

## Hopf algebra structure $H^{\sigma-R}$ with two sided invertible 2-cocycle

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*Abstract.* In this paper, we study the  $H^{\sigma-R}$  type Hopf algebras and present its braided and quasitriangular Hopf algebra structure. This generalizes well-known results on  $H^{\sigma}$  and  $H^R$  type Hopf algebras. Finally, the classification of  $H^{\sigma-R}$  type Hopf algebras is given.

*Keywords:* Hopf algebra, 2-cocycle, braided Hopf algebra

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The cocycle deformation  $H^{\sigma}$  of a bialgebra  $H$  was introduced and the Hopf algebra structure and the braided bialgebra structure of  $H^{\sigma}$  were given by Doi [1]. Doi and Takeuchi [2] showed that the Drinfeld double  $D(H)$  for a finite-dimensional Hopf algebra  $H$  is a cocycle deformation of  $(H^*)^{cop} \otimes H$ . This suggests that cocycle deformations will play an important role in quantum group theory. The bialgebra  $H^{\sigma}$  has the same underlying coalgebra  $H$ , so that they have the same corepresentation theory. In fact, these are well known classes of 2-cocycles which appear as the duals of the Reshetikhin-type twists  $H^R$  ([8]) which originate from particular solutions of the QYBE. The Reshetikhin-type twist  $H^R$  also plays an important role in quantum group theory. The bialgebra  $H^R$  has the same underlying algebra  $H$ , so that they have the same representation theory. In this paper we consider the mixed type twist  $H^{\sigma-R}$  and give Hopf algebra structure of  $H^{\sigma-R}$ , and its braided and quasitriangular Hopf algebra structure are also established. This generalizes well-known results on  $H^{\sigma}$  type Hopf algebras and  $H^R$  type Hopf algebras. Finally, the classification of  $H^{\sigma-R}$  type Hopf algebras is given.

Throughout this paper, we refer to [3] and [5] for full details on Hopf algebras, and in particular we use the sigma notation:  $\Delta(h) = \sum h_1 \otimes h_2$ .

Let  $H$  be a  $\sigma$ -Hopf algebra, that is, there is a bilinear map  $\sigma : H \times H \rightarrow k$  such that for all  $x, y, z \in H$  we have

$$(1) \quad \sum \sigma(x_1, y_1)\sigma(x_2y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2z_2),$$

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$$(2) \quad \sigma(x, 1) = \sigma(1, x) = \varepsilon(x)1.$$

If  $\sigma$  is invertible, its inverse is denoted by  $\sigma^{-1} : H \times H \rightarrow k$ , then by [1, Theorem 1.6(a)], we have that for all  $x, y, z \in H$

$$(3) \quad \sum \sigma^{-1}(x_1 y_1, z) \sigma^{-1}(x_2, y_2) = \sum \sigma^{-1}(x, y_1 z_1) \sigma^{-1}(y_2, z_2),$$

$$(4) \quad \sigma^{-1}(x, 1) = \sigma^{-1}(1, x) = \varepsilon(x)1.$$

By [1, Theorem 1.6(b)],  $H$  has a new algebra structure:

$$(A) \quad h \odot l = \sum \sigma(h_1, l_1) h_2 l_2 \sigma^{-1}(h_3, l_3) \quad \text{for all } h, l \in H$$

and  $(H, \odot, \Delta_H)$  is a Hopf algebra.

Dually, let  $H$  be a  $R$ -Hopf algebra, that is, there is an element  $R = \sum R^{(1)} \otimes R^{(2)} \in H \otimes H$  which satisfies the following conditions ( $r = R$ ):

$$(1') \quad \sum R^{(1)} r^{(1)}_1 \otimes R^{(2)} r^{(1)}_2 \otimes r^{(2)} = \sum r^{(1)} \otimes R^{(1)} r^{(2)}_1 \otimes R^{(2)} r^{(2)}_2,$$

$$(2') \quad \sum \varepsilon(R^{(1)}) R^{(2)} = \sum R^{(1)} \varepsilon(R^{(2)}) = 1.$$

If  $R$  is invertible with inverse  $U = \sum U^{(1)} \otimes U^{(2)} \in H \otimes H$ , we have by [8]:  $U = u$

$$(3') \quad \sum U^{(1)} \otimes U^{(2)}_1 u^{(1)} \otimes U^{(2)}_2 u^{(2)} = \sum U^{(1)}_1 u^{(1)} \otimes U^{(1)}_2 u^{(2)} \otimes U^{(2)},$$

$$(4') \quad \sum \varepsilon(U^{(1)}) U^{(2)} = \sum U^{(1)} \varepsilon(U^{(2)}) = 1.$$

By [3],  $H$  has a new comultiplication:

$$(B) \quad \widetilde{\Delta}(h) = \sum R^{(1)} h_1 U^{(1)} \otimes R^{(2)} h_2 U^{(2)} \quad \text{for all } h \in H$$

and  $(H, m_H, \widetilde{\Delta})$  is a Hopf algebra.

**Remark.** Condition (1') was first obtained by setting  $C = k$  in the crossed co-product  $C \times_{\sigma} H$  (see [7]), then we found that condition (1') is dual to condition (1).

**1. Hopf algebra structure of type  $H^{\sigma-R}$**

In this section, we assume that  $H$  is  $\sigma$ -Hopf and  $R$ -Hopf algebra. When  $\sigma$  and  $R$  are compatible, we prove that  $(H, \odot, \tilde{\Delta})$  is a Hopf algebra, and denote it by  $H^{\sigma-R}$ . In the following we always assume that  $\sigma$  has convolution inverse and  $R$  has inverse  $U = \sum U^{(1)} \otimes U^{(2)} \in H \otimes H$ .

**Definition 1.1.** Let  $H$  be a  $\sigma$ -Hopf and  $R$ -Hopf algebra.  $H$  will be called a  $\sigma - R$  compatible Hopf algebra, if  $\sigma$  and  $R$  satisfy the following conditions:

$$R = \tilde{R} = r = \tilde{r}, h, l \in H$$

- (i)  $\sum \sigma(R^{(1)}hr^{(1)}, \tilde{R}^{(1)}l\tilde{r}^{(1)})R^{(2)} \otimes r^{(2)} \otimes \tilde{R}^{(2)} \otimes \tilde{r}^{(2)} = \sigma(h, l)(1 \otimes 1 \otimes 1 \otimes 1)$ ;
- (ii)  $\sum \sigma(R^{(2)}hr^{(2)}, \tilde{R}^{(2)}l\tilde{r}^{(2)})R^{(1)} \otimes r^{(1)} \otimes \tilde{R}^{(1)} \otimes \tilde{r}^{(1)} = \sigma(h, l)(1 \otimes 1 \otimes 1 \otimes 1)$ ;
- (iii)  $\sum \sigma(h, R^{(1)}_2)R^{(1)}_1 \otimes R^{(2)} = \sum \sigma(R^{(1)}_2, h)R^{(1)}_1 \otimes R^{(2)} = \sum (R^{(1)} \otimes R^{(2)})\varepsilon(h)$ ;
- (iv)  $\sum \sigma^{-1}(R^{(1)}h, r^{(1)}l)R^{(2)} \otimes r^{(2)} = \sum \sigma^{-1}(R^{(2)}h, r^{(2)}l)R^{(1)} \otimes r^{(1)} = \sum \sigma(h, l)(1 \otimes 1)$ .

Now we give some properties of  $\sigma - R$  compatible Hopf algebras:

**Proposition 1.2.** Let  $H$  be an  $\sigma - R$  compatible Hopf algebra, then

- (a)  $\sum R^{(1)} \otimes R^{(1)}_1\sigma(R^{(2)}_2, h) = \sum R^{(1)} \otimes R^{(2)}_1\sigma(h, R^{(2)}_2) = \sum (R^{(1)} \otimes R^{(2)})\varepsilon(h)$ ;
- (b)  $\sum \sigma(R^{(1)}_1, l)R^{(1)}_2 \otimes R^{(2)} = \sum \sigma(l, R^{(1)}_1)R^{(1)}_2 \otimes R^{(2)} = \sum (R^{(1)} \otimes R^{(2)})\varepsilon(l)$ ;
- (c)  $\sum \sigma(h, R^{(2)}_1)R^{(2)}_2 \otimes R^{(1)} = \sum \sigma(R^{(2)}_1, h)R^{(2)}_2 \otimes R^{(1)} = \sum (R^{(2)} \otimes R^{(1)})\varepsilon(h)$ ;
- (d)  $\sum \sigma(U^{(1)}hu^{(1)}, \tilde{U}^{(1)}l\tilde{u}^{(1)})U^{(2)} \otimes u^{(2)} \otimes \tilde{U}^{(2)} \otimes \tilde{u}^{(2)} = \sigma(h, l)(1 \otimes 1 \otimes 1 \otimes 1)$ ;
- (e)  $\sum \sigma(U^{(2)}hu^{(2)}, \tilde{U}^{(2)}l\tilde{u}^{(2)})U^{(1)} \otimes u^{(1)} \otimes \tilde{U}^{(1)} \otimes \tilde{u}^{(1)} = \sigma(h, l)(1 \otimes 1 \otimes 1 \otimes 1)$ ;
- (f)  $\sum \sigma(U^{(1)}_1, l)U^{(1)}_2 \otimes U^{(2)} = \sum \sigma(l, U^{(1)}_1)U^{(1)}_2 \otimes U^{(2)} = \sum (U^{(1)} \otimes U^{(2)})\varepsilon(l)$ ;
- (g)  $\sum U^{(1)} \otimes U^{(2)}_1\sigma(U^{(2)}_2, l) = \sum U^{(1)} \otimes U^{(2)}_1\sigma(l, U^{(2)}_2) = \sum (U^{(1)} \otimes U^{(2)})\varepsilon(l)$ .

