_n = h_n(M_n)$ be normal and closed subsets of $h_n(M)$ and $h_{n,n+1|B_n}$ be a homeomorphism onto a closed subset of B_{n+1}. Then the image $H_{/\mathcal{M}}$ of a natural quotient mapping h of M is a T_1 normal space.*

Proof of Lemma 1: For any $x \in M$, $h^{-1}(h(x)) = \cup\{h_n^{-1}(h_n(x)) : n \in \omega\}$. For each $i \in \omega$, $h_{n+i}^{-1}(h_{n+i}(x)) \cap M_n = h_n^{-1}(h_n(x)) \cap M_n$, therefore $h^{-1}(h(x)) \cap M_n = h_n^{-1}(h_n(x)) \cap M_n$. The latter set is closed in M_n, hence $h^{-1}(h(x))$ is closed in M and $M_{/\mathcal{M}}$ is a T_1 space.

Let F, G be disjoint closed subsets of M such that $h^{-1}(h(F)) = F$, $h^{-1}(h(G)) = G$. Let O_0 and U_0 be functionally disjoint in B_0 neighborhoods of $h_0(F_0)$ and $h_0(G_0)$ respectively. The sets $V_0 = h_0^{-1}(O_0) \cap M_0$ and $W_0 = h^{-1}(U_0) \cap M_0$ satisfy the following conditions for $n = 0$:
(1) $h_n^{-1}(h_n(V_n)) \cap M_n = V_n$, $h_n^{-1}(h_n(W_n)) \cap M_n = W_n$;
(2) $F_n \subset V_n$ and $G_n \subset W_n$ where $F_n = F \cap M_n$ and $G_n = G \cap M_n$;
(3) $\overline{h_n(V_n)}^{B_n} \cap \overline{h_n(W_n)}^{B_n} = \emptyset$;
(4) $V_n \supset V_{n-1}$ and $W_n \supset W_{n-1}$ for all $n \geq 1$.

Let V_n, W_n be constructed for all $n < k$, $k \geq 1$, and satisfy (1)–(4). By (3) $h_{k-1,k}(\overline{h_{k-1}(V_{k-1})}^{B_{k-1}}) \cap h_{k-1,k}(\overline{h_{k-1}(W_{k-1})}^{B_{k-1}}) = \emptyset$. From the definition of F and G and by (1), (2) $h_{k-1,k}(\overline{h_{k-1}(V_{k-1})}^{B_{k-1}}) \cap h_k(G) = \emptyset$ and $h_k(F) \cap h_{k-1,k}(\overline{h_{k-1}(W_{k-1})}^{B_{k-1}}) = \emptyset$, then $\overline{h_k(V_{k-1} \cup F_k)}^{B_k} \cap \overline{h_k(W_{k-1} \cup G_k)}^{B_k} = \emptyset$, and these sets have functionally disjoint in B_k neighborhoods O_k and U_k respectively. Let $V_k = h_k^{-1}(O_k) \cap M_k$, $W_k = h_k^{-1}(U_k) \cap M_k$. V_k and W_k satisfy (1)–(4) for $n = k$, therefore the construction of V_n, W_n can be carried out for all $n \in \omega$.

Now let $V = \cup\{V_k : k \in \omega\}$ and $W = \cup\{W_k : k \in \omega\}$. V and W are open in M since M is an inductive limit of M_n. By (1) $h^{-1}(h(V)) = V$ and $h^{-1}(h(W)) = W$; by (2) $F \subset V$ and $G \subset W$. Lemma 1 is proved. □

