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$C_p(I)$ is not subsequential

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Abstract. If a separable dense in itself metric space is not a union of countably many nowhere dense subsets, then its C_p-space is not subsequential.

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0. Introduction

A subspace of a sequential space is called subsequential. Some time ago A.V. Arhangel'skii asked if $C_p(I)$ is subsequential. In [2] the author gave an example of a countable space which is not subsequential but can be embedded as a subspace in $C_p(2^\omega)$. In this note we prove several general propositions concerning non subsequentiality of C_p-spaces. We also give two simple examples of nonsubsequential subspaces of $C_p(2^\omega)$.

Recall that $C_p(X)$ denotes the space of real-valued continuous functions on X with pointwise convergence topology, I denotes the usual segment $[0,1]$. It is well known that $C_p(I)$ is not sequential (see, for example [1]).

The following proposition is in fact due to E.G. Pytke'ev [3].

Proposition 0.1. *Let X be subsequential, $x \notin A$, $x \in \overline{A}$. Then there exists a countable π-network at x of infinite subsets of A, i.e. there exists at x a countable family $\mathcal{A}$ of infinite subsets of A such that each neighbourhood of x contains an element of $\mathcal{A}$.*

1. Propositions

Here we prove that very often a C_p-space is not subsequential.

Proposition 1.1. *Let X be a separable metric space and $\mathcal{P}$ a countable family of infinite subsets of X. Then there exists an open ω-cover $\mathcal{V}$ of X with the property*

($\mathcal{P}_s$). Suppose $\mathcal{K}$ is an infinite subfamily of $\mathcal{V}$, then $\bigcap\{\overline{V} : V \in \mathcal{K}\}$ does not contain any element of $\mathcal{P}$.

Proof: We can assume that the metric d of X is totally bounded, i.e. for every $\delta > 0$ there exists a finite cover of balls of diameter less than δ. Let $\{P_i : i \in \omega\}$ be an enumeration of elements of $\mathcal{P}$. Now we need a very simple

Lemma 1.2. *Suppose M is an infinite subset of a metric space. Then for every $n \in \omega$ there exists $\delta > 0$ such that M cannot be covered by a union of n balls of diameter less than δ.*

Proof of Lemma: Let d be the metric on the space under consideration. As M is infinite, we can find an $N \subset M$, $|N| = n+1$. Then $\delta = \min\{d(x,y) : x, y \in N, x \neq y\}$ is the desired number. $\square$

Further, using this lemma we can construct a decreasing sequence of positive reals δ_i, $i \in \omega$, such that for every $i \in \omega$ every P_k, $k \leq i$, cannot be covered by a union of i closed balls of diameter less than δ_i. Now we find a sequence of finite open covers $\mathcal{W}_i$, $i \in \omega$, of balls of diameter less than δ_i. Further let $\mathcal{V}_i = \{\bigcup T : T \subset \mathcal{W}_i, |T| \leq i\}$. It is clear that $\mathcal{V}_i$ is finite. Let $\mathcal{V} = \bigcup\{\mathcal{V}_i : i \in \omega\}$. Let us prove that $\mathcal{V}$ is an open ω-cover of X with property $(\mathcal{P}_s)$. Let Z be a finite subset of X. Let us take an $i \in \omega$, $i \geq |Z|$. There is an element of $\mathcal{V}_i$ that covers Z. We proved that $\mathcal{V}$ is an ω-cover of X. Now let us finish the proof of Proposition 1.1. Let $P_k \in \mathcal{P}$. If $T \in \mathcal{V}$ and $T \supset P_k$, then $T \in \mathcal{V}_i$ with $i \leq k$. So, there are only finitely many elements of $\mathcal{V}$ that contain the given P_k. The proof of 1.1 is complete. $\square$

Proposition 1.3. *Let X be a separable metric space. Let $\mathcal{P}$ be a countable family of infinite subsets of X. Then $C_p(X)$ has an infinite subspace F, $1 \notin F$, $1 \in \overline{F}$ with the property*

$(\mathcal{P}_c)$. *Suppose K is an infinite subset of F, then $\bigcap\{f^{-1}[1/2, 3/2] : f \in K\}$ does not contain any element of $\mathcal{P}$.*

Proof: Let $\{P_i : i \in \omega\}$ be an enumeration of elements of $\mathcal{P}$ and let $\{V_i : i \in \omega\}$ be an enumeration of elements of $\mathcal{V}$ from Proposition 1.1. It is clear from the proof of Proposition 1.1 that there is a function $f : \omega \to \omega$ such that $P_k \not\subset \overline{V}_i$ if $i \geq f(k)$. For every $i \in \omega$ we can easily construct a real-valued continuous function f_i such that $f_i^{-1}(1) \supset V_i$ and $P_k \not\subset f_i^{-1}[1/2, 3/2]$ for every $i \geq f(k)$. $\square$

Now it remains to check that $F = \{f_i : i \in \omega\}$ is the desired subset of $C_p(X)$.