PROOF: (a)

$$\begin{aligned} & \sum R^{(1)} \otimes R^{(1)}_1\sigma(R^{(2)}_2, h) \\ & \stackrel{(ii)^+(2')}{=} \sum R^{(1)} \otimes r^{(1)}R^{(2)}_1\sigma(r^{(2)}R^{(2)}_2, h) \\ & \stackrel{(1')}{=} \sum r^{(1)}R^{(1)}_1 \otimes r^{(2)}R^{(1)}_2\sigma(R^{(2)}, h) \\ & \stackrel{(ii)^+(2')}{=} \sum R^{(1)} \otimes R^{(2)}\sigma(1, h) \\ & \stackrel{(2)}{=} \sum (R^{(1)} \otimes R^{(2)})\varepsilon(h). \end{aligned}$$

Similarly, we can prove the second formula of (a), (b) and (c).

By (i) and (ii) of Definition 1.1, and  $RU = UR = 1 \otimes 1$ , we can prove (d) and (e).

$$\begin{aligned} & \sum \sigma(U^{(1)}_1, l)U^{(1)}_2 \otimes U^{(2)} \\ \stackrel{(d)+(4')}{=} & \sum \sigma(U^{(1)}_1 u^{(1)}, l)U^{(1)}_2 u^{(2)} \otimes U^{(2)} \\ \stackrel{(3')}{=} & \sum \sigma(U^{(1)}, l)U^{(2)}_1 u^{(1)} \otimes U^{(2)}_2 u^{(2)} \\ \stackrel{(d)+(4')}{=} & \sum \sigma(1, l)U^{(1)} \otimes U^{(2)} \\ \stackrel{(2)}{=} & \sum (U^{(1)} \otimes U^{(2)})_\varepsilon(l). \end{aligned}$$

Analogously, we can prove the second formula of (f) and (g). □

The following proposition can be proved by direct computation.

**Proposition 1.3.** *Let  $H$  be an  $\sigma - R$  compatible Hopf algebra, then we have*

- (h)  $\sum R^{(1)} \otimes R^{(2)}_1 \sigma^{-1}(R^{(2)}_2, l) = \sum R^{(1)} \otimes R^{(2)}_1 \sigma^{-1}(l, R^{(2)}_2) = \sum (R^{(1)} \otimes R^{(2)})_\varepsilon(l);$
- (i)  $\sum R^{(1)}_1 \sigma^{-1}(R^{(1)}_2, l) \otimes R^{(2)} = \sum R^{(1)}_1 \sigma^{-1}(l, R^{(1)}_2) \otimes R^{(2)} = \sum (R^{(1)} \otimes R^{(2)})_\varepsilon(l);$
- (j)  $\sum \sigma^{-1}(R^{(1)}_1, l)R^{(1)}_2 \otimes R^{(2)} = \sum \sigma^{-1}(l, R^{(1)}_1)R^{(1)}_2 \otimes R^{(2)} = \sum (R^{(1)} \otimes R^{(2)})_\varepsilon(l);$
- (k)  $\sum \sigma^{-1}(h, R^{(2)}_1)R^{(2)}_2 \otimes R^{(1)} = \sum \sigma^{-1}(R^{(2)}_1, h)R^{(2)}_2 \otimes R^{(1)} = \sum (R^{(2)} \otimes R^{(1)})_\varepsilon(h);$
- (l)  $\sum U^{(1)} \otimes U^{(2)}_1 \sigma^{-1}(U^{(2)}_2, l) = \sum U^{(1)} \otimes U^{(2)}_1 \sigma^{-1}(l, U^{(2)}_2) = \sum (U^{(1)} \otimes U^{(2)})_\varepsilon(l);$
- (m)  $\sum \sigma^{-1}(U^{(1)}_1, l)U^{(1)}_2 \otimes U^{(2)} = \sum \sigma^{-1}(l, U^{(1)}_1)U^{(1)}_2 \otimes U^{(2)} = \sum (U^{(1)} \otimes U^{(2)})_\varepsilon(l);$
- (n)  $\sum \sigma^{-1}(R^{(2)}, l)R^{(1)} = \sum \sigma^{-1}(l, R^{(2)})R^{(1)} = \varepsilon(l);$
- (o)  $\sum \sigma^{-1}(R^{(1)}, l)R^{(2)} = \sum \sigma^{-1}(l, R^{(1)})R^{(2)} = \varepsilon(l);$
- (p)  $\sum \sigma^{-1}(U^{(2)}, l)U^{(1)} = \sum \sigma^{-1}(l, U^{(2)})U^{(1)} = \varepsilon(l);$
- (q)  $\sum \sigma^{-1}(U^{(1)}, l)U^{(2)} = \sum \sigma^{-1}(l, U^{(1)})U^{(2)} = \varepsilon(l);$
- (r)  $\sum \sigma^{-1}(U^{(1)}h, l)U^{(2)} = \sum \sigma^{-1}(hU^{(1)}, l)U^{(2)} = \sigma^{-1}(h, l);$
- (s)  $\sum \sigma^{-1}(h, lU^{(2)})U^{(1)} = \sum \sigma^{-1}(hU^{(2)}, l)U^{(1)} = \sigma^{-1}(h, l).$

**Theorem 1.4.** *Let  $H$  be an  $\sigma - R$  compatible bialgebra, then  $(H, \odot, \widetilde{\Delta})$  is a Hopf algebra which will be denoted by  $H^{\sigma-R}$ .*

PROOF: Clearly,  $\tilde{\Delta}(1) = 1 \otimes 1$ . For all  $h, l \in H$ , we have

$$\begin{aligned}
& \tilde{\Delta}(h)\tilde{\Delta}(l) \\
\stackrel{(B)}{=} & \sum (R^{(1)}h_1U^{(1)}) \odot (r^{(1)}l_1u^{(1)}) \otimes (R^{(2)}h_2U^{(2)}) \odot (r^{(2)}l_2u^{(2)}) \\
\stackrel{(A)}{=} & \sum \sigma(R^{(1)}_1h_1U^{(1)}_1, r^{(1)}_1l_1u^{(1)}_1)(R^{(1)}_2h_2U^{(1)}_2r^{(1)}_2l_2u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3h_3U^{(1)}_3, r^{(1)}_3l_3u^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1h_4U^{(2)}_1, r^{(2)}_1l_4u^{(2)}_1)(R^{(2)}_2h_5U^{(2)}_2r^{(2)}_2l_5u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_3h_6U^{(2)}_3, r^{(2)}_3l_6u^{(2)}_3) \\
\stackrel{(b)}{=} & \sum \sigma(r^{(1)}_1, l_1U^{(1)}_1) \\
& \sigma(R^{(1)}_1h_1U^{(1)}_1, r^{(1)}_2l_2u^{(1)}_2)(R^{(1)}_2h_2U^{(1)}_2r^{(1)}_2l_2u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3h_3U^{(1)}_3, r^{(1)}_3l_3u^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1h_4U^{(2)}_1, r^{(2)}_1l_4u^{(2)}_1)(R^{(2)}_2h_5U^{(2)}_2r^{(2)}_2l_5u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_3h_6U^{(2)}_3, r^{(2)}_3l_6u^{(2)}_3) \\
\stackrel{(1)}{=} & \sum \sigma(R^{(1)}_1h_1U^{(1)}_1, r^{(1)}_1) \\
& \sigma(R^{(1)}_2h_2U^{(1)}_2r^{(1)}_2, l_1u^{(1)}_1)(R^{(1)}_3h_3U^{(1)}_3r^{(1)}_3l_2u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_4h_4U^{(1)}_4, r^{(1)}_4l_3u^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1h_5U^{(2)}_1, r^{(2)}_1l_4u^{(2)}_1)(R^{(2)}_2h_6U^{(2)}_2r^{(2)}_2l_5u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_3h_7U^{(2)}_3, r^{(2)}_3l_6u^{(2)}_3) \\
\stackrel{(b)}{=} & \sum \sigma(R^{(1)}_1h_1U^{(1)}_1r^{(1)}_1, l_1u^{(1)}_1)(R^{(1)}_2h_2U^{(1)}_2r^{(1)}_2l_2u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3h_3U^{(1)}_3, r^{(1)}_3l_3u^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1h_4U^{(2)}_1, r^{(2)}_1l_4u^{(2)}_1)(R^{(2)}_2h_5U^{(2)}_2r^{(2)}_2l_5u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_3h_6U^{(2)}_3, r^{(2)}_3l_6u^{(2)}_3) \\
\stackrel{(I)}{=} & \sum \sigma(R^{(1)}_1h_1U^{(1)}_1r^{(1)}_1, l_1u^{(1)}_1)(R^{(1)}_2h_2U^{(1)}_2r^{(1)}_2l_2u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3h_3U^{(1)}_3, r^{(1)}_3l_3u^{(1)}_3)\sigma^{-1}(r^{(1)}_4, l_4u^{(1)}_4) \\
& \otimes \sigma(R^{(2)}_1h_4U^{(2)}_1, r^{(2)}_1l_5u^{(2)}_1)(R^{(2)}_2h_5U^{(2)}_2r^{(2)}_2l_5u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_3l_6U^{(2)}_3, r^{(2)}_3l_7u^{(2)}_3) \\
\stackrel{(3)}{=} & \sum \sigma(R^{(1)}_1h_1U^{(1)}_1r^{(1)}_1, l_1u^{(1)}_1)(R^{(1)}_2h_2U^{(1)}_2r^{(1)}_2l_2u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3h_3U^{(1)}_3r^{(1)}_3, l_3u^{(1)}_3)\sigma^{-1}(r^{(1)}_4h_4u^{(1)}_4, r^{(1)}_4)
\end{aligned}$$

