Vanishing of sections of vector bundles on 0-dimensional schemes

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Abstract. Here we give conditions and examples for the surjectivity or injectivity of the restriction map $H^0(X,F) \to H^0(Z,F\,|\,Z)$, where X is a projective variety, F is a vector bundle on X and Z is a "general" 0-dimensional subscheme of X, Z union of general "fat points".

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Let F be a rank r vector bundle on a projective variety X, F spanned by its global sections. Hence the pair $(F, H^0(X,F))$ induces a morphism f from X to the Grassmannian $G(r,v)$, $v := h^0(X,F)$, of r-dimensional quotients of $H^0(X,F)$; the morphism f is uniquely determined, up to a choice of a basis of $H^0(X,F)$. The geometry of $f(X)$ depends heavily on the rank of the restriction map $r_{F,Z} : H^0(X,F) \to H^0(Z,F\,|\,Z)$ for suitable 0-dimensional subschemes of X. For instance the existence of hyperosculating points of $f(X)$ or the existence of high order degenerate points for the differential of f may be translated in terms of $r_{F,z}$ for suitable Z. In this paper we study rank$\,(r_{F,Z})$ for a general union of so-called "fat points". The reader may find in [G], [H3], [I1l], [I2] and [AH] references and motivations for the line bundle case. We just remark that this is a generalization of the following interpolation problem: how many "functions" (belonging to a fixed finite-dimensional vector space of "functions") are there with given Taylor expansion (up to a certain prescribed order) at a certain number of points? What happens if the points are general? We will show that often $r_{F,Z}$ has maximal rank, i.e. it is injective or surjective.

Let X be an integral projective variety, m an integer > 0 and $P \in X_{reg}$. Set $n := \dim(X)$. The $(m-1)$-th infinitesimal neighborhood of P in X will be denoted with mP; hence mP has $(\boldsymbol{I}_{X,P})^m$ as ideal sheaf. Often mP is called a fat point; m is the multiplicity of mP and $(n+m-1)!/(n!(m-1)!) = mP = h^0(mP, \boldsymbol{O}_{mP})$ its degree. If $s, m_1, \dots, m_s$ are integers > 0 and $P_1, \dots, P_s$ are distinct points of X_{reg} the 0-dimensional scheme $Z := \bigcup_{1\le i\le s} m_iP_i$ is called a multi jet of X with multiplicity $\max\{m_i\}$, type $(s; m_1, \dots, m_s)$ and degree $h^0(Z, \boldsymbol{O}_Z)$. For a fixed type $(s; m_1, \dots, m_s)$ the set of all multi-jets of type $(s; m_1, \dots, m_s)$ on X is an

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integral variety of dimension ns. Hence we may speak of the general multi-jet of type $(s; m_1, \dots, m_s)$.

Fix a vector bundle E on X and a very ample $L \in \mathrm{Pic}(X)$. For every integer $m > 0$ consider the following property (Condition $(\$; m)$ or Property $(\$; m)$) which the triple (X, E, L) may have:

Condition ($): There is an integer $a(m, X, E, L)$ such for all integers $k \geq a(m, X, E, L)$ and all types $(s; m_1, \dots, m_s)$ with multiplicity $\leq m$ a general multi-jet Z of type $(s; m_1, \dots, m_s)$ the restriction map $r_{E\otimes L^{\otimes k}, Z} : H^0(X, E \otimes L^{\otimes k}) \to H^0(Z, E \otimes L^{\otimes k} \mid Z)$ has maximal rank.

We say that the triple (X, E, L) satisfies Condition ($) (or that it has Property ($)) if (X, E, L) satisfies $(\$; m)$ for all $m > 0$. In the range of integers in which we will consider the restriction map $r_{E\otimes L^{\otimes k}, Z}$ we will have $H^i(X, E \otimes L^{\otimes k}) = 0$ for $i > 0$ and hence if $H^0(X, E \otimes L^{\otimes k})$ has maximal rank, then its rank will be either $\deg(Z)$ or $\chi(E \otimes L^{\otimes k})$ (which is uniquely determined by k and the numerical invariants of X, E and L).