Let $M = g_{i-1}(\mathcal{K}')$ and $M_n = g_{i-1}(G_n)$. Let h_n be a natural quotient mapping for the decomposition $\mathcal{M}_n$ of the space $g_{i-1}(\mathcal{K}')$, where for $x \in M_n$, $x\mathcal{M}_n y \Leftrightarrow xE_i'y$ and for $x \notin M_n$, $x\mathcal{M}_n y \Rightarrow x = y$. Since any element of $\mathcal{M}_n$ is a subset of some element of $\mathcal{M}_{n+1}$, the composition mapping $h_{n-1,n} = h_n \circ h_{n-1}^{-1}$ also is a quotient mapping. $M = g_{i-1}(\mathcal{K}')$ is an inductive limit of compacta M_n since $\mathcal{K}'$ is an inductive limit of compacta G_n' and g_{i-1} is a quotient mapping. Since $\mathcal{M}_{n|M_n} \equiv \mathcal{M}_{n+1|M_n}$, $h_{n,n+1|h_n(M_n)}$ is a homeomorphism for any $n \in \omega$. All conditions of the lemma are satisfied, therefore h maps M onto a normal space $\mathcal{M}_{/M_n} \equiv E'_{i/M_n}$, $n \in \omega$. $\cup\{M_n : n \in \omega\} = M = g_{i-1}(\mathcal{K}')$ and $M = Dom(\mathcal{M})$, $g_{i-1}(\mathcal{K}') = Dom(E_i')$, thus $\mathcal{M} \equiv E_i'$ and the quotient mappings H and g_i (which are generated by $\mathcal{M}$ and E_i') coincide. Therefore $g_i(\mathcal{K}')$ is a T_1 normal space. Let us prove properties (b)–(e). For $U_{i0} = g_i(q(\tilde{f}^{-1}[0; i + \frac{4}{3})))$ and $U_{ij} = g_i(q(\tilde{f}^{-1}(b_{a_{ij}} - \frac{1}{3}; b_{a_{ij}} + \frac{1}{3})))$ for $j \geq 1$, the family $\{U_{in} : n \in \omega\}$ satisfies (b). Equality $D_{i+1} = \cup\{D'_{i+1,n} : n \in \omega\}$ and (c) follow from (b) and the fact that each subset $D'_{i+1,n}$ is compact, and therefore $g_i(D'_{i+1,n})$ is closed in $g_i(\mathcal{K}')$. Each $D'_{j,n}$ is compact and $E'_{i|D'_{j,n}}$ is a trivial decomposition into singletons, therefore (d) is true. (e) follows from (b)–(d).

Therefore, $g_{i-1,i}$ and g_i can be constructed for all $i \in \omega$ and satisfy (a)–(e). Let us prove (f) and (g) for $i \geq 1$. $B_i = g_i(\mathcal{K}) = g_i(G_i')$, hence B_i is compact. Map $g_{i,i+1}$ is defined by the decomposition E'_{i+1}, $E'_{i+1|B_i}$, which is a decomposition into singletons, therefore $g_{i,i+1|B_i}$ is a homeomorphism.

Now let $M = \mathcal{K}'$, $h_n = g_n$, $h_{n,n+1} = g_{n,n+1}$ and $M_n = D_n$ for $n \in \omega$. Conditions of the lemma follows from (f), (g). The resulting mapping g is defined by the decomposition E' of $\mathcal{K}$: $xE'y \Leftrightarrow g_i(x) = g_i(y)$ for some $i \in \omega$, and g maps $\mathcal{K}'$ onto a T_1 regular σ-compact space.

The conclusion of Theorem 1 follows from the following properties:

(h) $B_i \subset g_i(\mathcal{X}')$;

(k) $g_{i|\mathcal{X}'}$ is a condensation.

Assume the contrary to (h). Then there is the minimal $i_0 \in \omega$ such that for some $x \in \mathcal{C}_{i_0} \setminus \mathcal{X}$, $g_{i_0}(q(x)) \neq g_i(x')$. If $i_0 = a_{i_0 k_0}$ and $x \in \tilde{H}_{j_0,k_0}$, then $\tilde{\psi}_{i_0 k_0}(x) \in \mathcal{C}_{i_0}$, $j_0 < i_0$ and by the assumption $g_{i_0}(q(x)) \in g_{i_0}(q(\mathcal{C}_{j_0} \setminus \mathcal{X})) \subset g_{i_0}(x')$. That contradicts the minimality of i_0. If $x \notin \cup\{\tilde{H}_{jk} : j < j_0, k \in \omega\}$, then $x \in \mathcal{C}_{i_0 j_0}$ for some $j_0 \in \omega$. Since $\psi_{i_0\ j_0+1}$ maps $H_{i_0 j_0}$ onto $\mathcal{C}_{i_0 j_0}$ and from the definition of E'_{i_0}, $g_{i_0}(q(x)) \subset g_{i_0}(q(H_{i_0\ j_0+1})) \subset g_{i_0}(x')$ and (h) is proved.

Suppose it is proved that $g_{i|\mathcal{X}'}$ is a condensation for all $i < k$, $k \in \omega$. Since $g_k = g_{k-1,k} \circ g_{k-1}$, it is sufficient to prove that $g_{k-1,k|g_{k-1}(\mathcal{X}')}$ is a condensation.

