Absolute countable compactness of products and topological groups

Yan-Kui Song

Abstract. In this paper, we generalize Vaughan's and Bonanzinga's results on absolute countable compactness of product spaces and give an example of a separable, countably compact, topological group which is not absolutely countably compact. The example answers questions of Matveev [8, Question 1] and Vaughan [9, Question (1)].

Keywords: compact, countably compact, absolutely countably compact, hereditarily absolutely countably compact, topological group

Classification: 54D20, 54B10, 54D55

§1. Introduction

By a space, we mean a topological space. Matveev [7] defined a space X to be *absolutely countably compact* (= *acc*) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a finite subset $F \subseteq D$ such that $\mathrm{St}(F,\mathcal{U}) = X$, where $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. He also defined a space X to be *hereditarily absolutely countably compact* (= *hacc*) if all closed subspaces of X are acc. Obviously, all compact spaces are hacc and all hacc spaces are acc. Moreover, it is known ([7]) that all acc spaces are countably compact (cf. also [5]). For an infinite cardinal κ, a space X is called *initially κ-compact* if every open cover of X with the cardinality $\leq \kappa$ has a finite subcover. The purpose of this paper is to prove Theorem 1 and Theorem 2 below.

Theorem 1. *Let κ be an infinite cardinal. Let X be an initially κ-compact T_3-space, Y a compact T_2-space with $t(Y) \leq \kappa$ and A a closed subspace of $X \times Y$. Assume that $A \cap (X \times \{y\})$ is acc for each $y \in Y$ and the projection $\pi_Y : X \times Y \to Y$ is a closed map. Then, the subspace A is acc.*

Vaughan [11] proved that

(i) if X is an acc T_3-space and Y is a sequential, compact T_2-space, then $X \times Y$ is acc, and

(ii) if X is an ω-bounded, acc T_3-space and Y is a compact T_2-space with $t(Y) \leq \omega$, then $X \times Y$ is acc.

Further, Bonanzinga [2] proved that the above theorems (i) and (ii) remain true if "acc" is replaced by "hacc". In Section 2, we prove Theorem 1 and show that Vaughan's theorems (i), (ii) and Bonanzinga's theorems are deduced from

Theorem 1. Matveev [8] asked if there exists a separable, countably compact, topological group which is not acc. Vaughan [10] asked the same question and showed that the answer is positive if there is a separable, sequentially compact T_2-group which is not compact. Form this point of view, he also asked if there exists a separable, sequentially compact T_2-group which is not compact. Theorem 2 below, which is a joint work with Ohta, answers the former question positively and show that the latter question has a positive answer under extra set theoretic assumptions. The latter question remains open in ZFC. Let $\mathfrak{s}$ denote the splitting number, i.e., $\mathfrak{s} = \min\{\kappa : \text{the power } 2^\kappa \text{ is not sequentially compact}\}$ (cf. [3, Theorem 6.1]).

Theorem 2 (Ohta-Song)**.** *There exists a separable, countably compact T_2-group which is not acc. If $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$, then there exists a separable, sequentially compact T_2-group which is not acc.*

It was shown in the proof [3, Theorem 5.4] that the assumption that $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$ is consistent with ZFC. Theorem 2 will be proved in Section 3.

Remark 1. Matveev kindly informed Ohta that a similar theorem to Theorem 2 above was proved independently by W. Pack in his Ph. D. thesis at the University of Oxford (1997).

For a set A, $|A|$ denotes the cardinality of A. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. Other terms and symbols will be used as in [4].

§2. Proof of Theorem 1 and corollaries

Throughout this section, κ stands for an infinite cardinal. For a set A, let $[A]^{\le\kappa} = \{B : B \subseteq A, |B| \le \kappa\}$ and $[A]^{<\kappa} = \{B : B \subseteq A, |B| < \kappa\}$. Let A be a subset of a space X. Arhangel'skii [1] defined the *κ-closure* of A in X by $\kappa\text{-cl}_X A = \cup\{\text{cl}_X B : B \in [A]^{\le\kappa}\}$. A subset A is said to be *κ-closed* in X if $A = \kappa\text{-cl}_X A$. By the definition, $\kappa\text{-cl}_X A$ is κ-closed in X. We omit an easy proof of the following lemma.