In Section 2 we will prove the following criterion "reduction to the restriction to a general curve section" to obtain Property ($) for a triple (X, E, L) on a variety of dimension > 1.

Theorem 0.1. *Fix integers $n > 0$, $m > 0$ and $r > 0$. Let X be an integral n-dimensional projective variety, E a rank r vector bundle on X and L a very ample line bundle on X. Assume the existence of integers $a_1, \dots, a_{n-1}$ with $a_i > 0$ for all i and with the following property. Take general $D_i(a_i) \in |L^{\otimes a_i}|$. For every integer k with $1 \leq k \leq n-1$ set $D[k; a_1, \dots, a_k] := \bigcap_{1\leq i\leq k} D_i(a_i)$. Assume that $E \mid D[n-1; a_1, \dots, a_{m-1}]$ satisfies Condition ($). Assume that r divides both $a := \deg(L)$ and $p_a(D[n-1; 1, \dots, 1]) - 1$. Assume that (X, E, L) satisfies Condition $(\$; 1)$. Then (X, E, L) satisfies Condition $(\$; m)$.*

The proof of Theorem 0.1 will use heavily the proofs in [AH]. In our opinion the paper [AH] was a revolution on this topic: it contains an extremely powerful improvement of a method previously introduced by the authors, the statements proved there are very interesting and the loose ends left for the reader are very stimulating. In Section 3 we will show for a huge number of Chern classes the existence of rank 2 reflexive sheaves on $\mathbf{P}^3$ with Property ($). Using heavily the results and proofs of [H2] we will prove the following theorem.

Theorem 0.2. *Fix integers c_1, c_2 and c_3 with $c_1, c_2 \equiv c_3 \bmod (2)$, $0 \leq c_3 \leq 4c_2 - {c_1}^2 - 4$. If $4c_2 - {c_1}^2 = 7$ or 15, assume $c_3 \neq 0$. If c_1 is even and c_2 is odd, assume $c_3 \leq 4c_2 - {c_1}^2 - 6$. Then there exists a rank 2 stable reflexive sheaf F on $\mathbf{P}^3$ with $c_i(F) = c_i$ for $i = 1, 2, 3$ and with Property ($). Furthermore, if $c_3 = 0$ and c_1 is even, then Condition ($) is satisfied by the general stable bundle in the irreducible component of the moduli space of rank 2 vector bundles with Chern classes c_1 and c_2 containing the real instanton bundles.*

In the first section we will consider briefly the case in which X is a smooth curve. We work over an algebraically closed field K. In Sections 2 and 3 we will assume char$(\mathbf{K}) = 0$. It is impossible to follow the proof of Theorem 0.1 (resp. 0.2) without having on the table a copy of [AH] (resp. [H2]).

1. Vector bundles on curves

In this section we consider the case in which the variety is a smooth projective curve C of genus $g \geq 0$ and we do not make any restriction on char$(\mathbf{K})$. By the classification of line bundles and vector bundles on curves of genus ≤ 1, everything is well known for $g \leq 1$. We will repeat here the classification to show its relation with Property ($) and that we need to make strong cohomological restrictions to be sure that a vector bundle of rank > 1 has Property ($).

Example 1.1. Every vector bundle F on $\mathbf{P}^1$ is the direct sum of line bundles, say $F \cong \boldsymbol{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \boldsymbol{O}_{\mathbf{P}^1}(a_r)$ with $a_1 \geq \cdots \geq a_r$, and the isomorphism class of F is uniquely determined by the integers $a_1, \ldots, a_r$. For every effective divisor Z of $\mathbf{P}^1$ with deg$(Z) = z$, we have $h^0(\mathbf{P}^1, \boldsymbol{I}_Z \otimes \boldsymbol{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \boldsymbol{O}_{\mathbf{P}^1}(a_r)) = \sum_{1 \leq i \leq r} \max\{a_i + 1 - z, 0\}$. Hence $\boldsymbol{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \boldsymbol{O}_{\mathbf{P}^1}(a_r)$ has Property ($) if and only if $a_1 = a_r$, i.e. if and only if it is semistable. Furthermore F has Property ($\$; m$) for some integer $m \geq 1$ if and only if it is semistable.