By (d) $g_{k|D'_{kj}}$ is a homeomorphism for any $i \in \omega$. It is sufficient to prove that for any $j_0, j_1 \in \omega$, $0 < j_0 < j_1$, and $x_0 \in D_{k,j_0} \cap \mathcal{X}$, $x_1 \in D_{k,j_1} \cap \mathcal{X}$ and $y \in D_{k_0} \cap \mathcal{X}$ the following inequalities hold: $g_k(q(x_0)) \neq g_k(q(x_1)) \neq g_k(q(y)) \neq g_k(q(x_0))$. $\psi_{k,j_0}(x_0) \in \mathcal{C}_{k,j_0-1}$, $\psi_{k,j_1}(x_1) \in \mathcal{C}_{k,j_1-1}$, therefore $g_k(q(x_0)) \neq g_k(q(x_1))$ since $\mathcal{C}_{k,j_0-1} \cap \mathcal{C}_{k,j_1-1} = \emptyset$. From the definition of ψ_{ij}, $\tilde{\psi}_{kj_0}$ maps $D'_{kj_0} \cap \mathcal{X}$ in $\mathcal{C}_{k,j_0-1} \in D'_{k_0} \setminus \mathcal{X}$ and $\tilde{\psi}_{kj_1}$ maps $D'_{kj_1} \cap \mathcal{X}$ in $\mathcal{C}_{k,j_1-1} \in D'_{k_1} \setminus \mathcal{X}$. Hence other inequalities also hold. □

A cardinal μ is called τ-measurable, if there is a τ-centered ultrafilter on μ, so the Ulam-measurable cardinals are exactly those which are ω-centered. The same method allows us to prove the following

Theorem 2. *Let μ_0 be a non τ-measurable cardinal and for every $k \in \omega$, $\mu_{k+1} = exp(\mu_k)$ and $\mu = sup\{\mu_k : k \in \omega\}$. Let X_0 be a Tychonoff non-pseudocompact space and $\{X_\alpha : 1 \leq \alpha \leq \mu\}$ be a family of spaces such that $ext(X_\alpha) \geq \tau$ for $1 \leq \alpha < \tau$ and $|X_\alpha| < \mu$ for $0 \leq \alpha < \mu$. Then $\prod\{X_\alpha : \alpha < \mu\}$ can be condensed onto a regular σ-compact space.*

2. A case of τ-compact spaces

For any cardinal τ, let $\tilde{\tau}$ be the set of all isolated ordinals less then τ. A space X is called τ-compact if each of its subsets of cardinality τ has a complete accumulation point in X. For any space X, a compactification cX, and cardinals τ_1, τ_2 let $M(X, cX, \tau_1, \tau_2) = ((\tau_1+1)\times(\tau_2+1)\times cX)\setminus(\tilde{\tau}_1\times\tilde{\tau}_2\times(cX\setminus X))$. This construction is related to the space $((\tau+1)\times\beta X)\setminus(\tau\times(\beta X\setminus X))$ for certain X and τ which was described by R. Buzyakova in [4].

We have shown in Section 1 that for many spaces X there are certain powers μ, which depend on X, such that X^μ can be condensed onto a σ-compact space. The original space can be as bad as we wish and fail all the properties of σ-compact spaces. Thus, in that situation condensations can improve topological properties of powers. In this section we prove somewhat reverse result by producing examples of good spaces M whose (small) powers are so bad that they cannot even be improved by condensations. Let μ be an ordinal, and let τ_i, $i = 1, 2, 3, 4$, be cardinals which depend on τ and on the size of X as it is stated in Theorem 3. We denote $M = M(X, cX, \tau_1, \tau_2) \bigoplus M(X, cX, \tau_3, \tau_4)$ and $M_\nu \approx M$ for $\nu < \mu$. M consists of a compact "skeleton" $K = \{[((\tau_1+1)\times(\tau_2+1))\setminus(\tilde{\tau}_1\times\tilde{\tau}_2)] \bigoplus [((\tau_3+1)\times(\tau_4+1))\setminus(\tilde{\tau}_3\times\tilde{\tau}_4)]\}\times cX$ and of many clopen copies of X. If $f : M^\mu \to Z$ is a condensation, then $f_{|K^\mu}$ is a homeomorphism since K^μ is compact. K^μ is only a part of M^μ, but the copies of X are inserted in M in such a way that this restriction influences the whole map F and we can ultimately find clopen copies X_ν of X in M_ν for all $\nu < \mu$ such that f restricted to $\prod\{X_\nu : \nu < \mu\}$ is a homeomorphism onto a closed subset of Z. Now suppose that X^μ is not normal (paracompact, etc.). Then Z is not normal (paracompact, etc.) either. This means that M^μ cannot be condensed onto a normal (paracompact, etc.) space.