$$\begin{aligned}
& \otimes \sigma(R^{(2)}_1 h_5 U^{(2)}_1, r^{(2)}_1 l_4 u^{(2)}_1) (R^{(2)}_2 h_6 U^{(2)}_2 r^{(2)}_2 l_5 u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_3 h_7 U^{(2)}_3, r^{(2)}_3 l_6 u^{(2)}_3) \\
\stackrel{(I)}{=} & \sum \sigma(R^{(1)}_1 h_1 U^{(1)}_1 r^{(1)}_1, l_1 u^{(1)}_1) (R^{(1)}_2 h_2 U^{(1)}_2 r^{(1)}_2 l_2 u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3 h_3 U^{(1)}_3 r^{(1)}_3, l_3 u^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1 h_4 U^{(2)}_1, r^{(2)}_1 l_4 u^{(2)}_1) (R^{(2)}_2 h_5 U^{(2)}_2 r^{(2)}_2 l_5 u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_3 h_6 U^{(2)}_3, r^{(2)}_3 l_6 u^{(2)}_3) \\
\stackrel{(c)}{=} & \sum \sigma(R^{(1)}_1 h_1 U^{(1)}_1 r^{(1)}_1, l_1 u^{(1)}_1) (R^{(1)}_2 h_2 U^{(1)}_2 r^{(1)}_2 l_2 u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3 h_3 U^{(1)}_3 r^{(1)}_3, l_3 u^{(1)}_3) \otimes \sigma(r^{(2)}_1, l_4 u^{(2)}_1) \\
& \sigma(R^{(2)}_1 h_4 U^{(2)}_1, r^{(2)}_2 l_5 u^{(2)}_2) (R^{(2)}_2 h_5 U^{(2)}_2 r^{(2)}_3 l_6 u^{(2)}_3) \\
& \sigma^{-1}(R^{(2)}_3 h_6 U^{(2)}_3, r^{(2)}_4 l_7 u^{(2)}_4) \\
\stackrel{(1)}{=} & \sum \sigma(R^{(1)}_1 h_1 U^{(1)}_1 r^{(1)}_1, l_1 u^{(1)}_1) (R^{(1)}_2 h_2 U^{(1)}_2 r^{(1)}_2 l_2 u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3 h_3 U^{(1)}_3 r^{(1)}_3, l_3 u^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1 h_4 U^{(2)}_1, r^{(2)}_1) \\
& \sigma(R^{(2)}_2 h_5 U^{(2)}_2 r^{(2)}_2, l_4 u^{(2)}_1) (R^{(2)}_3 h_6 U^{(2)}_3 r^{(2)}_3 l_5 u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_4 h_7 U^{(2)}_4, r^{(2)}_4 l_6 u^{(2)}_3) \\
\stackrel{(h)^+(c)}{=} & \sum \sigma(R^{(1)}_1 h_1 U^{(1)}_1 r^{(1)}_1, l_1 u^{(1)}_1) (R^{(1)}_2 h_2 U^{(1)}_2 r^{(1)}_2 l_2 u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3 h_3 U^{(1)}_3 r^{(1)}_3, l_3 u^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1 h_4 U^{(2)}_1 r^{(2)}_1, l_4 u^{(2)}_1) (R^{(2)}_2 h_5 U^{(2)}_2 r^{(2)}_2 l_5 u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_3 h_6 U^{(2)}_3, r^{(2)}_3 l_6 u^{(2)}_3) \sigma^{-1}(R^{(2)}_4, l_7 u^{(2)}_4) \\
\stackrel{(3)^+(h)}{=} & \sum \sigma(R^{(1)}_1 h_1 U^{(1)}_1 r^{(1)}_1, l_1 u^{(1)}_1) (R^{(1)}_2 h_2 U^{(1)}_2 r^{(1)}_2 l_2 u^{(1)}_2) \\
& \sigma^{-1}(R^{(1)}_3 h_3 U^{(1)}_3 r^{(1)}_3, l_3 u^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1 h_4 U^{(2)}_1 r^{(2)}_1, l_4 u^{(2)}_1) (R^{(2)}_2 h_5 U^{(2)}_2 r^{(2)}_2 l_5 u^{(2)}_2) \\
& \sigma^{-1}(R^{(2)}_3 h_6 U^{(2)}_3 r^{(2)}_3, l_6 u^{(2)}_3) \\
\stackrel{\text{R invertible}}{=} & \sum \sigma(R^{(1)}_1 h_1, l_1 U^{(1)}_1) (R^{(1)}_2 h_2 l_2 u^{(1)}_2) \sigma^{-1}(R^{(1)}_3 h_3, l_3 u^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1 h_4, l_4 U^{(2)}_1) (R^{(2)}_2 h_5 l_5 u^{(2)}_2) \sigma^{-1}(R^{(2)}_3 h_6, l_6 u^{(2)}_3) \\
\stackrel{(i)}{=} & \sum \sigma(\tilde{R}^{(1)} R^{(1)}_1 h_1, l_1 u^{(1)}_1) (\tilde{R}^{(2)} R^{(1)}_2 h_2 l_2 u^{(1)}_2) \\
& \sigma^{-1}(\tilde{R}^{(2)} R^{(1)}_3 h_3, l_3 u^{(1)}_3)
\end{aligned}$$