Example 1.2. By Atiyah's classification of vector bundles on an elliptic curve X ([A]) every vector bundle on X is a direct sum of semistable vector bundles and a vector bundle on X has Property ($) if and only if it has Property ($\$, m$) for some integer $m \geq 1$ and this is the case if and only if it is semistable.

From now on we assume $g \geq 2$. It is easy to check (see [N, Lemma 2.6]) that for any integer $s \geq g$ and any choice of s non-zero integers $a_1, \ldots, a_s$ the map $\tau : C^{(a_1)} \times \cdots \times C^{(a_1)} \to \mathrm{Pic}^a(C)$, $a := \sum_{1 \leq i \leq s} a_i$, given by $\tau((P_1, \ldots, P_s)) := \boldsymbol{O}_C(\sum_{1 \leq i \leq s} a_i P_i)$ is surjective. Hence the original asymptotic problem for the vector bundle E is equivalent to the fact that for every integer x and for a general $M \in \mathrm{Pic}^x(C)$, either $h^0(C, E \otimes M) = 0$ or $h^1(C, E \otimes M) = 0$. This problem was considered for the first time by Raynaud ([R]), at least when deg(E) is divisible by rank(E); the general case may easily be reduced to this case using elementary transformations. This condition (call it Condition (R) or Property (R)) is obviously satisfied if rank$(E) = 1$. If Condition (R) is true for E, then E must be semistable. If E is a stable bundle with rank 2, then E satisfies Condition (R) (see [R, Proposition 1.6.2], and use elementary transformations to reduce the case deg(E) odd to the case deg(E) even considered in [R]). If E is a general stable bundle (for its degree and rank), then E satisfies Condition (R) (see [R, Proposition 1.8.1] if rank(E) divides deg(E) and use elementary transformations to reduce the general case to the case considered in [R] or, if char$(\mathbf{K}) = 0$, see [H1, Theorem 1.2], for much more). If E has a Krull-Schmidt filtration whose graded subquotients have the same slope and satisfy Condition (R), then E satisfies Condition ($); for instance this is the case if E has rank 2 and it is semistable but not stable. For every smooth curve C of genus $g \geq 2$ and for every integer $x \geq 2$ there is a semistable bundle E of rank x^g without Property (R) (see [R, 3.1]); obviously at least one of the stable subquotients of E in a Krull-Schmidt filtration of E cannot have Property (R).

2. Proof of Theorem 0.1

In this section we prove Theorem 0.1.

Remark 2.1. By the adjunction formula we have $2p_a(D[n-1;1,\dots,1])-2 = K\cdot L\cdot\ldots\cdot L+\deg(L)$. Hence (again by the adjunction formula or by the genus formula for reducible curves) if r divides both $\deg(L)$ and $p_a(D[n-1;1,\dots,1])-1$, then it divides $p_a(D[n-1;b_1,\dots,b_{n-1}])-1$ for all integers $b_i>0$. If $L\cong A^{\otimes r}$ for some $A\in \mathrm{Pic}(X)$ and either $\dim(X)\geq 3$ or r odd, then this divisibility condition is satisfied. If r is even and $\dim(X)=2$ the divisibility condition is satisfied if $L\cong A^{\otimes 2r}$ for some $A\in \mathrm{Pic}(X)$.