Lemma 3. *Let X be a space. Then, $t(X) \le \kappa$ if and only if every κ-closed set in X is closed.*

Lemma 4. *Let X and Y be spaces such that $\pi_Y : X \times Y \to Y$ is a closed map. Then, $\pi_Y(A)$ is κ-closed in Y for each κ-closed set A in $X \times Y$.*

Proof: Let A be a κ-closed set in $X \times Y$. To show that $\pi_Y(A)$ is κ-closed in Y, let $y \in \kappa\text{-cl}_Y \pi_Y(A)$. Then, there is $B \in [\pi_Y(A)]^{\le\kappa}$ such that $y \in \text{cl}_Y B$. Choose a point $\langle x_z, z\rangle \in A$ for each $z \in B$ and let $C = \{\langle x_z, z\rangle : z \in B\}$. Since $C \in [A]^{\le\kappa}$ and A is κ-closed in $X \times Y$, $\text{cl}_{X\times Y} C \subseteq A$. Since $\pi_Y(C) = B$ and π_Y is closed, then $y \in \text{cl}_Y B = \pi_Y(\text{cl}_{X\times Y} C) \subseteq \pi_Y(A)$. Hence, $\kappa\text{-cl}_Y(\pi_Y(A)) = \pi_Y(A)$. □

Proof of Theorem 1: The proof is a slight variation of Vaughan's proofs [11, Theorems 1.3 and 1.4]. Suppose on the contrary that A is not acc. Then, there exist an open cover $\mathcal{U}$ of A and a dense subset D of A such that $A \not\subseteq \mathrm{St}(B,\mathcal{U})$ for each $B \in [D]^{<\omega}$. Since $X \times Y$ is initially κ-compact, A is initially κ-compact, which implies that $A \not\subseteq \mathrm{St}(B,\mathcal{U})$ for each $B \in [D]^{\leq\kappa}$. For each $B \in [D]^{\leq\kappa}$, define $F_B = \pi_Y(A \setminus \mathrm{St}(B,\mathcal{U}))$. Since π_Y is closed, F_B is closed in Y. Thus, $\mathcal{F} = \{F_B : B \in [D]^{\leq\kappa}\}$ is a filter base of closed subsets in Y. By compactness of Y, there exists a point $y \in \bigcap\{F_B : B \in [D]^{\leq\kappa}\}$. Let $L = A \cap (X \times \{y\})$. Then,

$$(1) \qquad L \not\subseteq \mathrm{St}(B,\mathcal{U}) \text{ for each } B \in [D]^{\leq\kappa}.$$

Further, let $K = (\kappa\text{-cl}_{X\times Y} D) \cap (X \times \{y\})$. We show that K is not dense in L. To show this, suppose that K is dense in L. Since L is acc by the assumption, there is $E \in [K]^{<\omega}$ such that $L \subseteq \mathrm{St}(E,\mathcal{U})$. For each $p \in E$, since $p \in K \subseteq \kappa\text{-cl}_{X\times Y} D$, there is $A_p \in [D]^{\leq\kappa}$ such that $p \in \mathrm{cl}_{X\times Y} A_p$. Let $B_0 = \cup\{A_p : p \in E\}$. Then, $B_0 \in [D]^{\leq\kappa}$ and $L \subseteq \mathrm{St}(E,\mathcal{U}) \subseteq \mathrm{St}(B_0,\mathcal{U})$, which contradicts (1). Hence, K is not dense in L. Thus, we can find an open set V in X such that

$$(2) \qquad (V \times \{y\}) \cap A \neq \emptyset$$

and $(V \times \{y\}) \cap (\kappa\text{-cl}_{X\times Y} D) = \emptyset$. Since X is a T_3-space, we may assume that

$$(3) \qquad (\mathrm{cl}_X V \times \{y\}) \cap (\kappa\text{-cl}_{X\times Y} D) = \emptyset.$$

Let $Z = \pi_Y((\mathrm{cl}_X V \times Y) \cap (\kappa\text{-cl}_{X\times Y} D))$. Since π_Y is closed, it follows from Lemma 4 that Z is κ-closed in Y. Since $t(Y) \leq \kappa$, Z is closed in Y by Lemma 3. Moreover, $y \notin Z$ by (3). Hence, there is a neighborhood W of y in Y such that $W \cap Z = \emptyset$. By (2), there is a point $\langle x, y\rangle \in (V \times \{y\}) \cap A$. Since

$$\pi_Y^{-1}(W) \cap ((\mathrm{cl}_X V \times Y) \cap (\kappa\text{-cl}_{X\times Y} D)) = \emptyset,$$

$(V \times W) \cap D = \emptyset$. Since $V \times W$ is a neighborhood of $\langle x, y\rangle \in A$, this contradicts the fact that D is dense in A. □

The following corollary directly follows from Theorem 1.