The fact that M is good itself when X is so follows from Lemma 2. Hence M is the desired example.

Lemma 2. *Let X be a Tychonoff space and let cX be a compactification of X. Let $M = M(X, cX, \tau_1, \tau_2) \bigoplus M(X, cX, \tau_3, \tau_4)$ for some cardinals τ_i, $i = 1, 2, 3, 4$. Then M is normal (τ-paracompact, realcompact) iff X is so and M^μ is pseudocompact iff X^μ is so.*

Let a property $\mathcal{P}$ be invariant of continuous mappings, of inverse perfect mappings and suppose $\mathcal{P}$ is inherited by clopen subsets. Then M^μ satisfies $\mathcal{P}$ iff so does X^μ. In particular, $l(M^\mu) = \tau$ (M^μ is τ-initially compact, σ-compact, τ is regular and M^μ is τ-compact, respectively) iff the same is true for X^μ.

Proof: $K = \{[((\tau_1+1)\times(\tau_2+1))\setminus(\tilde{\tau}_1\times\tilde{\tau}_2)] \bigoplus [((\tau_3+1)\times(\tau_4+1))\setminus(\tilde{\tau}_3\times\tilde{\tau}_4)]\}\times cX$ is compact and any neighborhood of K in M contains a neighborhood U such that $M \setminus U$ is a union of finitely many clopen copies of X. This proves the first part of the lemma.

$K_1 = ((\tau_1 + 1) \times (\tau_2 + 1)) \bigoplus ((\tau_3 + 1) \times (\tau_4 + 1))$ is compact and $K_1 \times X$ is dense in M. Therefore $(K_1)^\mu \times X^\mu$ is dense in M^μ. Some clopen subset of M^μ can be projected onto X. By these reasons M^μ is pseudocompact iff so is X^μ.

The space $M/(K \times cX)$ is obtained from M by identifying a closed subset $K \times cX$ to a single point (see [5]). $K \times cX$ is compact, so the corresponding quotient map $q : M \to M/(K \times cX)$ is perfect. Let p be a restriction of q to $K_1 \times X$, then $p(K_1 \times X) = q(M)$. Let p_α, q_α be the α-th "copies" of p, q, $\alpha < \mu$ and $\mathbf{p} = \Delta\{p_\alpha : \alpha < \mu\}$, $\mathbf{q} = \Delta\{q_\alpha : \alpha < \mu\}$, then $M^\mu = \mathbf{q}^{-1}(\mathbf{p}((K_1 \times X)^\mu))$. □

Theorem 3. *Let X^μ be τ-compact and let τ, τ_i be regular cardinals, $i = 1, 2, 3, 4$, such that $\tau_1 > \tau_2 > \tau_3 > \tau_4 > max\{|cX|, \tau\}$. Then for $M = M(X, cX, \tau_1, \tau_2) \bigoplus M(X, cX, \tau_3, \tau_4)$, $Y = M^\mu$ and any condensation $f : Y \to Z$ there is a closed subset F of Y homeomorphic to X^μ such that $f_{|F}$ is a homeomorphism onto a closed subset of Z. Also, any continuous function on $f(F)$ that can be extended to a function on $(cX)^\mu$ (when $f(F)$ is naturally embedded in $(cX)^\mu$) can be extended on Z. In particular, if X^μ is pseudocompact and $cX = \beta X$, then $f(F)$ is C-embedded in Z.*