$$\begin{aligned}
& \otimes \sigma(R^{(2)}_3 h_4, l_4 U^{(2)}_1)(R^{(2)}_4 h_5 l_5 u^{(1)}_2) \sigma^{-1}(R^{(2)}_5 h_6, l_6 u^{(2)}_3) \\
\stackrel{(1')}{=} & \sum \sigma(R^{(1)} h_1, l_1 u^{(1)}_1)(\tilde{R}_1^{(1)} R^{(2)}_1 h_2 l_2 u^{(1)}_2) \\
& \sigma^{-1}(\tilde{R}_2^{(1)} R^{(2)}_2 h_3, l_3 u^{(1)}_3) \\
& \otimes \sigma(\tilde{R}_1^{(2)} R^{(2)}_3 h_4, l_4 U^{(2)}_1)(\tilde{R}_2^{(2)} R^{(2)}_4 h_5 l_5 u^{(1)}_2) \\
& \sigma^{-1}(\tilde{R}_3^{(2)} R^{(2)}_5 h_6, l_6 u^{(2)}_3) \\
\stackrel{(i)}{=} & \sum \sigma(h_1, l_1 u^{(1)}_1)(\tilde{R}_1^{(1)} h_2 l_2 u^{(1)}_2) \sigma^{-1}(\tilde{R}_2^{(1)} h_3, l_3 u^{(1)}_3) \\
& \otimes \sigma(\tilde{R}_1^{(2)} h_4, l_4 u^{(2)}_1)(\tilde{R}_2^{(2)} h_5 l_5 u^{(1)}_2) \sigma^{-1}(\tilde{R}_3^{(2)} h_6, l_6 u^{(2)}_3) \\
\stackrel{(d)}{=} & \sum \sigma(h_1, l_1 u^{(1)}_1 \tilde{u}^{(1)})(\tilde{R}_1^{(1)} h_2 l_2 u^{(1)}_2 \tilde{u}^{(1)}_2) \sigma^{-1}(\tilde{R}_2^{(1)} h_3, l_3 u^{(1)}_3 \tilde{u}^{(1)}_3) \\
& \otimes \sigma(\tilde{R}_1^{(2)} h_4, l_4 u^{(2)}_1)(\tilde{R}_2^{(2)} h_5 l_5 u^{(2)}_2) \sigma^{-1}(\tilde{R}_3^{(2)} h_6, l_6 u^{(2)}_3) \\
\stackrel{(3')}{=} & \sum \sigma(h_1, l_1 u^{(1)})(\tilde{R}_1^{(1)} h_2 l_2 u^{(2)}_1 \tilde{u}^{(1)}_1) \sigma^{-1}(\tilde{R}_2^{(1)} h_3, l_3 u^{(2)}_2 \tilde{u}^{(1)}_2) \\
& \otimes \sigma(\tilde{R}_1^{(2)} h_4, l_4 u^{(2)}_2 \tilde{u}^{(2)}_2)(\tilde{R}_2^{(2)} h_5 l_6 u^{(2)}_3 \tilde{u}^{(2)}_3) \sigma^{-1}(\tilde{R}_3^{(2)} h_6, l_6 u^{(2)}_4 \tilde{u}^{(2)}_4) \\
\stackrel{(d)}{=} & \sum \sigma(h_1, l_1)(\tilde{R}_1^{(1)} h_2 l_2 \tilde{u}^{(1)}_1) \sigma^{-1}(\tilde{R}_2^{(1)} h_3, l_3 \tilde{u}^{(1)}_3) \\
& \otimes \sigma(\tilde{R}_1^{(2)} h_4, l_4 \tilde{u}^{(2)}_1)(\tilde{R}_2^{(2)} h_5 l_5 \tilde{u}^{(2)}_2) \sigma^{-1}(\tilde{R}_3^{(2)} h_6, l_6 \tilde{u}^{(2)}_3) \\
\stackrel{(n)}{=} & \sum \sigma(h_1, l_1)(R^{(1)} \tilde{R}_1^{(1)} h_2 l_2 \tilde{u}^{(1)}_1) \\
& \sigma^{-1}(R^{(2)}, \tilde{R}_2^{(1)} h_3 l_3 \tilde{u}^{(1)}_2) \sigma^{-1}(\tilde{R}_3^{(1)} h_4, l_4 \tilde{u}^{(1)}_3) \\
& \otimes \sigma(\tilde{R}_1^{(2)} h_5, l_5 \tilde{u}^{(2)}_1)(\tilde{R}_2^{(2)} h_6 l_6 \tilde{u}^{(2)}_2) \sigma^{-1}(\tilde{R}_3^{(2)} h_7, l_7 \tilde{u}^{(2)}_3) \\
\stackrel{(3)}{=} & \sum \sigma(h_1, l_1)(R^{(1)} \tilde{R}_1^{(1)} h_2 l_2 \tilde{u}^{(1)}_1) \\
& \sigma^{-1}(R^{(2)}_1 \tilde{R}_2^{(1)} h_3, l_3 \tilde{u}^{(1)}_2) \sigma^{-1}(R^{(2)}_2, \tilde{R}_3^{(1)} h_4) \\
& \otimes \sigma(\tilde{R}_1^{(2)} h_5, l_4 \tilde{u}^{(2)}_1)(\tilde{R}_2^{(2)} h_6 l_5 \tilde{u}^{(2)}_2) \sigma^{-1}(\tilde{R}_3^{(2)} h_7, l_6 \tilde{u}^{(2)}_3) \\
\stackrel{(h)}{=} & \sum \sigma(h_1, l_1)(R^{(1)} \tilde{R}_1^{(1)} h_2 l_2 \tilde{u}^{(1)}_1) \sigma^{-1}(R^{(2)} \tilde{R}_2^{(1)} h_3, l_3 \tilde{u}^{(1)}_3) \\
& \otimes \sigma(\tilde{R}_1^{(2)} h_4, l_4 \tilde{u}^{(2)}_1)(\tilde{R}_2^{(2)} h_5 l_5 \tilde{u}^{(2)}_2) \sigma^{-1}(\tilde{R}_3^{(2)} h_6, l_6 \tilde{u}^{(2)}_3) \\
\stackrel{(1')}{=} & \sum \sigma(h_1, l_1)(\tilde{R}^{(1)} h_2 l_2 \tilde{u}^{(1)}_1) \sigma^{-1}(R^{(1)} \tilde{R}_1^{(2)} h_3, l_3 \tilde{u}^{(1)}_3) \\
& \otimes \sigma(R^{(2)}_1 \tilde{R}_2^{(2)} h_4, l_4 u^{(2)}_1)(R^{(2)}_2 \tilde{R}_3^{(2)} h_5 l_5 \tilde{u}^{(2)}_2) \sigma^{-1}(R^{(2)}_3 \tilde{R}_4^{(2)} h_6, l_6 \tilde{u}^{(2)}_3) \\
\stackrel{(iv)}{=} & \sum \sigma(h_1, l_1)(\tilde{R}^{(1)} h_2 l_2 \tilde{u}^{(1)}_1) \sigma^{-1}(\tilde{R}_1^{(2)} h_3, l_3 \tilde{u}^{(1)}_3) \\
& \otimes \sigma(\tilde{R}_2^{(2)} h_4, l_4 \tilde{u}^{(2)}_1)(\tilde{R}_3^{(2)} h_5 l_5 \tilde{u}^{(2)}_2) \sigma^{-1}(\tilde{R}_4^{(2)} h_6, l_6 \tilde{u}^{(2)}_3)
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(p)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)}h_2l_2\tilde{u}^{(1)}{}_1u^{(1)}) \\
&\quad \sigma^{-1}(\tilde{R}_2^{(2)}h_4, l_4\tilde{u}_3^{(1)})\sigma^{-1}(\tilde{R}_1^{(1)}h_3l_3\tilde{u}^{(1)}{}_2, u^{(2)}) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)}h_5, l_5\tilde{u}^{(2)}{}_1)(\tilde{R}_3^{(2)}h_6l_6\tilde{u}_2^{(2)})\sigma^{-1}(\tilde{R}_4^{(2)}h_7, l_7\tilde{u}_3^{(2)}) \\
&\stackrel{(3)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)}h_2l_2\tilde{u}^{(1)}{}_1u^{(1)}) \\
&\quad \sigma^{-1}(\tilde{R}_1^{(2)}h_3, l_3\tilde{u}_2^{(1)}{}_1u^{(2)}{}_1)\sigma^{-1}(l_4\tilde{u}_3^{(1)}, u^{(2)}{}_2) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)}h_4, l_5\tilde{u}^{(2)}{}_1)(\tilde{R}_3^{(2)}h_6l_6\tilde{u}_2^{(2)})\sigma^{-1}(\tilde{R}_4^{(2)}h_7, l_7\tilde{u}_3^{(2)}) \\
&\stackrel{(l)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)}h_2l_2\tilde{u}^{(1)}{}_1u^{(1)})\sigma^{-1}(\tilde{R}_1^{(2)}h_3, l_3\tilde{u}_2^{(1)}{}_1u^{(2)}) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)}h_4, l_4\tilde{u}^{(2)}{}_1)(\tilde{R}_3^{(2)}h_5l_5\tilde{u}_2^{(2)})\sigma^{-1}(\tilde{R}_4^{(2)}h_6, l_6\tilde{u}_3^{(2)}) \\
&\stackrel{(3')}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)}h_2l_2\tilde{u}^{(1)})\sigma^{-1}(\tilde{R}_1^{(2)}h_3, l_3\tilde{u}_1^{(2)}u^{(1)}) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)}h_4, l_4\tilde{u}^{(2)}{}_2u_1^{(2)})(\tilde{R}_3^{(2)}h_5l_5\tilde{u}_3^{(2)}u_3^{(2)})\sigma^{-1}(\tilde{R}_4^{(2)}h_6, l_6\tilde{u}_4^{(2)}u_3^{(2)}) \\
&\stackrel{(r)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)}h_2l_2\tilde{u}^{(1)})\sigma^{-1}(\tilde{R}_1^{(2)}h_3, l_3\tilde{u}_1^{(2)}) \\
&\quad \otimes \sigma(\tilde{R}_2^{(2)}h_4, l_4\tilde{u}^{(2)}{}_2)(\tilde{R}_3^{(2)}h_5l_5\tilde{u}_3^{(2)})\sigma^{-1}(\tilde{R}_4^{(2)}h_6, l_6\tilde{u}_4^{(2)}) \\
&= \sum \sigma(h_1, l_1)(\tilde{R}^{(1)}h_2l_2\tilde{u}^{(1)}) \otimes \tilde{R}_1^{(2)}h_3l_3\tilde{u}^{(2)}{}_1\sigma^{-1}(\tilde{R}_2^{(2)}h_4, l_4\tilde{u}_2^{(2)}) \\
&\stackrel{(iv)}{=} \sum \sigma(h_1, l_1)(\tilde{R}^{(1)}h_2l_2\tilde{u}^{(1)}) \otimes (R^{(1)}\tilde{R}_1^{(2)}h_3l_3\tilde{u}^{(2)}{}_1) \\
&\quad \sigma^{-1}(R^{(2)}\tilde{R}_2^{(2)}h_4, l_4\tilde{u}_2^{(2)}) \\
&\stackrel{(1')}{=} \sum \sigma(h_1, l_1)(R^{(1)}\tilde{R}_1^{(1)}h_2l_2\tilde{u}^{(1)}) \otimes (R^{(2)}\tilde{R}_2^{(1)}h_3l_3\tilde{u}^{(2)}{}_1) \\
&\quad \sigma^{-1}(\tilde{R}^{(2)}h_4, l_4\tilde{u}_2^{(2)}) \\
&\stackrel{(iv)}{=} \sum \sigma(h_1, l_1)(R^{(1)}h_2l_2\tilde{u}^{(1)}) \otimes (R^{(2)}h_3l_3\tilde{u}^{(2)}{}_1)\sigma^{-1}(h_4, l_4\tilde{u}_2^{(2)}) \\
&\stackrel{(s)}{=} \sum \sigma(h_1, l_1)(R^{(1)}h_2l_2\tilde{u}^{(1)}) \otimes (R^{(2)}h_3l_3\tilde{u}^{(2)}{}_1u^{(1)})\sigma^{-1}(h_4, l_4\tilde{u}_2^{(2)}u^{(2)}) \\
&\stackrel{(3')}{=} \sum \sigma(h_1, l_1)(R^{(1)}h_2l_2\tilde{u}_1^{(1)}u^{(1)}) \otimes (R^{(2)}h_3l_3\tilde{u}^{(1)}{}_2u^{(2)})\sigma^{-1}(h_4, l_4\tilde{u}^{(2)}) \\
&\stackrel{(s)}{=} \sum \sigma(h_1, l_1)(R^{(1)}h_2l_2u^{(1)}) \otimes (R^{(2)}h_3l_3u^{(2)})\sigma^{-1}(h_4, l_4)
\end{aligned}$$

$${}^{(A)\pm(B)}\tilde{\Delta}(h \odot l).$$

□

Now we will give the Hopf structure of  $H^{\sigma-R}$ , and we will always assume that the following two conditions hold:  $\forall h, l \in H$

- (t)  $\sum \sigma^{-1}(h, lS(U^{(1)}{}_1))S(U^{(1)}{}_2) \otimes U^{(2)} = \sum \sigma^{-1}(h, l)S(U^{(1)}) \otimes U^{(2)}$ ,
- (u)  $\sum \sigma^{-1}(S(U^{(2)})hS(R^{(2)}), lS(r^{(2)}))U^{(1)} \otimes R^{(1)} \otimes r^{(1)} = \sigma^{-1}(h, l)$ .