Remark 2.2. Assume $r=2$. If $E\mid D[n-1;a_1,\dots,a_{n-1}]$ satisfies Condition (\$), then obviously $E\mid D[n-1;a_1,\dots,a_{n-1}]$ must be semistable (see Section 1). If $D[n-1;a_1,\dots a_{n-1}]$ is smooth (i.e. if X is smooth in codimension ≤ 1) and $E\mid D[n-1;a_1,\dots,a_{n-1}]$ is stable and "sufficiently general" or with low rank (say $r\leq 2$), then $E\mid D[n-1;a_1,\dots,a_{n-1}]$ satisfies Condition (\$) by the discussion in Section 1. It is easy to check that the same is true even if $D[n-1;a_1,\dots,a_{n-1}]$ is singular. By the theory of semistability for reduced but reducible curves made in [HK] if $E\mid D[n-1;1,\dots,1]$ is semistable or stable, then $E\mid D[n-1;a_1,\dots,a_{n-1}]$ has the same property (see [HK, Theorem 2.4]).

Proof of Theorem 0.1: By induction on n we may assume that for all integers k and a_i with $1\leq k\leq n-1$ the triple $(D[k;a_1,\dots,a_k], E\mid D[k;a_1,\dots,a_k], L\mid D[k;a_1,\dots,a_k])$ satisfies Condition $(\$;m)$. By the divisibility condition all the calculations and constructions made in [AH, § 3, 4, 5, 6 and 7], work verbatim, just inserting a factor r in some of the estimates; however, to help the reader we will give a few details trying to use the language and, when not conflicting with previous use, the notations of [AH]. Section 3 of [AH] is just nomenclature; we just have to assume that in any (a,m)-configuration we want to use and in any (d,m,a)-candidate we want to use both the number of free points and the number of G_a-residues are divisible by r. Lemma 3.2 of [AH] follows just from the asymptotic estimate for $h^0(X,L^{\otimes d})$ for $d\gg 0$; as remarked in [AH], beginning of page 11 during the proof of 1.1 (the case $M\neq \boldsymbol{O}_X$), the same is true for $h^0(X,M\otimes L^{\otimes d})$, $M\in \mathrm{Pic}(X)$, M fixed; in our situation instead of M we have the rank r vector bundle E and this gives that the same asymptotic estimates for $\deg(\mathrm{Free}(Z))$ holds: the expected contribution of every zero-dimensional scheme is r times its length, while asymptotically, up to terms of order d^{n-1} (d^n in the notations of [AH] because their ambient variety has dimension $n+1$) we have $h^0(X,E\otimes L^{\otimes d})\approx r(h^0(X,L^{\otimes d}))$. Section 4 of [AH] just contains [AH, Lemma 4.2]; this lemma holds in our situation (with both the degree of free points and of the concentrated derivatives divisible by rank (E)) because its proof uses only [AH, Lemma 3.2], whose extension was discussed before. As remarked in the first lines of [AH, § 5], this would be sufficient (plus the corresponding assertion in lower dimension) if one could start the inductive procedure on X with respect to the degree of the zero-dimensional subscheme on X, i.e. if one had proved the theorem for varieties of dimension $\dim(X)$ but for zero-dimensional schemes of low degree; concerning [AH, § 5], we just need to use the concept of "concentrated derivative" and extend [AH, Lemma 5.2]; for this extension we need only that all integers $h^0(G_1, E\otimes L^{\otimes d} \mid G_1)$ are divisible by $\mathrm{rank}(E)$ to be sure that at each step both the numbers of free points on G_1 (resp. G_{a-1}) and the number of derivatives on G_1 (resp. G_{a-1}) are divisible by $\mathrm{rank}(E)$; see Remark 2.1 for this assertion; if instead of $G_1 \cup G_{a-1}$ we fix an integer α with $0 < \alpha < a$ and consider $G_\alpha \cup G_{a-\alpha}$ the same divisibility condition is satisfied for all cohomology groups appearing in [AH, § 6]. Section 7 of [AH] contains the reduction of [AH, Theorem 1.1], i.e. of our Theorem 0.1, to the proof of [AH, Proposition 7.1]. The discussion with a vector bundle E instead of $M \in \mathrm{Pic}(X)$ works because every relevant integer appearing therein is (under our assumptions) divisible by $\mathrm{rank}(E)$. Then the proof of the reduction of [AH, 1.1] to [AH, 7.1] goes on by induction on $\dim(X)$. The starting point of the induction on $\dim(X)$, i.e. the case of a curve ([AH, Proposition 7.2]) is one of the assumptions of Theorem 0.1. To conclude the proof it remains to justify the vector bundle extension of the key differential lemma [AH, Lemma 2.3]. We will reduce the vector bundle case to the line bundle case (see Lemma 2.3 below). This approach has the advantage that every improvement of [AH, Lemma 2.3] (e.g. any characteristic free proof or any extension to more general base rings) works verbatim. □