Proof: Assume that $cf(\mu) \neq \tau_1, \tau_2$. Let $Y = \prod\{Y_\alpha : \alpha < \mu\}$, where each Y_α is homeomorphic to M. We denote $\tilde{Y} = \beta Y$, $\tilde{Z} = \beta Z$; $\tilde{f}$ is a continuous extension of f from $\tilde{Y}$ to $\tilde{Z}$. For any $\alpha < \mu$, let $\pi_\alpha : Y \to Y_\alpha$ be a projection and let $\tilde{\pi}_\alpha$ be its extension from $\tilde{Y}$ onto $\tilde{Y}_\alpha = \beta Y_\alpha$. For $y \in \tilde{Y}_\alpha$ and $i = 1, 2, 3$, $\phi_i(y)$ is a projection onto $(\tau_1 + 1)$, $(\tau_2 + 1)$ or cX respectively if $y \in \overline{M(X, cX, \tau_1, \tau_2)}^{\tilde{Y}_\alpha}$ or onto $(\tau_3 + 1)$, $(\tau_4 + 1)$ or cX respectively if $y \in \overline{M(X, cX, \tau_3, \tau_4)}^{\tilde{Y}_\alpha}$. For $\alpha < \mu$ and $i = 1, 2, 3$, we denote $\psi_{\alpha,i} = \phi_i \circ \tilde{\pi}_\alpha$ and $\psi_3 = \Delta\{\psi_{\alpha,3} : \alpha < \mu\}$. For any combination i, j of indexes $1, 2, 3$, let $\phi_{ij} = \phi_i \Delta \phi_j$ and $\psi_{\alpha,ij} = \phi_{ij} \circ \tilde{\pi}_\alpha$. For $(\alpha, \beta) \in \tau_1 \times \tau_2$, let $Y_{\alpha\beta} = \{y \in \tilde{Y} : if\ \psi_{\gamma,3}(y) \in cX \setminus X$ for some $\gamma < \mu$, then $\psi_{\gamma,12}(y) = (\alpha, \beta)\}$. If $\gamma < \mu$ then let $Y^\gamma_{\alpha\beta} = \{y \in Y_{\alpha\beta} : \psi_{\gamma,3}(y) \in cX \setminus X\}$.

Now let $\gamma < \mu$ be fixed. For any $\beta' \in \tilde{\tau}_2$, let $A_{\beta'} = \{y \in Y^{\gamma}_{\alpha\beta'} : \alpha \in \tilde{\tau}_1$ and there is $y' \in Y^{\gamma}_{\alpha\beta'} \cup Y$ such that $\psi_{\gamma,3}(y) \neq \psi_{\gamma,3}(y')$ and $\tilde{f}(y) = \tilde{f}(y')\}$. Let $\tau' = max\{\tau, |cX|\}^+$, we claim that $|\{\psi_{\gamma,1}(A_{\beta'})\}| < \tau'$. For, assume the contrary. Then there is a monotonically increasing mapping ϕ from τ' in $\tilde{\tau}_1$, a point $c \in cX \setminus X$, sets $A = \{y_\delta : \delta < \tau'\}$ and $A' = \{y'_\delta : \delta < \tau'\}$ and a neighborhood U of c in $\tau_2 \times cX$ such that for any $\delta < \tau'$, $y_\delta \in Y^{\gamma}_{\phi(\delta)\beta'}$, $y'_\delta \in Y^{\gamma}_{\phi(\delta)\beta'} \cup Y$, $\psi_{\gamma,23}(y_\delta) = c$, $\psi_{\gamma,23}(y'_\delta) \notin U$, and $\tilde{f}(y_\delta) = \tilde{f}(y'_\delta)$ (it's all possible because $\psi_{\gamma,23}(A_{\beta'}) \subset \{\beta'\} \times cX$ and $\{\beta'\} \times cX$ is open in $\tau_2 \times cX$, so $\psi_{\gamma,23}(A_{\beta'})$ has a base of cardinality $\leq cX < \tau'$ in $\tau_2 \times cX$). For any $y_\delta \in A$, let $\tilde{y}_\delta$ be such a point from Y that for any $\nu < \mu$, $\pi_\nu(\tilde{y}_\delta) = \tilde{\pi}_\nu(y_\delta)$ if $\tilde{\pi}_\nu(y_\delta) \in Y_\nu$, otherwise let $\psi_{\nu,23}(\tilde{y}_\delta) = \psi_{\nu,23}(y_\delta)$ and $\psi_{\nu,1}(\tilde{y}_\delta) = \psi_{\nu,1}(y_\delta) + \omega$. Let $\tilde{A} = \{\tilde{y}_\delta : \delta < \tau'\}$. In the same way the set $\tilde{A}' = \{\tilde{y}'_\delta : \delta < \tau'\}$ is defined. The set $\{(\tilde{y}_\delta, \tilde{y}'_\delta) \in Y \times Y : \delta < \tau'\}$ has a complete accumulation point (a, a') in $Y \times Y$ ($Y \times Y \approx Y$ is τ-compact). From the constructions of $\tilde{A}$ and $\tilde{A}'$ from A and A', (a, a') is also a complete accumulation point of $\{(y_\delta, y'_\delta) \in \tilde{Y} \times \tilde{Y} : \delta < \tau'\}$, so from the continuity of f $f(a) = f(a')$. But $\psi_{\gamma,23}(a) \notin U$, so $a \neq a'$ — contradiction to the fact that f is a condensation. So $|\psi_{\gamma,1}(A_{\beta'})| \leq \tau \times |cX| < \tau_1$ and, since $\tau_2 < \tau_1$, there is an ordinal $\nu_\gamma < \tau_1$ such that $\psi_{\gamma,1}(A_{\beta'}) \subset \nu_\gamma$ for any $\beta' \in \tilde{\tau}_2$.