Corollary 5. *Let X be an initially κ-compact, acc (resp. hacc) T_3-space and Y a compact T_2-space with $t(Y) \leq \kappa$. Assume that $\pi_Y : X \times Y \to Y$ is a closed map. Then, $X \times Y$ is acc (resp. hacc).*

Since an acc space is countably compact (i.e., initially ω-compact), we have the following corollary from Corollary 5:

Corollary 6. *Let X be an acc (resp. hacc) T_3-space and Y a compact T_2-space with $t(Y) \leq \omega$. Assume that $\pi_Y : X \times Y \to Y$ is a closed map. Then, $X \times Y$ is acc (resp. hacc).*

It is known (cf. [4, Theorem 3.10.7]) that if X is countably compact and Y is sequential, then $\pi_Y : X \times Y \to Y$ is closed. Hence, we have the following corollary, which is Vaughan's theorem (i) stated in the introduction and Bonanzinga's theorem [2, Theorem 1.1]:

Corollary 7 (Vaughan [11] and Bonanzinga [2]). *Let X be an acc (resp. hacc) T_3-space and Y a sequential, compact T_2-space. Then, $X \times Y$ is acc (resp. hacc).*

Recall that a space X is *κ-bounded* if $\mathrm{cl}_X A$ is compact for each $A \in [X]^{\leq\kappa}$. It is known (cf. [9]) that all κ-bounded spaces are initially κ-compact, and Kombarov [6] proved that if X is κ-bounded and $t(Y) \leq \kappa$, then $\pi_Y : X \times Y \to Y$ is closed. Hence, we have the following corollary, which generalizes Vaughan's theorem (ii) stated in the introduction and Bonanzinga's theorem [2, Theorem 2.1].

Corollary 8. *Let X be a κ-bounded, acc (resp. hacc) T_3-space and Y a compact T_2-space with $t(Y) \leq \kappa$. Then, $X \times Y$ is acc (resp. hacc).*

§3. Proof of Theorem 2

We give two lemmas before proving Theorem 2.

Lemma 9. *Let X be a space and Y a space having at least one pair of disjoint non-empty closed subsets. Assume that $X \times Y^\kappa$ is acc for an infinite cardinal κ. Then, X is initially κ-compact.*

Proof: Let $\mathcal{U} = \{U_\gamma : \gamma < \kappa\}$ be an open cover of X. By the assumption, there are disjoint non-empty closed subsets E and F of Y. Let $D = \{f \in Y^\kappa : |\{\alpha < \kappa : f(\alpha) \notin E\}| < \omega\}$; then D is dense in Y^κ. Let $V = Y \setminus E$ and $I = F^\kappa$. For each $A \in [\kappa]^{<\omega}$, let $V_A = \bigcap_{\alpha \in A} \pi_\alpha^{-1}(V)$, where $\pi_\alpha : Y^\kappa \to Y$ is the α-th projection. Then, V_A is an open neighborhood of I in Y^κ. Let $\mathcal{V} = \{V_A : A \in [\kappa]^{<\omega}\}$. Observe that, for each $f \in D$, $f \in V_A$ implies that $A \subseteq \{\alpha < \kappa : f(\alpha) \notin E\}$. This means that $\mathcal{V}$ is point-finite at each point of D. Enumerate the family $\mathcal{V}$ as $\{V_\gamma : \gamma < \kappa\}$ and let $\mathcal{W} = \{U_\gamma \times V_\gamma : \gamma < \kappa\} \cup \{(X \times Y^\kappa) \setminus (X \times I)\}$. Since $I \subseteq V_\gamma$ for all $\gamma < \kappa$, $\mathcal{W}$ is an open cover of $X \times Y^\kappa$. Since $X \times Y^\kappa$ is acc, there exists a finite subset M of $X \times D$ such that $X \times Y^\kappa = \mathrm{St}(M, \mathcal{W})$. Let $J = \{\gamma < \kappa : (U_\gamma \times V_\gamma) \cap M \neq \emptyset\}$. Then, $X \times I \subseteq \bigcup\{U_\gamma \times V_\gamma : \gamma \in J\}$. Since $\mathcal{V}$ is point-finite at each point of D, J is finite. Hence, $\mathcal{U}$ has a finite subcover $\{U_\gamma : \gamma \in J\}$. □

We consider $2 = \{0, 1\}$ the discrete group of integers modulo 2. Then, 2^κ is a topological group under coordinatewise addition. The following lemma seems to be well known (see [9, 3.5] for the first statement), but we include it here for the sake of completeness.