Lemma 2.3. *Let X be an integral n-dimensional projective variety over K and F a rank r reflexive sheaf on X whose non locally free locus* $\mathrm{Sing}(F)$ *is finite. Let H be an effective, reduced and irreducible Cartier divisor on X such that $H \cap \mathrm{Sing}(F) = \emptyset$. Let W be a zero dimensional subscheme of X with $W \cap \mathrm{Sing}(F) = \emptyset$, and let a, d be positive integers. Assume $h^0(H, F \mid H) - \deg(W \mid H) = ry \geq 0$ with y integer. Fix y positive integers $m_1, \dots, m_y$ such that $\deg(W) + \sum_{1\leq i\leq y} r(m_i+n)!/m_i!n! \geq h^0(X, F)$. Let $P_1, \dots, P_y$ be generic points of Y and $Q_1, \dots, Q_y$ generic points of H. Let $D_{m_i}(Q_i)$ be the simple residue of m_iQ_i with respect to H and $D := \bigcup_{1\leq i\leq y} D_{m_i}(Q_i)$. Set $Q\{m\} := \sum_{1\leq i\leq y} m_iQ_i$, $T := W \cup (\sum_{1\leq i\leq y} m_iP_i)$, $T' := \mathrm{Res}_H(W) \cup D$ and $T" := (W \mid H) \cup (\bigcup_{1\leq i\leq r} Q_i)$. Assume $H^1(X, \boldsymbol{I}_{Q\{m\}}F(-H)) = H^0(X, \boldsymbol{I}_{T'} \otimes F(-H)) = H^0(H, \boldsymbol{I}_{T''} \otimes (F \mid H)) = 0$. Then $H^0(X, \boldsymbol{I}_T \otimes F) = 0$.*

Proof: Let $\pi : \mathbf{P}(F) \to X$ be the projection. Since $\boldsymbol{O}_{\mathbf{P}(F)}(1)$ is relatively very ample, there is $R \in \mathrm{Pic}(X)$ such that $M := \pi^*(R) \otimes \boldsymbol{O}_{\mathbf{P}(F)}(1)$ is very ample. We take a general complete intersection A of $r-2$ hypersurfaces in the linear system $|M|$ and of an element of $|M^{\otimes r}|$. In particular, we assume that $\pi \mid A$ is étale in a neighborhood of $\pi^{-1}(Q_1 \cup \cdots \cup Q_y)$ and of $\pi^{-1}(W_{\mathrm{red}})$. Set $\{Q_{ij}\}_{1\leq j\leq r} := \pi^{-1}(Q_i) \cap A$. Set $W(\pi) := \pi^{-1}(W) \cap A$ and $H(\pi) := \pi^{-1}(H) \cap A$. Note that $H^0(X, F) \cong H^0(\mathbf{P}(F), \boldsymbol{O}_{\mathbf{P}(F)}(1))$. We want to apply [AH, Lemma 2.3] to $W(\pi)$ and the points Q_{ij}. The points Q_{ij} are not generic on $H(\pi)$ because $\pi(Q_{ij}) = \pi(Q_{it})$ even if $j \neq t$. Nevertheless, the proof of [AH, § 9, 10, 11, 12] works in this situation. However, just the application of the statement of [AH, Lemma 2.3] would give ry generic points $P_{ij} \in A$, while we want points $P'_{ij} \in A$ with $\pi(P'_{ij}) = \pi(P_{it})$ for all i, j, t and generic with this property. This is possible because, since $\pi \mid A$ is étale in a neighborhood of $\pi^{-1}(Q_1 \cup \cdots \cup Q_y)$ we may pass from the formal lemma to an effective degeneration of the points Q_{ij}, $1 \leq j \leq r$, preserving the condition of being in the same fiber of $\pi \mid A$. We take $P_i := \pi(P'_{i1})$ and conclude. □

We state explicitly the last part of the proof of Lemma 2.3, because it seems to be useful even in the rank 1 case.