In the same way, for any $\gamma < \mu$ and $\alpha' < \tau_1$ there is an ordinal $\beta^{\gamma}_{\alpha'} < \tau_2$ such that $\psi_{\gamma,2}(A_{\alpha'}) \subset \beta^{\gamma}_{\alpha'}$ where $A_{\alpha'} = \{y \in Y^{\gamma}_{\alpha'\beta} : \beta \in \tilde{\tau}_2$ and there is $y' \in Y^{\gamma}_{\alpha'\beta} \cup Y$ such that $\psi_{\gamma,3}(y) \neq \psi_{\gamma,3}(y')$ and $\tilde{f}(y) = f(y')\}$.

Since $cf(\mu) \neq \tau_1$, there is $\tilde{\alpha} < \tau_1$ and $\Gamma_1 \subset \mu$ such that $|\Gamma_1| = \mu$ and for any $\gamma \in \Gamma_1$, $\nu_\gamma \leq \tilde{\alpha}$. Since also $cf(\mu) \neq \tau_2$, there is $\tilde{\beta} < \tau_2$ and $\Gamma_2 \subset \Gamma_1$ such that $|\Gamma_2| = \mu$ and for any $\gamma \in \Gamma_2$, $\beta^{\gamma}_{\tilde{\alpha}+1} \leq \tilde{\beta}$. Now let $y \in Y$; for any $\gamma \in \Gamma_2$ we define $F_\gamma = (\tilde{\alpha} + 1) \times (\tilde{\beta} + 1) \times X$ and for any $\gamma \in \mu \setminus \Gamma_2$, $F_\gamma = \pi_\gamma(y)$. The set $F = \prod\{F_\gamma : \gamma \in \mu\}$ is homeomorphic to X^μ and $f_{|F}$ is a homeomorphism onto a closed subset $f(F)$ of Z. Let g be a continuous function on $(cX)^\mu$ and let h be a map from $\overline{F}^{\tilde{Y}}$ onto $(cX)^\mu$ such that $h(y) = \{\psi_{\gamma,3}(y) : \gamma \in \Gamma_2\}$, $y \in \overline{f}^{\tilde{Y}}$. Then $h \circ f^{-1}{}_{|f(F)}$ is a natural embedding of $f(F)$ in $X^\mu \subset (cX)^\mu$ by the properties of $f_{|F}$. Since $\tilde{f}(h^{-1}(x_1)) \cap \tilde{f}(h^{-1}(x_2)) = \emptyset$ for $x_1 \neq x_2$, $x_1, x_2 \in (cX)^\mu$ by the choice of F, $h \circ f^{-1}$ is a continuous function from $\overline{f(F)}^{\tilde{Z}}$ onto $(cX)^\mu$. Therefore g can be lifted to a continuous function on $\overline{f(F)}^{\tilde{Z}}$ and extended to a function on $\tilde{Z}$.
If $cf(\mu) = \tau_1$ or $cf(\mu) = \tau_2$, all the preceding arguments remain valid if τ_1 and τ_2 are replaced everywhere with τ_3 and τ_4 respectively. □

Corollary 1. a. *For any Tychonoff space X and any cardinal ν there is a larger space M which preserves many properties of X listed in Lemma 2 and* *such that for any $\mu \leq \nu$ and a condensation $f : M^\mu \to Z$, Z contains a closed subset homeomorphic to X^μ; if X^μ is pseudocompact, then this subset is also C-embedded in Z. In particular, M^μ cannot be condensed onto a normal (Lindelöf, σ-compact, etc.) space if X^μ is not normal (Lindelöf, σ-compact, etc.).*

b. *If X is countably compact in all powers or if there is a $|X|$-measurable cardinal, then M satisfies the above properties for all ν.*