**Theorem 1.5.** *Let  $H$  be a  $\sigma - R$  compatible Hopf algebra. Suppose that conditions (t), (u) hold. Then  $(H, \odot, \tilde{\Delta})$  is a Hopf algebra with the antipode:*

$$S^{\sigma-R}(h) = \sum \sigma(h_1, S(h_2))R^{(1)}S(U^{(1)}h_3R^{(2)})U^{(2)}\sigma^{-1}(S(h_4), h_5).$$

PROOF: By Theorem 1.4,  $H^{\sigma-R}$  is a bialgebra. It remains only to verify that  $S^{\sigma-R}$  is the antipode of  $H^{\sigma-R}$ . For all  $h \in H$

$$\begin{aligned} & (I \star S^{\sigma-R})(h) \\ & \stackrel{(B)}{=} \sum (\tilde{R}^{(1)}h_1\tilde{U}^{(1)}) \odot S^{\sigma-R}(\tilde{R}^{(2)}h_2\tilde{U}^{(2)}) \\ & \stackrel{(A)}{=} \sum \sigma(\tilde{R}^{(1)}h_1\tilde{U}^{(1)}_1, S^{\sigma-R}(\tilde{R}^{(2)}h_4\tilde{U}^{(2)}_1)) \\ & \quad [(\tilde{R}^{(1)}h_2\tilde{U}^{(1)}_2)S^{\sigma-R}(\tilde{R}^{(2)}h_4\tilde{U}^{(2)}_2)] \\ & \quad \sigma^{-1}(\tilde{R}^{(1)}h_3\tilde{U}^{(1)}_3, S^{\sigma-R}(\tilde{R}^{(2)}h_4\tilde{U}^{(2)}_3)) \\ & = \sum \sigma(\tilde{R}^{(1)}h_1\tilde{U}^{(1)}_1, (R^{(1)}S(U^{(1)}\tilde{R}_3^{(2)}h_6\tilde{U}_3^{(2)}R^{(2)})U^{(2)}_1)) \\ & \quad \sigma(\tilde{R}^{(2)}h_4\tilde{U}^{(2)}_1, S(\tilde{R}_2^{(2)}h_5\tilde{U}_2^{(2)})) \\ & \quad [\tilde{R}^{(1)}h_2\tilde{U}^{(1)}_2(R^{(1)}S(U^{(1)}\tilde{R}_3^{(2)}h_6\tilde{U}^{(2)}_3R^{(2)})U^{(2)}_2)] \\ & \quad \sigma^{-1}(\tilde{R}^{(1)}h_3\tilde{U}^{(1)}_3, (R^{(1)}S(U^{(1)}\tilde{R}^{(2)}_3h_6\tilde{U}^{(2)}_3R^{(2)})U^{(2)}_3)) \\ & \quad \sigma^{-1}(S(\tilde{R}^{(2)}h_7\tilde{U}_4^{(2)}), \tilde{R}^{(2)}h_8\tilde{U}^{(2)}_5) \\ & \stackrel{(i)+(2')}{=} \sum \sigma(r^{(1)}\tilde{R}^{(1)}h_1\tilde{U}^{(1)}_1, R^{(1)}S(U^{(1)}\tilde{R}_3^{(2)}h_6\tilde{U}_3^{(2)}R^{(2)})U^{(2)}_1) \\ & \quad \sigma(\tilde{R}^{(2)}h_4\tilde{U}^{(2)}_1, S(\tilde{R}_2^{(2)}h_5\tilde{U}_2^{(2)})) \\ & \quad [r^{(2)}\tilde{R}^{(1)}h_2\tilde{U}^{(1)}_2R^{(1)}S(U^{(1)}\tilde{R}_3^{(2)}h_6\tilde{U}_3^{(2)}R^{(2)})U^{(2)}_2] \\ & \quad \sigma^{-1}(r^{(2)}\tilde{R}^{(1)}h_3\tilde{U}^{(1)}_3, R^{(1)}S(U^{(1)}\tilde{R}^{(2)}_3h_6 \\ & \quad \tilde{U}^{(2)}_3R^{(2)})U^{(2)}_3)\sigma^{-1}(S(\tilde{R}^{(2)}h_7\tilde{U}^{(2)}_4), \tilde{R}^{(2)}h_8\tilde{U}^{(2)}_5) \\ & \stackrel{(1')}{=} \sum \sigma(\tilde{R}^{(1)}h_1\tilde{U}^{(1)}_1, R^{(1)}S(U^{(1)}r^{(2)}\tilde{R}^{(2)}_3h_6\tilde{U}^{(2)}_3R^{(2)})U^{(2)}_1) \\ & \quad \sigma(r^{(2)}\tilde{R}^{(1)}h_4\tilde{U}^{(2)}_1, S(r^{(2)}\tilde{R}^{(2)}_3h_5\tilde{U}^{(2)}_2)) \\ & \quad [r^{(1)}\tilde{R}^{(2)}h_2\tilde{U}^{(1)}_2R^{(1)}S(U^{(1)}r^{(2)}\tilde{R}^{(2)}_3h_6\tilde{U}^{(2)}_3R^{(2)})U^{(2)}_2] \\ & \quad \sigma^{-1}(r^{(1)}\tilde{R}^{(2)}h_3\tilde{U}^{(1)}_3, R^{(1)}_3 \\ & \quad S(U^{(1)}r^{(2)}\tilde{R}^{(2)}_4h_6\tilde{U}^{(2)}_3R^{(2)})U^{(2)}_3) \\ & \quad \sigma^{-1}(S(r^{(2)}\tilde{R}^{(2)}_5h_7\tilde{U}^{(2)}_4), r^{(2)}\tilde{R}^{(2)}_6h_8\tilde{U}^{(2)}_5) \\ & \stackrel{(2')+(i)}{=} \sum \sigma(h_1\tilde{U}^{(1)}_1, R^{(1)}S(U^{(1)}r^{(2)}h_6\tilde{U}^{(2)}_3R^{(2)})U^{(2)}_1) \end{aligned}$$

$$\begin{aligned}
 & \sigma(r^{(2)}_1 h_4 \tilde{U}^{(2)}_1, S(r^{(2)}_2 h_5 \tilde{U}^{(2)}_2)) \\
 & [r^{(1)}_1 h_2 \tilde{U}^{(1)}_2 R^{(1)}_2 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 R^{(2)}_2) U^{(2)}_2] \\
 & \sigma^{-1}(r^{(1)}_2 h_3 \tilde{U}^{(1)}_3, R^{(1)}_3 S(U^{(1)}_r^{(2)}_3 h_6 \\
 & \tilde{U}^{(2)}_3 R^{(2)}_3) U^{(2)}_3) \sigma^{-1}(S(r^{(2)}_4 h_7 \tilde{U}^{(2)}_4), r^{(2)}_5 h_8 \tilde{U}^{(2)}_5) \\
 (i)^+ \stackrel{(2')}{=} & \sum \sigma(h_1 \tilde{U}^{(1)}_1, \tilde{r}^{(1)} R^{(1)}_1 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 R^{(2)}_1) U^{(2)}_1) \\
 & \sigma(r^{(2)}_1 h_4 \tilde{U}^{(2)}_1, S(r^{(2)}_2 h_5 \tilde{U}^{(2)}_2)) \\
 & [r^{(1)}_1 h_2 \tilde{U}^{(1)}_2 \tilde{r}^{(2)}_1 R^{(1)}_2 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 R^{(2)}_2) U^{(2)}_2] \\
 & \sigma^{-1}(r^{(1)}_2 h_3 \tilde{U}^{(1)}_3, \tilde{r}^{(2)}_2 R^{(1)}_3 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 R^{(2)}_3) U^{(2)}_3) \\
 & \sigma^{-1}(S(r^{(2)}_4 h_7 \tilde{U}^{(2)}_4), r^{(2)}_5 h_8 \tilde{U}^{(2)}_5) \\
 (1') \stackrel{=}{=} & \sum \sigma(h_1 \tilde{U}^{(1)}_1, R^{(1)}_1 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 \tilde{r}^{(2)} R^{(2)}_3) U^{(2)}_1) \\
 & \sigma(r^{(2)}_1 h_4 \tilde{U}^{(2)}_1, S(r^{(2)}_2 h_5 \tilde{U}^{(2)}_2)) \\
 & [r^{(1)}_1 h_2 \tilde{U}^{(1)}_2 \tilde{r}^{(1)}_1 R^{(2)}_1 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 \tilde{r}^{(2)} R^{(2)}_3) U^{(2)}_2] \\
 & \sigma^{-1}(r^{(1)}_2 h_3 \tilde{U}^{(1)}_3, \tilde{r}^{(1)}_2 R^{(2)}_2 \\
 & S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 \tilde{r}^{(2)} R^{(2)}_3) U^{(2)}_3) \\
 & \sigma^{-1}(S(r^{(2)}_4 h_7 \tilde{U}^{(2)}_4), r^{(2)}_5 h_8 \tilde{U}^{(2)}_5) \\
 (2')^+ \stackrel{(i)}{=} & \sum \sigma(h_1 \tilde{U}^{(1)}_1, S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 \tilde{r}^{(2)}_1) U^{(2)}_1) \sigma(r^{(2)}_1 h_4 \tilde{U}^{(2)}_1, \\
 & S(r^{(2)}_2 h_5 \tilde{U}^{(2)}_2)) \\
 & [r^{(1)}_1 h_2 \tilde{U}^{(1)}_2 \tilde{r}^{(1)}_1 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 \tilde{r}^{(2)}_2) U^{(2)}_2] \\
 & \sigma^{-1}(r^{(1)}_2 h_3 \tilde{U}^{(1)}_3, \tilde{r}^{(1)}_2 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 \tilde{r}^{(2)}_3) U^{(2)}_3) \\
 & \sigma^{-1}(S(r^{(2)}_4 h_7 \tilde{U}^{(2)}_4), r^{(2)}_5 h_8 \tilde{U}^{(2)}_5) \\
 (d)^+ \stackrel{(4')}{=} & \sum \sigma(h_1 \tilde{U}^{(1)}_1 u^{(1)}_1, S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 \tilde{r}^{(2)}_1) U^{(2)}_1) \\
 & \sigma(r^{(2)}_1 h_4 \tilde{U}^{(2)}_1, S(r^{(2)}_2 h_5 \tilde{U}^{(2)}_2)) \\
 & [r^{(1)}_1 h_2 \tilde{U}^{(1)}_2 u^{(2)}_1 \tilde{r}^{(1)}_1 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 \tilde{r}^{(2)}_2) U^{(2)}_2] \\
 & \sigma^{-1}(r^{(1)}_2 h_3 \tilde{U}^{(1)}_3 u^{(2)}_2, \tilde{r}^{(1)}_2 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 \tilde{r}^{(2)}_3) U^{(2)}_3) \\
 & \sigma^{-1}(S(r^{(2)}_4 h_7 \tilde{U}^{(2)}_4), r^{(2)}_5 h_8 \tilde{U}^{(2)}_5) \\
 (3') \stackrel{=}{=} & \sum \sigma(h_1 \tilde{U}^{(1)}_1, S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_4 u^{(2)}_3 \tilde{r}^{(2)}_1) U^{(2)}_1) \\
 & \sigma(r^{(2)}_1 h_4 \tilde{U}^{(2)}_2 u^{(2)}_1, S(r^{(2)}_2 h_5 \tilde{U}^{(2)}_3 u^{(2)}_2)) \\
 & [r^{(1)}_1 h_2 \tilde{U}^{(2)}_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_3 u^{(2)}_3 \tilde{r}^{(2)}_0) U^{(2)}_2]
 \end{aligned}$$

