# Pervasive algebras on planar compacts

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*Abstract.* We characterize compact sets $X$ in the Riemann sphere $\mathbb{S}$ not separating $\mathbb{S}$ for which the algebra $A(X)$ of all functions continuous on $\mathbb{S}$ and holomorphic on $\mathbb{S} \setminus X$, restricted to the set $X$, is pervasive on $X$.

*Keywords:* compact Hausdorff space $X$, the sup-norm algebra $C(X)$ of all complex-valued continuous functions on $X$, its closed subalgebras (called function algebras), pervasive algebras; the algebra $A(X)$ of all functions continuous on $\mathbb{S}$ and holomorphic on $\mathbb{S} \setminus X$

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Let $X$ be a compact Hausdorff topological space. Denote by $C(X)$ the commutative Banach algebra consisting of all continuous complex-valued functions on $X$ (with respect to usual point-wise algebraic operations) endowed with the sup-norm:

$$|f| = \sup_{x \in X} |f(x)|, \quad f \in C(X).$$

By a *function algebra* on $X$ we mean any closed subalgebra of $C(X)$ which contains constant functions on $X$ and which separates points of $X$. (The last property, more precisely, means: whenever $x, y$ is a couple of distinct points in $X$, then there exists a function $f \in A$ such that $f(x) \neq f(y)$.)

A function algebra on $X$ is said to be *pervasive* whenever for any nonvoid proper closed subset $F$ of $X$ the algebra $A/F$ of all restrictions $f/F$ of functions $f \in A$ to the set $F$ is dense in $C(F)$ with respect to $|.|_F$, the sup-norm on $F$.

The notion "pervasive" is due to Hoffman and Singer [1].

In whole the following text let $X$ be a compact subset of the complex plane $\mathbb{C}$ which *does not separate* $\mathbb{C}$, i.e. such that the complement of $X$ in $\mathbb{C}$ *is connected.* Let $A(X)$ be the algebra consisting of all functions which are continuous on the Riemann sphere $\mathbb{S}$ and holomorphic on $\mathbb{S} \setminus X$. It is well known (see Gamelin [2, 1.6]) that if $A(X)$ contains at least one non-constant function, then it separates points of $\mathbb{S}$ (and is a function algebra on $\mathbb{S}$).

By the Maximum Modulus Principle, every function $f \in A(X)$ attains its maximum modulus on $X$; it follows that $\tilde{A}(X) := A(X)/X$, the algebra of all restrictions to the set $X$, is a function algebra on $X$ which is in a natural way isometrically isomorphic to $A(X)$.

Fuka [3] has proved that if $X$ is the two-dimensional Cantor discontinuum, then $\tilde{A}(X)$ is pervasive. His idea is as follows.

Let $F$ be a nonvoid proper closed subset of $X$ and $x \in X \setminus F$. Because $\mathbb{C} \setminus X$ a nd then also $\mathbb{C} \setminus F$ is connected, any function in $C(F)$ is a uniform limit of complex polynomials on the set $F$; it follows from the Mergelyan Theorem (see [2, 9.1]). Now it is enough to approximate the function $Z$, $Z(z) = z$, uniformly on $F$ by functions in $A$. But it follows from the classical Runge Theorem (see Saks and Zygmund [4, 2.1]) that $Z/F$ is on $F$ a uniform limit of a sequence $f_n$, $f_n(z) = p_n(\frac{1}{z-x})$ where $p_n$ are appropriate polynomials. Hence it is sufficient to approximate only the function $R$, $R(z) = \frac{1}{z-x}$, uniformly on $F$ by functions from $\tilde{A}(X)$.

The last part of Fuka's proof is based on the nice Urysohn construction [5]: let us denote by $C_n$ the set of all centres of all $4^n$ partial squares of the rank $n$, in obvious sense, of the two-dimensional Cantor set in $\mathbb{C}$. Then the sequence of functions

$$f_n(z) = \sum_{c \in C_n} \frac{1}{z - c}$$

is pointwise convergent to a continuous nonconstant function $f$ on the whole $\mathbb{S}$ and the convergence is locally uniform in $\mathbb{C} \setminus X$. It follows that $f \in A(X)$. But the Cantor set is a fractal — it is similar, in usual geometrical sense, to its intersection with any partial square of rank $n$. Thus it is possible to construct the Urysohn's sequence in any partial square. Fuka has shown that, whenever $F$ is a closed subset of $X$, $x \in X \setminus F$, we can take a partial square containing $x$ so small that Urysohn's limit function is near to the function $R(z) = \frac{1}{z-x}$ in the norm $|.|_F$.

Now we shall show that this idea holds in a **more general setting**, not only in the case of the two-dimensional Cantor set: whenever $X$ is a compact perfect set in the complex plane with a connected complement which is nowhere dense in $\mathbb{C}$, the existence of a function in $\tilde{A}(X)$ which is rather big in $x$ and rather small on $F$ implies that it is possible to approximate the function $R$ by functions from $A(X)$. More precisely, we shall prove the following

**Theorem.** *Let $X$ be a compact perfect subset of the complex plane $\mathbb{C}$ with connected complement $\mathbb{C} \setminus X$ which is nowhere dense in $\mathbb{C}$ (or, which has empty interior). Let $A(X)$ and $\tilde{A}(X)$ be the algebras defined above. Then the following two properties are equivalent:*

(1) *$\tilde{A}(X)$ is pervasive on $X$;*
(2) *whenever $F$ is a closed subset of $X$ and $x$ a point in $X \setminus F$, then there exists a function $f \in \tilde{A}(X)$ such that*

(∗) $f(x) = 1, \quad |f|_F < 1.$

For the proof we shall use the following Proposition (see [2, 1.8]):

**Proposition.** *Let $f$ be a continuous function on the Riemann sphere $\mathbb{S}$, which is holomorphic on an open subset $U$ of $\mathbb{S}$. Let $z_0 \in \mathbb{S}$. Then there is a sequence*

$\{f_n\}_{n=1}^{\infty}$ *of continuous functions on* $\mathbb{S}$ *such that* $f_n$ *is holomorphic on* $U$, $f_n$ *is holomorphic in a neighbourhood of* $z_0$, *and* $f_n \to f$ *uniformly on* $\mathbb{S}$.