PROOF: **a.** Let $\tau = |\beta X^\nu|^+$ and $\tau_1 = \tau^+$, $\tau_{i+1} = \tau_i^+$, $i = 1, 2, 3$. Clearly, X^μ is τ-compact for any $\mu \leq \nu$, so $M = M(X, \beta X, \tau_1, \tau_2) \bigoplus M(X, \beta X, \tau_3, \tau_4)$ is a required space.

b. If X is countably compact in all powers, let $\tau = |\beta X|^+$, $\tau_1 = \tau^+$, and for $i = 1, 2, 3$, $\tau_{i+1} = \tau_i^+$. Then $M = M(X, \beta X, \tau_1, \tau_2) \bigoplus M(X, \beta X, \tau_3, \tau_4)$ is as desired. If τ is the first $|X|$-measurable cardinal, then all powers of X are τ-compact, hence for $\tau_1 = \tau^+$, $\tau_{i+1} = \tau_i^+$, $i = 1, 2, 3$, $M = M(X, \beta X, \tau_1, \tau_2) \bigoplus M(X, \beta X, \tau_3, \tau_4)$ is as required. □

Corollary 2. *For any infinite compactum K there is a normal space X such that $X \times K$ cannot be condensed onto a normal space.*

PROOF: Let Y be a Dowker space and $\tau = max\{|\beta Y|, |K|\}^+$, $\tau_1 = \tau^+$, $\tau_{i+1} = \tau_i^+$, $i = 1, 2, 3$. The space $X = M(Y, \beta Y, \tau_1, \tau_2) \bigoplus M(Y, \beta Y, \tau_3, \tau_4)$ is normal by Lemma 2. $X \times K$ cannot be condensed onto a normal space by Theorem 3 since $X \times K = M(Y \times K, \beta Y \times K, \tau_1, \tau_2) \bigoplus M(Y \times K, \beta Y \times K, \tau_3, \tau_4)$. □

From Theorem 1 and Corollary 1 we derive the following

Corollary 3. *The following are equivalent:*

(1) *for any Tychonoff non-pseudocompact space X there is μ such that X^μ can be condensed onto a normal space;*
(2) *for any Tychonoff non-pseudocompact space X there is μ such that X^μ can be condensed onto a regular σ-compact space;*
(3) *there is no measurable cardinal.*

Acknowledgments. The author is grateful to A.V. Arhangel'skii for attention to this paper.

References

[1] Arhangel'skii A.V., *Some problems and lines of investigation in general topology*, Comment. Math. Univ. Carolinae **29.4** (1988), 611–629.
[2] Bourbaki N., *General Topology*, Addison-Wesley, 1966.
[3] Buzyakova R.Z., *On the product of normal spaces* (in Russian), Vestnik Moskov. Univ. Ser. 1 Mat. Mekh. 1994, no. 5, 81–82; translation in Moscow Univ. Math. Bull. **49.5** (1994), 52–53.
[4] Buzyakova R.Z., *On the condensation of Cartesian products onto normal spaces* (in Russian), Vestnik Moskov. Univ. Ser. 1 Mat. Mekh. 1996, no. 1, 17–19; translation in Moscow Univ. Math. Bull. **51.1** (1996), 13–14.
[5] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[6] Kelley J., *General Topology*, Springer-Verlag, New York, 1975.
[7] Kunen K., *Set Theory*, North-Holland, Amsterdam, 1980.
[8] Kuratowski K., *Topology, Vol. 2*, Academic Press, New York, 1968.
[9] Pytkeev E.G., *The upper bounds of topologies* (in Russian), Mat. Zametki **20** (1976), 489–500; translation in Math. Notes **20** (1976), 831–837.
[10] Yakivchik A.N., *On tightenings of a product of finally compact spaces* (in Russian), Vestnik Moskov. Univ. Ser. 1 Mat. Mekh. 1989, no. 4, 84–86; translation in Moscow Univ. Math. Bull. **44.4** (1989), 86–88.

Ohio University, Athens, OH 45701, USA

(*Received* October 6, 1997, *revised* June 1, 1998)