$$\begin{aligned}
& \sigma^{-1}(r^{(1)}_2 h_3 \tilde{U}^{(2)}_2 u^{(1)}_2, \tilde{r}^{(1)}_2 S(U^{(1)}_r^{(2)}_3 h_6 \tilde{U}^{(2)}_4 u^{(2)}_3 \tilde{r}^{(2)}_3) U^{(2)}_3) \\
& \sigma^{-1}(S(r^{(2)}_4 h_7 \tilde{U}^{(2)}_5 u^{(2)}_4), r^{(2)}_5 h_8 \tilde{U}^{(2)}_6 u^{(2)}_5) \\
\stackrel{(d)+(4')}{=} & \sum \sigma(h_1, S(U^{(1)}_r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1) U_1^{(2)} \tilde{u}^{(1)}) \\
& \sigma(r^{(2)}_1 h_4 u^{(2)}_1, S(r^{(2)}_2 h_5 u^{(2)}_2)) \\
& [r^{(1)}_1 h_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(U^{(1)}_r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_2) U^{(2)}_2 \tilde{u}^{(2)}_1] \\
& \sigma^{-1}(r^{(1)}_2 h_3 u^{(1)}_2, \tilde{r}^{(1)}_2 S(U^{(1)}_r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_3) U^{(2)}_3 \tilde{u}^{(2)}_2) \\
& \sigma^{-1}(S(r^{(2)}_4 h_7 u^{(2)}_4), r^{(2)}_5 h_8 u^{(2)}_5) \\
\stackrel{(3')}{=} & \sum \sigma(h_1, S(U^{(1)}_1 \tilde{u}^{(1)} r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1) U_2^{(1)} \tilde{u}^{(2)}) \\
& \sigma(r^{(2)}_1 h_4 u^{(2)}_1, S(r^{(2)}_2 h_5 u^{(2)}_2)) \\
& [r^{(1)}_1 h_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(U^{(1)}_{(1)} \tilde{u}^{(1)} r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_2) U^{(2)}_1] \\
& \sigma^{-1}(r^{(1)}_2 h_3 u^{(1)}_2, \tilde{r}^{(1)}_2 S(U^{(1)}_1 \tilde{u}^{(1)} r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_3) U^{(2)}_2) \\
& \sigma^{-1}(S(r^{(2)}_4 h_7 u^{(2)}_4), r^{(2)}_5 h_8 u^{(2)}_5) \\
\stackrel{(e)+(4')}{=} & \sum \sigma(h_1, S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1) U_2^{(1)} \tilde{u}^{(2)}) \\
& \sigma(r^{(2)}_1 h_4 u^{(2)}_1, S(r^{(2)}_2 h_5 u^{(2)}_2)) \\
& [r^{(1)}_1 h_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_2) S(U^{(1)}_2) U^{(2)}_1] \\
& \sigma^{-1}(r^{(1)}_2 h_3 u^{(1)}_2, \tilde{r}^{(1)}_2 S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_3) S(U^{(1)}_1) U^{(2)}_2) \\
& \sigma^{-1}(S(r^{(2)}_4 h_7 u^{(2)}_4), r^{(2)}_5 h_8 u^{(2)}_5) \\
\stackrel{(s)}{=} & \sum \sigma(h_1, S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1) U_2^{(1)} \tilde{u}^{(2)}) \\
& \sigma(r^{(2)}_1 h_4 u^{(2)}_1, S(r^{(2)}_2 h_5 u^{(2)}_2)) \\
& [r^{(1)}_1 h_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_2) S(U^{(1)}_2) U^{(2)}_1 \tilde{u}^{(1)}] \\
& \sigma^{-1}(r^{(1)}_2 h_3 u^{(1)}_2, \tilde{r}^{(1)}_2 S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_3) S(U^{(1)}_1) U^{(2)}_2 \tilde{u}^{(2)}) \\
& \sigma^{-1}(S(r^{(2)}_4 h_7 u^{(2)}_4), r^{(2)}_5 h_8 u^{(2)}_5) \\
\stackrel{(3')}{=} & \sum \sigma(h_1, S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1) U_2^{(1)} \tilde{u}^{(2)}) \\
& \sigma(r^{(2)}_1 h_4 u^{(2)}_1, S(r^{(2)}_2 h_5 u^{(2)}_2)) \\
& [r^{(1)}_1 h_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_2) S(U^{(1)}_2 \tilde{u}^{(1)}_2) U^{(1)}_3 \tilde{u}^{(2)}] \\
& \sigma^{-1}(r^{(1)}_2 h_3 u^{(1)}_2, \tilde{r}^{(1)}_2 S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_3) S(U^{(1)}_1 \tilde{u}^{(1)}_1) U^{(2)}_2) \\
& \sigma^{-1}(S(r^{(2)}_4 h_7 u^{(2)}_4), r^{(2)}_5 h_8 u^{(2)}_5)
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(s)}{=} \sum \sigma(h_1, S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1) U_2^{(1)} \tilde{u}^{(2)}) \\
&\quad \sigma(r^{(2)}_1 h_4 u^{(2)}_1, S(r^{(2)}_2 h_5 u^{(2)}_2)) \\
&\quad [r^{(1)}_1 h_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_2) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(r^{(1)}_2 h_3 u^{(1)}_2, \tilde{r}^{(1)}_2 S(r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_3) S(\tilde{u}^{(1)}_1)) \\
&\quad \sigma^{-1}(S(r^{(2)}_4 h_7 u^{(2)}_4), r^{(2)}_5 h_8 u^{(2)}_5) \\
&\stackrel{(iv)}{=} \sum \sigma(h_1, S(R^{(1)}_3 r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1) U_2^{(1)} \tilde{u}^{(2)}) \\
&\quad \sigma(R^{(1)}_1 r^{(2)}_1 h_4 u^{(2)}_1, S(R^{(1)}_2 r^{(2)}_2 h_5 u^{(2)}_2)) \\
&\quad [r^{(1)}_1 h_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(R^{(1)}_3 r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_2) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(r^{(1)}_2 h_3 u^{(1)}_2, \tilde{r}^{(1)}_2 S(R^{(1)}_3 r^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_3) S(\tilde{u}^{(1)}_1)) \\
&\quad \sigma^{-1}(S(R^{(1)}_4 r^{(2)}_4 h_7 u^{(2)}_4), R^{(2)}_5 h_8 u^{(2)}_5) \\
&\stackrel{(1')}{=} \sum \sigma(h_1, S(R^{(2)}_3 r^{(1)}_5 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1) U_2^{(1)} \tilde{u}^{(2)}) \\
&\quad \sigma(R^{(2)}_1 r^{(1)}_3 h_4 u^{(2)}_1, S(R^{(2)}_2 r^{(1)}_4 h_5 u^{(2)}_2)) \\
&\quad [R^{(1)}_1 r^{(1)}_1 h_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(R^{(2)}_3 r^{(1)}_5 h_6 u^{(2)}_3 \tilde{r}^{(2)}_2) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(R^{(1)}_2 r^{(1)}_2 h_3 u^{(1)}_2, \tilde{r}^{(1)}_2 S(R^{(2)}_3 r^{(1)}_5 h_6 u^{(2)}_3 \tilde{r}^{(2)}_3) S(\tilde{u}^{(1)}_1)) \\
&\quad \sigma^{-1}(S(R^{(2)}_4 r^{(1)}_6 h_7 u^{(2)}_4), r^{(2)}_5 h_8 u^{(2)}_5) \\
&\stackrel{(iv)^+(s)}{=} \sum \sigma(h_1, S(R^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1) U^{(1)}_2 \tilde{u}^{(2)}) \\
&\quad \sigma(R^{(2)}_1 h_4 u^{(2)}_1 U^{(1)}_1, S(R^{(2)}_2 h_5 u^{(2)}_2 U^{(1)}_2)) \\
&\quad [R^{(1)}_1 h_2 u^{(1)}_1 \tilde{r}^{(1)}_1 S(R^{(2)}_3 h_6 u^{(2)}_3 U^{(1)}_3 \tilde{r}^{(2)}_2) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(R^{(1)}_2 h_3 u^{(1)}_2, \tilde{r}^{(1)}_2 S(R^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_3) S(\tilde{u}^{(1)}_1)) \\
&\quad \sigma^{-1}(S(R^{(2)}_4 h_7 u^{(2)}_4 U^{(1)}_4), h_8 u^{(2)}_5 U^{(2)}) \\
&\stackrel{(3')}{=} \sum \sigma(h_1, S(R^{(2)}_3 h_6 u^{(2)}_3 \tilde{r}^{(2)}_1)) \\
&\quad \sigma(R^{(2)}_1 h_4 u^{(1)}_3 U^{(2)}_1, S(R^{(2)}_2 h_5 u^{(1)}_4 U^{(2)}_2)) \\
&\quad [R^{(1)}_1 h_2 u^{(1)}_1 U^{(1)}_1 \tilde{r}^{(1)}_1 S(R^{(2)}_3 h_6 u^{(1)}_5 U^{(2)}_3 \tilde{r}^{(2)}_2) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
&\quad \sigma^{-1}(R^{(1)}_2 h_3 u^{(1)}_2 U^{(1)}_2, \tilde{r}^{(1)}_2 S(R^{(2)}_3 h_6 u^{(1)}_5 U^{(2)}_3 \tilde{r}^{(2)}_3) S(\tilde{u}^{(1)}_1)) \\
&\quad \sigma^{-1}(S(R^{(2)}_4 h_7 u^{(1)}_6 U^{(2)}_4), h_8 u^{(2)}) \\
&\stackrel{(s)^+(iv)}{=} \sum \sigma(h_1, S(R^{(2)}_5 h_8 U^{(2)}_5 \tilde{r}^{(2)}_3)) \\
&\quad \sigma(R^{(2)}_1 h_4 U^{(2)}_1, S(R^{(2)}_2 h_5 U^{(2)}_2))
\end{aligned}$$