Proof of the Theorem: Let $\tilde{A}(X)$ be pervasive on $X$; take a closed proper subset $F$ and a point $x \in X \setminus F$. Put $H = F \cup \{x\}$; then $H$ is closed in $X$ and it is a proper subset of $X$ because $x$ is not isolated in $X$. The function $h$ which is equal to 1 at $x$ and to 0 on $F$ is continuous on $H$. Then $\tilde{A}(X)$ being pervasive contains a function $g$ such that

$$|h - g|_H < \frac{1}{2}\,.$$

If we put $f = \frac{g}{g(x)}$, then $f$ fulfills $(*)$.

Conversely, suppose that the condition (2) is valid. Let $F$ be a proper closed subset of $X$, $x \in X \setminus F$, $\varepsilon > 0$. From the above considerations it follows that it is enough to approximate the function $R$ where $R(z) = \frac{1}{z-x}$ or to find a function $g \in \tilde{A}(X)$ such that

$$|g - R|_F < \varepsilon.$$

Now put $\eta = \frac{1}{3} \cdot \varepsilon \cdot \operatorname{dist}(x, F)$. Let $f \in \tilde{A}(X)$ be a function satisfying $(*)$; let $n$ be a natural number so great that

$$|f|_F^n < \eta.$$

Denote by $\tilde{f}$ the function in $A(X)$ for which $\tilde{f}/F = f$ and put $v = 1 - \tilde{f}^n$. Then for $v \in A(X)$ we have

$$v(x) = 0, \quad |v - 1|_F < \eta.$$

The existence of a function $w \in A(X)$ which is holomorphic at the point $x$ and satisfies

$$|v - w|_X < \eta$$

follows from Proposition; if we put $\tilde{w} = w - w(x)$ we have $\tilde{w} \in A(X)$, $\tilde{w}(x) = 0$ and

$$|v - \tilde{w}|_X \le |v - w|_X + |w(x)| < 2\eta;$$

moreover $\tilde{w}$ is holomorphic at the point $x$. It follows that the function $g$ defined by $g(z) = \frac{\tilde{w}(z)}{z-x}$ is also in $A(X)$ and for any $z \in F$ we have

$$\begin{aligned} |g(z) - R(z)| &= \left|\frac{\tilde{w}(z)}{z-x} - \frac{1}{z-x}\right| \\ &\le \left|\frac{\tilde{w}(z)}{z-x} - \frac{v(z)}{z-x}\right| + \left|\frac{v(z)}{z-x} - \frac{1}{z-x}\right| < \frac{3\eta}{\operatorname{dist}(x, F)} = \varepsilon, \end{aligned}$$

hence Theorem is proved. □

**Remark** that in the case $X$ is not perfect the notion “to be pervasive” does not make any reasonable sense: in [1] it is proved that for any abstract Hausdorff space $X$ every pervasive function algebra $A$ on $X$ which is a proper part of $C(X)$ is *analytic* which means: whenever a function in $A$ vanishes on a nonvoid open subset it must vanish identically. Then any function which vanishes on the isolated point of $X$ is zero. It follows that $C(X)$ has no proper pervasive function subalgebras.

**Remark also** that the so called *classical disc algebra*, i.e. the algebra $A$ consisting of all restrictions to the unit circle $X$ of all functions continuous on the closed unit disc and holomorphic on its interior, fulfills both conditions of our theorem:

In fact, it follows immediately from the Mergelyan Theorem that $A$ is pervasive on $X$. Let $z_0 \in X$; it is enough to find a function $f \in A$ such that $f(z_0) = 1$, $|f(z)| < 1$ for any $z \in X \setminus \{z_0\}$. Put $f(z) = \frac{1}{2z_0}(z_0 + z)$.

**Remark at last** that the two conditions of the Theorem are **not** equivalent one to the other in general case, neither for $X$ being a compact subset of the complex plane: McKissick [6] has constructed a function algebra $A$ on a planar compact $X$ which is a proper subset of $C(X)$ and which is *normal* on $X$; it means that for any disjoint couple $F, H$ of closed subsets of $X$ there is a function in $A$ which is equal to 1 on $F$ and to 0 on $H$. It is clear that every normal function algebra on $X$ fulfills the condition (2) of the Theorem, but it is not pervasive (because it is not analytic) whenever $X$ contains more than one point.

## References

[1] Hoffman K., Singer I.M., *Maximal algebras of continuous functions*, Acta Math. **103** (1960), 217–241.

[2] Gamelin T.W., *Uniform Algebras*, Prentice Hall, Inc., Englewood Cliffs, N.J., 1969.

[3] Fuka J., *A remark to maximality of several function algebras* (in Russian), Čas. Pěst. Mat. **93** (1968), 346–348.

[4] Saks S., Zygmund A., *Analytic Functions*, Polskie Towarzystwo Matematyczne, Warszawa, 1952.

[5] Urysohn P.S., *Sur une fonction analytique partout continue*, Fund. Math. **4** (1922), 144–150.

[6] McKissick R., *A nontrivial normal sup norm algebra*, Bull. Amer. Math. Soc. **69** (1963), 391–395.

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