Remark 2.4. We use the notations of the statements of Lemma 2.3. Assume that a subset S of $\{1, \ldots, y\}$ and every $i \in S$, $Qi \in D_i$ with D_i integral curve intersecting transversally H at Q_i; we allow the case $D_i = D_j$ for some $(i, j) \in S \times S$ with $i \neq j$. Then in the statement of Lemma 2.3 for every $i \in S$ we may take as P_i a general point of D_i.

3. Proof of Theorem 0.2

In this section we consider the case in which $X = \mathbf{P}^3$ and prove Theorem 0.2. Here we prove the existence of rank 2 stable vector bundles (and of non-locally free reflexive sheaves) with Property ($) for a large number of Chern classes c_i, $1 \leq i \leq 3$. For all (c_1, c_2, c_3) covered by the statement of Theorem 0.2 we will show that Condition ($) is satisfied by the general member of the irreducible component, $M(c_1, c_2, c_3)$, of the moduli space of rank 2 stable reflexive sheaves such that in [HH] and [H2] it was proved that a general $E \in M(c_1, c_2, c_3)$ has semi-natural cohomology in the sense of [HH]. Recall that a rank 2 reflexive sheaf E on $\mathbf{P}^3$ has semi-natural cohomology if for all integers $t \geq -2 - c_1(E)/2$ at most one the cohomology groups $H^i(\mathbf{P}^3, E(t))$, $0 \leq i \leq 3$, is not zero.

To explain the proof of Theorem 0.2 and the approach of [HH] and [H2] to the proof of the existence of reflexive sheaves with semi-natural cohomology we will consider first the following toy case.

Proposition 3.1. *Let X be a smooth projective 3-fold, $A, B, L \in \mathrm{Pic}(X)$ with L very ample and a 1-dimensional subscheme of X. Fix an integer $s \geq 0$ and assume that for a general surjection $f : A \otimes L^{\otimes s} \oplus B \otimes L^{\otimes s}$, $\mathrm{Ker}\,(f)$ is the flat limit of a family of reflexive sheaves parametrized by an integral variety. Call F the generic member of this family. By semicontinuity F has a good cohomological property (e.g. Property ($)) if $\mathrm{Ker}\,(f)$ has the same property. We assume that the map $h(f(t)) : H^0(X, A \otimes L^{\otimes(s+t)} \oplus B \otimes L^{\otimes(s+t)}) \to H^0(Y, \boldsymbol{O}_Y \otimes L^{\otimes(s+t)})$ is surjective for all $t \geq 0$, that $h(f(0))$ is bijective and that $h^i(X, A \otimes L^{\otimes(s+t)}) = h^i(X, B \otimes L^{\otimes(s+t)}) = h_i(Y, \boldsymbol{O}_Y \otimes L^{\otimes(s+t)}) = 0$ for every $i > 0$ and every $t \geq 0$. Assume that for all integers $t > 0$, the integers $h^0(X, A \otimes L^{\otimes(s+t)}) - h^0(X, A \otimes L^{\otimes(s+t-1)})$,*

$h^0(X, B\otimes L^{\otimes(s+t)})-h^0(X, B\otimes L^{\otimes(s+t-1)})$ and $h^0(Y, \boldsymbol{O}_Y\otimes L^{\otimes(s+t)})-h^0(Y, \boldsymbol{O}_Y\otimes L^{\otimes(s+t-1)})$ are even; this is always the case if $L\cong M^{\otimes 2}$ for some $M\in \mathrm{Pic}(X)$. Then Ker (f) *and F have Property* (\$) *with respect to L.*