$$\begin{aligned}
& [R^{(1)}_1 h_2 U^{(1)}_1 \tilde{R}^{(1)} \tilde{r}^{(1)}_1 S(R^{(2)}_4 h_7 U^{(2)}_4 \tilde{r}^{(2)}_2) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
& \sigma^{-1}(R^{(1)}_2 h_3 U^{(1)}_2, \tilde{R}^{(2)} \tilde{r}^{(1)}_2 S(R^{(2)}_3 h_6 U^{(2)}_3 \tilde{r}^{(2)}_1) S(\tilde{u}^{(1)}_1)) \\
& \sigma^{-1}(S(R^{(2)}_4 h_9 U^{(2)}_4), h_{10}) \\
\stackrel{(1')}{=} & \sum \sigma(h_1, S(R^{(2)}_5 h_8 U^{(2)}_5 \tilde{R}^{(2)}_3 \tilde{r}^{(2)}_4)) \\
& \sigma(R^{(2)}_1 h_4 U^{(2)}_1, S(R^{(2)}_2 h_5 U^{(2)}_2)) \\
& [R^{(1)}_1 h_2 U^{(1)}_1 \tilde{r}^{(1)} S(R^{(2)}_4 h_7 U^{(2)}_4 \tilde{R}^{(2)}_2 \tilde{r}^{(2)}_3) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
& \sigma^{-1}(R^{(1)}_2 h_3 U^{(1)}_2, \tilde{R}^{(1)} \tilde{r}^{(2)}_1 S(R^{(2)}_3 h_6 U^{(2)}_3 \tilde{R}^{(2)}_1 \tilde{r}^{(2)}_2) \\
& S(\tilde{u}^{(1)}_1)) \sigma^{-1}(S(R^{(2)}_4 h_9 U^{(2)}_4), h_{10}) \\
\stackrel{(iv)^+(s)}{=} & \sum \sigma(h_1, S(R^{(2)}_5 h_8 U^{(2)}_5 \tilde{r}^{(2)}_2)) \sigma(R^{(2)}_1 h_4 U^{(2)}_1, S(R^{(2)}_2 h_5 U^{(2)}_2)) \\
& [r^{(1)} R^{(1)}_1 h_2 U^{(1)}_1 u^{(1)} \tilde{r}^{(1)} S(R^{(2)}_4 h_7 U^{(2)}_4 \tilde{r}^{(2)}_1) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
& \sigma^{-1}(r^{(2)} R^{(1)}_2 h_3 U^{(1)}_2 u^{(2)}, \\
& S(R^{(2)}_3 h_6 U^{(2)}_3) S(\tilde{u}^{(1)}_1)) \sigma^{-1}(S(R^{(2)}_6 h_9 U^{(2)}_4), h_{10}) \\
\stackrel{(1')^+(3')}{=} & \sum \sigma(h_1, S(r^{(2)}_5 R^{(2)}_6 h_8 U^{(2)}_6 u^{(2)}_5 \tilde{r}^{(2)}_2)) \\
& \sigma(r^{(2)}_1 R^{(2)}_2 h_4 U^{(2)}_2 u^{(2)}_1, S(r^{(2)}_2 R^{(2)}_3 h_5 U^{(2)}_3 u^{(2)}_2)) \\
& [R^{(1)}_1 h_2 U^{(1)}_1 \tilde{r}^{(1)} S(r^{(2)}_4 R^{(2)}_5 h_7 U^{(2)}_5 u^{(2)} \tilde{r}^{(2)}_1) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
& \sigma^{-1}(r^{(1)} R^{(2)}_1 h_3 U^{(2)}_1 u^{(1)}, S(r^{(2)}_3 R^{(2)}_4 h_6 U^{(2)}_4 u^{(2)}_3) S(\tilde{u}^{(1)}_1)) \\
& \sigma^{-1}(S(r^{(2)}_6 R^{(2)}_7 h_9 U^{(2)}_7 u^{(2)}_6), h_{10}) \\
\stackrel{(r)^+(iv)}{=} & \sum \sigma(h_1, S(R^{(2)}_6 h_8 U^{(2)}_6 \tilde{r}^{(2)}_2)) \sigma(R^{(2)}_2 h_4 U^{(2)}_2, S(R^{(2)}_3 h_5 U^{(2)}_3)) \\
& [R^{(1)}_1 h_2 U^{(1)}_1 \tilde{r}^{(1)} S(R^{(2)}_5 h_7 U^{(2)}_5 \tilde{r}^{(2)}_1) S(\tilde{u}^{(1)}_2) \tilde{u}^{(2)}] \\
& \sigma^{-1}(R^{(2)}_1 h_3 U^{(2)}_1, S(R^{(2)}_4 h_6 U^{(2)}_4) S(\tilde{u}^{(1)}_1)) \\
& \sigma^{-1}(S(R^{(2)}_7 h_9 U^{(2)}_7), h_{10}) \\
\stackrel{(t)^+(u)}{=} & \sum \sigma(h_1, S(R^{(2)}_2 h_4 U^{(2)}_2 \tilde{u}^{(1)}_2 \tilde{r}^{(2)}_2)) \\
& [R^{(1)}_1 h_2 U^{(1)}_1 \tilde{r}^{(1)} S(R^{(2)}_1 h_3 U^{(2)}_1 \tilde{u}^{(1)}_1 \tilde{r}^{(2)}_1) S(u^{(1)}) u^{(2)}] \\
& \sigma^{-1}(S(R^{(2)}_3 h_5 U^{(2)}_3 \tilde{u}^{(2)}), h_6) \\
\stackrel{(3')^+(u)}{=} & \sum \sigma(h_1, S(R^{(2)}_2 h_4 \tilde{u}^{(2)}_2 \tilde{r}^{(2)}_2)) [R^{(1)}_1 h_2 \tilde{u}^{(1)}_1 \tilde{r}^{(1)} S(R^{(2)}_1 h_3 \\
& \tilde{u}^{(2)}_1 \tilde{r}^{(2)}_1) S(u^{(1)}) u^{(2)}] \sigma^{-1}(S(h_5) S(R^{(2)}_3), h_6) \\
\stackrel{(u)^+(1')}{=} & \sum \sigma(h_1, S(h_2) S(\tilde{R}^{(2)})) [R^{(1)}_1 \tilde{R}^{(1)}_1 S(R^{(2)}_2 \tilde{R}^{(1)}_2) S(u^{(1)}) u^{(2)}]
\end{aligned}$$

$$\begin{aligned} & \sigma^{-1}(S(h_3), h_4) \\ (u)+([1, \underline{\text{Thm}}1.6]) & \sum \epsilon(h)[R^{(1)}r^{(1)}_1U^{(1)}_1S(u^{(1)}R^{(2)}r^{(1)}_2U^{(1)}_2)u^{(2)}r^{(2)}U^{(2)}] \\ & \sigma^{-1}(S(h_3), h_4) \\ (1')+\underline{(2')} & \sum \epsilon(h) \end{aligned}$$

Similarly, we can prove that  $(S^{\sigma-R} \star I)(h) = \epsilon(h)$ . □

**Corollary 1.6** ([1, Theorem 1.6]). *Let  $A$  be  $\sigma$ -Hopf algebra. Then  $A^\sigma$  is a Hopf algebra.*

PROOF: Take  $R = 1 \otimes 1$  in Theorem 1.5. Then  $A$  is  $\sigma - R$  compatible Hopf algebra. This completes the proof. □

**Corollary 1.7.** *Let  $H$  be  $R$ -Hopf algebra. Then  $A^R$  is a Hopf algebra.*

PROOF: Take  $\sigma$  be trivial in Theorem 1.5. Then  $H$  is  $\sigma - R$  compatible Hopf algebra. This completes the proof. □

**2. Braided structure over  $H^{\sigma-R}$**

The main purpose of this section is to generalize [1, Theorem 1.6(c)].