Lemma 10. *There exists a separable, countably compact, non-compact subgroup G_1 of $2^{\mathfrak{c}}$. If $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$, then there exists a separable, sequentially compact, non-compact subgroup G_2 of 2^{ω_1}.*

Proof: For each $S \subseteq 2^{\mathfrak{c}}$, we define a subgroup $G(S)$ of $2^{\mathfrak{c}}$ as follows: Choose an accumulation point x_A of A in $2^{\mathfrak{c}}$ for each $A \in [S]^\omega$. Define $G(S)$ to be the smallest subgroup of $2^{\mathfrak{c}}$ including the set $S \cup \{x_A : A \in [S]^\omega\}$. Note that if $|S| \leq \mathfrak{c}$, $|G(S)| \leq \mathfrak{c}$. By transfinite induction, we can define $S_\alpha \subseteq 2^{\mathfrak{c}}$ for each $\alpha < \omega_1$ as follows: Let S_0 be a countable dense subset of $2^{\mathfrak{c}}$. Now, assume that $0 < \alpha < \omega_1$ and S_β has been defined for all $\beta < \alpha$. If α is a limit, let $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$. If $\alpha = \beta + 1$, let $S_\alpha = G(S_\beta)$. Define $G_1 = \bigcup_{\alpha<\omega_1} S_\alpha$. Then, G_1 is a separable, countably compact subgroup of $2^{\mathfrak{c}}$. Since $|G_1| = \mathfrak{c}$, G_1 is a proper dense subset of $2^{\mathfrak{c}}$. Hence, G_1 is not compact. Next, assume that $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$. The construction of G_2 is similar to that of G_1. The only difference is in the definition of x_A. Since $\omega_1 < \mathfrak{s}$, 2^{ω_1} is sequentially compact. Hence, we can choose x_A as a limit point of a sequence in A. Then, $G_2 = \bigcup_{\alpha<\omega_1} S_\alpha$ becomes sequentially compact. Since $|G_2| = \mathfrak{c}$ and $2^\omega < 2^{\omega_1}$, G_2 is not compact. □

Proof of Theorem 2: Let G_1 be the group in Lemma 10. Then, $G_1 \times 2^{\mathfrak{c}}$ is a separable, countably compact T_2-group. Since G_1 is not compact and $w(G_1) \leq \mathfrak{c}$, G_1 is not initially $\mathfrak{c}$-compact. Hence, it follows from Lemma 9 that $G_1 \times 2^{\mathfrak{c}}$ is not acc. Next, assume that $2^\omega < 2^{\omega_1}$ and $\omega_1 < \mathfrak{s}$, and let G_2 be the group in Lemma 10. Since $\omega_1 < \mathfrak{s}$, 2^{ω_1} is sequentially compact. Hence, $G_2 \times 2^{\omega_1}$ is a separable, sequentially compact T_2 group which is not compact. Since $w(G_2) = \omega_1$, $G_2 \times 2^{\omega_1}$ is not acc by Lemma 9. □

Acknowledgments. The author would like to thank H. Ohta for his allowing him to include Theorem 2 in this paper. He is also indebted to K. Chiba for her helpful suggestions.

References

[1] Arhangel'skii A.V., *On bicompacta hereditarily satisfying Suslin's condition tightness and free sequences*, Soviet Math. Dokl. **12** (1971), 1253–1257.
[2] Bonanzinga M., *On the product of a compact space with an hereditarily absolutely countably compact space*, Comment. Math. Univ. Carolinae **38** (1997), 557–562.
[3] van Douwen E.K., *The Integer and Topology*, Handbook of Set-theoertic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 111–167.
[4] Engelking R., *General Topology, Revised and completed edition*, Heldermann Verlag, Berlin, 1989.
[5] Fleishman W.M., *A new extension of countable compactness*, Fund. Math. **67** (1970), 1–9.
[6] Kombarov A.P., *On the product of normal spaces*, Soviet. Math. Dokl. **13** (4) (1972), 1068–1071.
[7] Matveev M.V., *Absolutely countably compact spaces*, Topology Appl. **58** (1994), 81–92.
[8] Matveev M.V., *A countably compact topological group which is not absolutely countably compact*, Questions Answers Gen. Topology **11** (1993), 173–176.
[9] Stephenson R.M., Jr., *Initially κ-Compact and Related Space*, Handbook of Set-theoertic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 603–632.
[10] Vaughan J.E., *A countably compact, separable space which is not absolutely countably compact*, Comment. Math. Univ. Carolinae **34** (1995), 197–200.

[11] Vaughan J.E., *On the product of a compact space with an absolutely countably compact space*, Annals of New York Acad. Soc. **788** (1996), 203–208.
[12] Vaughan J.E., *Countably Compact and Sequentially Compact Spaces*, Handbook of Set-theoertic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 569–602.

Department of Mathematics, Faculty of Science, Shizuoka University, 836 Ohya, Shizuoka, 422-8529 Japan

(*Received* April 8, 1998, *revised* July 2, 1998)