Proof: By semicontinuity it is sufficient to prove that Ker (f) has Property (\$). Let $\mathbf{V}$ be the total space of the vector bundle $A\oplus B$ and call $\pi : \mathbf{V}\to X$ the projection. The surjection $f(0)$ induces an embedding $\mathbf{i} : Y\to \mathbf{V}$. We fix the integer $m>0$, a large integer n (how large it will be clear later), a type $(x; m_1,\dots,m_x)$ for multi-jets with multiplicity $\leq m$ and a generic multi-jet Z of type $(x; m_1,\dots,m_x)$. If $m_x\leq m_i$ for $i\leq x$, we may assume $2\deg(Z)-(m_x+3)(m_x+2)(m_x+1)/6+(m_x+2)(m_x+1)m_x/6 < h^0(X, A\otimes L^{\otimes(s+n)})+h^0(X, B\otimes L^{\otimes(s+n)})-h^0(Y, \boldsymbol{O}_Y\otimes L^{\otimes(s+n)}) = \dim(\mathrm{Ker}(f(n)))\leq 2\deg(Z)+(m_x+3)(m_x+2)(m_x+1)/6-(m_x+2)(m_x+1)m_x/6$. Adding simple points, we will even assume $2\deg(Z)\geq \dim(\mathrm{Ker}(f(n)))$. Then we apply the reduction steps in [AH, § 3, 4, 5 and 6] to reduce the case of multiplicity $\leq m$ to the case of multiplicity $\leq m-1$; here we work on $\pi^{-1}(T)$ with T generic in $|L^{\otimes a}|$ for some $a>0$. The difference with respect to [AH] is that now in the hypersurface $\pi^{-1}(T)$ of $\mathbf{V}$ we have also the $a\cdot\deg(L\,|\,Y)$ points $\pi^{-1}(T)\cap\mathbf{i}(Y)$. Since $Z_{red}\cap T$ is made by generic points of T and card $(Z_{red}\cap T)$ increases with order >1 as function of a, we may apply verbatim the asymptotic estimates in [AH, Lemma 4.2]; here of course we use the parity condition to pass from an assertion concerning Ker $(h(f(n)))$ to an assertion concerning Ker $(h(f(n-a)))$. Then we exploit a general $D\in|L^{\otimes n'}|$ to reduce the assertion to the bijectivity of $f(0)$; again, here we use the parity condition. □

Remark 3.2. In the case $A=B$ the proof of [H2, § 3] shows how to reduce the search of pairs (s, Y) with $h(f(0))$ of maximal rank to the search of curves $Y'\subset X$ with good postulation, i.e. to a problem usually much easier.

Proof of Theorem 0.2: We divide the proof into 4 steps.

Step 1. We follow the notations of the proof of 3.1. Again we reduce to the case $m=1$ (for some integer $n'\leq n$ with $n'-n$ even) taking always generic hypersurfaces $T\in|L^{\otimes a}|$ with a even and degenerating T to the generic union $T'\cup T''$ with $T'\in|L^{\otimes(a-2)}|$, $T''\in|L^{\otimes 2}|$, T' and T'' generic, instead of taking $T'\in|L^{\otimes(a-1)}|$ and $T''\in|L|$. In this way we do not need the parity condition assumed in 3.1 to reduce to the critical case $m=1$.

Step 2. We follow the proof of [H2] and in particular the proofs in [H2, Sections 3, 4, 5 and 6]. We assume $m=1$, i.e. we consider only simple points. We have seen in Step 1 how to reduce the general case $m\geq 1$ to this case without using any parity condition. We do not have a curve, Y, for which a suitable map $f(0)$ (with $\deg(A)=0$ and $\deg(B)=-b$, $0\leq b\leq 3$) is bijective. In [H2] the corresponding scheme Y is the union of a smooth curve Y' and of $h^0(\mathbf{P}^3, A\otimes L^{\otimes s})+h^0(\mathbf{P}^3, B\otimes L^{\otimes s})-h^0(Y', \boldsymbol{O}_{Y'}(s))$ colinear points.