**Theorem 2.1.** *Let  $H$  be an  $\sigma - R$ -compatible Hopf algebra, and  $(H, \tau)$  a braided Hopf algebra. Then  $H^{\sigma-R}$  is a braided Hopf algebra with braided structure as follows: For all  $h, l \in H$*

$$\begin{aligned} \tilde{\sigma}(h, l) = & \sum \sigma(R^{(1)}(r^{(1)}_1h_1u_1^{(1)})U^{(1)}, \tilde{R}^{(1)}(\tilde{r}^{(1)}_1l_1\tilde{u}^{(1)}_1)\tilde{U}^{(1)}) \\ & \tau(\tilde{R}^{(2)}(r^{(1)}_2l_2\tilde{u}^{(1)}_2)\tilde{U}^{(2)}, R^{(2)}(r^{(1)}_2h_2u^{(1)}_2)U^{(2)}) \\ & \sigma^{-1}(\tilde{r}^{(2)}l_3\tilde{u}^{(2)}, r^{(2)}h_3u^{(2)}) \end{aligned}$$

PROOF: Firstly, the above braided structure of  $\tilde{\sigma}(h, l)$  will be simplified as following:

$$\begin{aligned} \tilde{\sigma}(h, l) & \stackrel{(i)}{=} \sum \sigma((r^{(1)}_1h_1u_1^{(1)})U^{(1)}, (\tilde{r}^{(1)}_1l_1\tilde{u}^{(1)}_1)\tilde{U}^{(1)}) \\ & \quad \tau((\tilde{r}^{(1)}_2l_2\tilde{u}^{(1)}_2)\tilde{U}^{(2)}, (r^{(1)}_2h_2u^{(1)}_2)U^{(2)})\sigma^{-1}(\tilde{r}^{(2)}l_3\tilde{u}^{(2)}, r^{(2)}h_3u^{(2)}) \\ & \stackrel{(d)}{=} \sum \sigma(r^{(1)}_1h_1u_1^{(1)}, \tilde{r}^{(1)}_1l_1\tilde{u}^{(1)}_1)\tau(\tilde{r}^{(1)}_2l_2\tilde{u}^{(1)}_2, r^{(1)}_2h_2u^{(1)}_2) \\ & \quad \sigma^{-1}(\tilde{r}^{(2)}l_3\tilde{u}^{(2)}, r^{(2)}h_3u^{(2)}) \\ & \stackrel{(iv)}{=} \sum \sigma(h_1u_1^{(1)}, l_1\tilde{u}^{(1)}_1)\tau(l_2\tilde{u}^{(1)}_2, h_2u^{(1)}_2)\sigma^{-1}(l_3\tilde{u}^{(2)}, h_3u^{(2)}) \\ & \stackrel{(s)}{=} \sum \sigma(h_1, l_1)\tau(l_2, h_2)\sigma^{-1}(l_3, h_3) \end{aligned}$$

Secondly,  $\tilde{\sigma}$  satisfies conditions in [1, Definition 1.1] by straightforward computations. □

**Remark.** This theorem shows that  $R$  does not affect the braided structure  $\tilde{\sigma}$ .

**Corollary 2.2.** *Let  $H$  be a  $R$ -Hopf algebra, and  $(H, \tau)$  a braided Hopf algebra. Then  $(H^{\sigma(\tau)}, \tilde{\sigma})$  is a braided Hopf algebra with braiding:*

$$\tilde{\sigma}(x, y) = \sum \sigma(x_1, y_1)\tau(y_2, x_2)\sigma^{-1}(y_3, x_3)$$

PROOF: Let  $R$  be trivial in Theorem 2.1. This completes the proof. □

**Corollary 2.3** ([1, Theorem 1.6(c)]). *Let  $H$  be a commutative  $\sigma$ -Hopf algebra. Then  $(H^\sigma, \tilde{\sigma})$  is a braided Hopf algebra with braiding:*

$$\tilde{\sigma}(x, y) = \sum \sigma(y_1, x_1)\sigma^{-1}(x_2, y_2)$$

PROOF: Let  $\tau$  be trivial in Theorem 2.2. This completes the proof. □

Dually, we have the following result:

**Theorem 2.4.** *Let  $H$  be an  $\sigma - R$ -compatible Hopf algebra, and  $(H, F)$  a quasitriangular Hopf algebra. Then  $H^{\sigma-R}$  is a quasitriangular Hopf algebra with the following quasitriangular structure:*

$$\tilde{R} = \sum U^{(2)}F^{(1)}R^{(1)} \otimes U^{(1)}F^{(2)}R^{(2)}.$$

### 3. Classification of $H^{\sigma-R}$ and examples

Let  $u \in Hom_k(k, H)$  be convolution invertible with  $\epsilon u = \epsilon$ , and suppose that  $H$  is not only a  $R$ -Hopf algebra but also a  $F$ -Hopf algebra satisfying:

$$(1) \quad \sum F^{(1)} \otimes F^{(2)} = \sum (u^{-1} \otimes u^{-1})(R^{(1)} \otimes R^{(2)})\Delta u.$$

This equivalent to

$$\sum G^{(1)} \otimes G^{(2)} = \sum \Delta u^{-1}(U^{(1)} \otimes U^{(2)})(u \otimes u)$$

where  $G$  is the inverse of  $F$  ( $G = \sum G^{(1)} \otimes G^{(2)}$ ). Now we have

**Theorem 3.1.** *Let  $H^{\sigma-R}$  and  $H^{\sigma-F}$  be two Hopf algebras satisfying condition (1). Then  $H^{\sigma-R} \cong H^{\sigma-F}$  (as coalgebra).*

PROOF: Define  $\psi : H^{\sigma-R} \rightarrow H^{\sigma-F}$  by  $h \rightarrow u^{-1}hu$  and  $\varphi : H^{\sigma-F} \rightarrow H^{\sigma-R}$  by  $l \rightarrow ulu^{-1}$ . We can prove the claim by direct computations. □

Dually, Let  $u \in H^*$  with a convolution inverse  $u^{-1}$  and  $u(1) = 1$ , and suppose that  $H$  is not only an  $\sigma$ -Hopf algebra but also a  $\tau$ -Hopf algebra satisfying:

$$(2) \quad \tau(h, l) = \sum u^{-1}(l_1) \otimes u^{-1}(h_1)\sigma(h_2, l_2)u(h_3, l_3).$$

Then we have the following theorem.

**Theorem 3.2.** *Let  $H^{\sigma-R}$  and  $H^{\tau-R}$  be two Hopf algebras satisfying condition (1). Then  $H^{\sigma-R} \cong H^{\tau-R}$  (as algebra).*

PROOF: Define the maps  $\psi : H^{\sigma-R} \rightarrow H^{\tau-R}$  by  $h \rightarrow u(h_1)h_2$  and  $\varphi : H^{\tau-R} \rightarrow H^{\sigma-R}$  by  $l \rightarrow u^{-1}(l_1)l_2$ . We can prove the claim by direct computations. □

**Example 1.** Let  $H_4$  be the Sweedler’s 4-dimensional Hopf algebra ([6]). Define  $\sigma : H_4 \otimes H_4 \rightarrow k$  as follows:

$$\begin{matrix} & 1 & x & y & z \\ 1 & 1 & 1 & 0 & 0 \\ x & 1 & -1 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & 0 \end{matrix}$$

and let  $R = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) \in H \otimes H$ . It is easy to prove that  $H_4$  is a  $\sigma - R$  compatible Hopf algebra.

**Example 2.** Let  $B \bowtie_{\tau} H$  be the Doi-Takeuchi’s Hopf algebra, and  $B \otimes_R H$  a Hopf algebra. Let  $R = R^{(1)} \otimes R^{(2)} \in B \otimes H$  satisfy  $\tau(R^{(1)}cr^{(1)}, h)R^{(2)} \otimes r^{(2)} = \tau(c, h)1 \otimes 1 = \tau(c, R^{(2)}hr^{(2)})R^{(1)} \otimes r^{(1)}$ . Then  $B \otimes H$  is a  $[\tau] - Q$  compatible Hopf algebra, where  $[\tau](b \otimes g, c \otimes h) = \epsilon(b)\epsilon(h)\tau(c, g)$  for all  $b, c \in B, g, h \in H$  and  $Q = \sum 1 \otimes R^{(2)} \otimes R^{(1)} \otimes 1$ .

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