Proposition 1.4. *Let X be a space which is not a union of countably many nowhere dense subsets, let X have a countable π-network $\mathcal{N}$ of infinite subsets. If $C_p(X)$ has a subspace F from Proposition 1.3 with the property $(\mathcal{N}_c)$, then $C_p(X)$ is not subsequential.*

Proof: We have $1 \in \overline{F}$. Let us prove that 1 has no countable π-network of infinite subsets of F. Let us suppose the contrary and let $\{P_j : j \in \omega\}$ be such a π-net. Let $O_x[1, \epsilon)$ denote a basic neighbourhood of 1 in $C_p(X)$, i.e. $O_x[1, \epsilon) = \{f \in C_p(X) : |f(x) - 1| < \epsilon\}$. Then for every $x \in X$ there is a $j_x \in \omega$ such that $P_{j_x} \subset O_x[1/2, 3/2]$. As X is not a union of countably many nowhere dense subsets, there exist $m \in \omega$ and $X_m \subset X$ such that X_m is not nowhere dense and $m = j_x$ for each $x \in X_m$. It is clear that $\overline{X}_m \subset f^{-1}[1/2, 3/2]$ for every $f \in P_m$, hence $\overline{X}_m \subset \bigcap\{f^{-1}[1/2, 3/2] : f \in P_m\}$. But $Int(\overline{X}_m) \neq \emptyset$, hence

$\overline{X}_m$ contains some element $N' \in \mathcal{N}$. Then $N' \subset (\bigcap\{f^{-1}[1/2, 3/2] : f \in P_m\})$. A contradiction is obtained. □

Combining Propositions 1.1, 1.3, 1.4 we obtain

Theorem 1.5. *If a separable dense in itself metric space is not a union of countably many nowhere dense subsets then its C_p-space is not subsequential.*

PROOF: It is enough to mention that a separable dense in itself metric space has a countable π-network of infinite subsets (moreover it has a countable base of nonempty open subsets which are infinite). □

Corollary 1.6. *$C_p(I)$ is not subsequential, $C_p(Y)$ is not subsequential for a nonscattered compactum Y.*

PROOF: A compactum Y is not scattered iff it maps continuously onto I. But in this case $C_p(I) \subset C_p(Y)$. □

Proposition 1.7 (Compact dixotomy)**.** *A C_p-space over a compactum either is sequential or is not subsequential.*

Corollary 1.8. *If a metric space contains a copy of 2^ω then its C_p-space is not subsequential. In particular, the C_p-space over an uncountable A-set in a metric space is not subsequential.*

Because in these cases C_p-space contains a copy of $C_p(2^\omega)$.

Theorem 1.5 and Corollary 1.8 allow us to raise the following conjecture:

Hypothesis 1.9 (General dixotomy)**.** *A C_p-space either is sequential or is not subsequential.*

2. Two concrete examples

Here we give two examples of nonsubsequential subspaces of $C_p(2^\omega)$.

2.1. The first example. It is the space Z introduced in [4]. We describe it here. Let $\{K_n : n \in \omega\}$ be disjoint finite subsets, $K = \bigcup\{K_n : n \in \omega\}$ and $* \notin K$. Let $Z = \{*\} \cup K$. Let all points of K be isolated and a typical neihgbourhood of $*$ be a set $\{*\} \cup (K \setminus L)$ where $|L \cap K_n| \leq m$ with the same m for every $k \in \omega$. In [2] it is proved that $*$ has no countable π-net of infinite subsets of K and it is proved that Z can be embedded as a subspace in $C_p(2^\omega)$.

2.2. The second example. We will work in 2^ω. Let us follow the general way described in Propositions 1.1, 1.3, 1.4. Let Ω_n denote the set of functions $f : n \to 2$ and let $\Omega = \bigcup\{\Omega_n : n \in \omega\}$. For every $f \in \Omega$ the subset $O(f) = \{x \in 2^\omega : x \supset f\}$ is a basic clopen subset in 2^ω.

For every $n \in \omega$, let $\mathcal{S}_n$ be the family $\{O(f) : f \in \Omega_{2^n}\}$ and $\mathcal{V}_n = \{\bigcup T : T \subset \mathcal{S}_n, |T| = n\}$.

Further, let $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \omega\}$. A little later we will prove that $\mathcal{V}$ is a clopen ω-cover of 2^ω with the following property:

Suppose $\mathcal{K}$ is a infinite subfamily of $\mathcal{V}$, then $Int(\bigcap \mathcal{K}) = \emptyset$.

It implies that a subspace F of characteristic functions of elements of this cover $\mathcal{V}$ is the same as in Proposition 1.3. Hence this subspace demonstrates nonsubsequentiality of $C_p(2^\omega)$.

Now the desired proof. Let Z be a finite subset of 2^ω. Let us take some $n \geq |Z|$. As $\mathcal{S}_n$ covers 2^ω, there is an element of $\mathcal{V}_n$ that contains Z. Now let W be a clopen subset of 2^ω. For our goal we can assume that $W = O(f)$ for some $f \in \Omega_n$. We see that $m(O(f)) = 2^{-n}$ and $m(W) = i * 2^{-2^i}$ for a $W \in \mathcal{V}_i$. Here m denotes Lebesgue measure on 2^ω. Therefore if $W \supset O(f)$ then $i \leq n$, i.e. only finitely many elements of $\mathcal{V}$ contain (f).

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