Step 3. If $Y = Y'$ and the corresponding sheaf has Chern classes c_i, then we have won. In the general case there is an integer, e, with $0 \leq e \leq s < s$ (see [H2, § 4, notations 4.0]) for the cases with $b \neq 0$, or integers e_i,$1 = 1, 2$, with $0 \leq e_i \leq s$ for the case $b = 0$ (see [H2, § 3]) and the union Y of suitable collinear points. A sheaf with seminatural cohomology will be associated to the integer s and to a union of integral components of Y' (case in which $H^0(\mathbf{P}^3, F(s)) \neq 0$) or to a curve containing Y' and a line containing the e collinear points (case in which $H^0(\mathbf{P}^3, F(s)) = 0$). We assume $n' > s + (s+1)^2$. This is true (for fixed m) for large n. We have an integer $y \geq 0$, a "suitable" general curve T, a general surjection $f(0) : \boldsymbol{O}_{\mathbf{P}^3}(s) \oplus \boldsymbol{O}_{\mathbf{P}^3}(s-b) \to \boldsymbol{O}_T(s)$; to conclude it would be sufficient to prove that for general $S \subset \mathbf{P}^3$ with card $(S) = y$ the induced map $f(0, W)$: $H^0(\mathbf{P}^3, \boldsymbol{I}_W \otimes \boldsymbol{O}_{\mathbf{P}^3}(s)) \oplus H^0(\mathbf{P}^3, \boldsymbol{I}_W \otimes \boldsymbol{O}_{\mathbf{P}^3}(s)) \to H^0(T, \boldsymbol{O}_T(s))$ has maximal rank. Since the local deformation spaces of the sheaves of type Ker $(f(0))$ is smooth, each of them is a flat limit of reflexive sheaves belonging to the irreducible component $M(c_1, c_2, c_3)$. Hence it is sufficient to check that for some integer $k \geq s$ with $k \leq n'$ there is $A \subset \mathbf{P}^3$, card $(A) = [(h^0(\mathbf{P}^3, \boldsymbol{O}_{\mathbf{P}^3}(k)) + h^0(\mathbf{P}^3, \boldsymbol{O}_{\mathbf{P}^3}(k-b)) - h^0(T, \boldsymbol{O}_T(k)))/2]$ the map $f(k-s, A)$: $H^0(\mathbf{P}^3, \boldsymbol{I}_A \otimes \boldsymbol{O}_{\mathbf{P}^3}(k)) \oplus H^0(\mathbf{P}^3, \boldsymbol{I}_A \otimes \boldsymbol{O}_{\mathbf{P}^3}(k-b)) \to H^0(T, \boldsymbol{O}_T(k))$ is surjective and for some $B \subset \mathbf{P}^3$ with card $(B) =$ card $(A) + 1$ the map $f(k-s, B) : H^0(\mathbf{P}^3, \boldsymbol{I}_B \otimes \boldsymbol{O}_{\mathbf{P}^3}(k)) \oplus H^0(\mathbf{P}^3, \boldsymbol{I}_B \otimes \boldsymbol{O}_{\mathbf{P}^3}(k-b)) \to H^0(T, \boldsymbol{O}_T(k))$ is injective. We start with a good configuration (a curve M union collinear points) for the integer $s-1$ constructed in [H2] (in § 3+b for the integer b, $0 \leq b \leq 3$). Then, instead of using it to obtain a good configuration for the integer s we add over a plane H (i.e. on $\mathbf{V}(\boldsymbol{O}_{\mathbf{P}^2}(-b))$ for $b \neq 0$ and on $\mathbf{P}^2 \times \mathbf{A}^2$ for $b = 0$) general points and a low degree curve which will be a union of components of the curve $T \setminus M$; we do this with the construction with nilpotents described in [H2, 4.5, 5.5 and 6.5]. However, since we may use up to $(s+1)^2 >$ deg $(T) -$ deg (M) steps, we are never forced to use more than 3 nilpotents at each step and hence the arithmetic simplifies drastically.

Step 4. For the last assertion, i.e. that $M(0, c_2, 0)$ contains the real instanton bundles, see the introduction of [HH]. □

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