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Centered-Lindelöfness versus star-Lindelöfness

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Abstract. We discuss various generalizations of the class of Lindelöf spaces and study the difference between two of these generalizations, the classes of star-Lindelöf and centered-Lindelöf spaces.

Keywords: star-Lindelöf, centered-Lindelöf, linked-Lindelöf, CCC-Lindelöf, metaLindelöf, paraLindelöf, weakly separable, CCC, $C_p(X)$

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1. Introduction and positive results

The aim of this paper is to compare two particular generalizations of Lindelöf spaces indicated in the title. Before we start, we have to find out which place these two classes, star-Lindelöf and centered-Lindelöf spaces, take among other classes of generalized Lindelöf spaces. The most well known consequences of the Lindelöf property are the countability of the extent (recall that $e(X) = \omega$ provided every closed discrete subset of X is countable) and the Discrete Countable Chain Condition, DCCC (recall that X has DCCC provided every discrete family of nonempty open subsets of X is at most countable, see, for example [11]; sometimes DCCC is called pseudoLindelöfness). It is clear that the countability of the extent implies DCCC and that the converse is true in collectionwise Hausdorff spaces. Other generalizations of Lindelöfness are based on the idea of declaring of the existence, rather than of a countable subcover, of a subcover that can be represented as the union of countably many subfamilies which are linked in some way (usually instead of a subcover one can speak about a refinement). All these properties are between the countability of the extent and DCCC.

Now we recall some notation and definitions and introduce some new. Let $\mathcal{U}$ be a cover of the space X. For a set $A \subset X$, we write $St(A,\mathcal{U}) = St^1(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and $St^{n+1}(A,\mathcal{U}) = St(St^n(A,\mathcal{U}),\mathcal{U})$. It is convenient to set also $St^0(A,\mathcal{U}) = A$. Thus $St^n(A,\mathcal{U})$ is defined for all $n \in \omega$. For $A = \{x\}$ we usually write $St(x,\mathcal{U})$ instead of $St(\{x\},\mathcal{U})$. Further, put $\mathcal{U}^n = \{St^n(x,\mathcal{U}) : x \in X\}$ and $\mathcal{U}^{n\frac{1}{2}} = \{St^n(U,\mathcal{U}) : U \in \mathcal{U}\}$. We denote $\tilde{\mathbb{N}}$ the set of all numbers of the form n or $n\frac{1}{2}$ where $n \in \mathbb{N} = \{1, 2, \dots\}$. A family of sets $\mathcal{F}$ is called *centered* if any finite subfamily of $\mathcal{F}$ has nonempty intersection; $\mathcal{F}$ is *linked* if any two elements of $\mathcal{F}$ have nonempty intersection. Naturally, a family is σ-centered (σ-linked) if it can be represented as the union of countably many centered (respectively, linked) subfamilies. Say that a family of sets $\mathcal{F}$ is CCC if every pairwise disjoint subfamily of $\mathcal{F}$ is at most countable.

Definition 1. *A space is:*

— *n-star-Lindelöf (where $n \in \tilde{\mathbb{N}}$) if for every open cover $\mathcal{U}$ the cover $\mathcal{U}^n$ contains a countable subcover,*
— *centered-Lindelöf if every open cover contains a σ-centered subcover,*
— *linked-Lindelöf if every open cover contains a σ-linked subcover,*
— *CCC-Lindelöf if every open cover contains a CCC subcover.*

1-star-Lindelöfness is called just *star-Lindelöfness* [15], [16], [18]. As it was noted by the referee, it would be nice to call star-Lindelöfness fixed-Lindelöfness (say that a family of sets is fixed if it has nonempty intersection; a σ-fixed family is the union of countably many fixed subfamilies); this would make the definitions of all properties under consideration more uniform. However, we decided not to change the notation after many papers published. For the same reason, in the definition of n-star-Lindelöfness, we follow the terminology of earlier papers rather than that of [11]. For $n \in \mathbb{N}$, n-star-Lindelöfness was defined in [15], [16], [18]; for n of the form $m\frac{1}{2}$ where $m \in \mathbb{N}$ it was defined in [11]. Centered-Lindelöfness was introduced in [8]. There are other generalizations of Lindelöfness, but here we do not intend to give a complete picture (the reader can find more details in [19]). Instead, we indicate some interrelations with some non-Lindelöf-type properties important for the constructions below.

First of all, it is easy to see that star-Lindelöfness follows not only from Lindelöfness, but also from separability and countable compactness. A Hausdorff space is countably compact iff it is starcompact ([11]) (a space X is starcompact if for every open cover $\mathcal{U}$ there is a finite $F \subset X$ such that $St(F,\mathcal{U}) = X$ ([14])). Another key idea is the notion of weak separability (=having a σ-centered base or π-base). Though the name "weakly separable" appeared only in [6], [7], this class of spaces was considered in [3], [9], [17].

Definition 2. *A space is weakly separable provided it has a σ-centered π-base.*

Theorem 1 ([9], [17])**.** *For a Tychonoff space X, the following conditions are equivalent:*

(1) *X is weakly separable,*
(2) *βX is separable,*
(3) *every Hausdorff compactification of X is separable,*
(4) *X has a separable Tychonoff extension.*

In other words, weakly separable Tychonoff spaces are just the dense subspaces of separable Tychonoff spaces. For example, all dense subspaces of I^κ (or D^κ, or $\mathbb{R}^\kappa$), where $\kappa \leq \mathbf{c}$, are weakly separable. Therefore, $C_p(X)$, the space of all real-valued continuous functions on X equipped with the pointwise convergence topology, is weakly separable whenever $|X| \leq \mathbf{c}$.

The implications in the following diagram either are straightforward or can be found in [11] or [19].

$$
\begin{tikzcd}
\text{Lindelöf} \arrow[dr] & & \text{countably compact} \arrow[dl] & & & \\
& \text{countable extent} \arrow[drr, "T_1"'] & & & \text{separable} \arrow[dl] \arrow[dr] & \\
& & & \text{star-Lindelöf} \arrow[d] & & \text{weakly separable} \arrow[dll] \arrow[dd] \\
& & & \text{centered-Lindelöf} \arrow[d] & & \\
& & & \text{linked-Lindelöf} \arrow[d] & & \text{CCC} \arrow[dll] \\
& & & \text{CCC-Lindelöf} \arrow[d, "T_1"'] & & \\
& & & 1\tfrac{1}{2}\text{-star-Lindelöf} \arrow[d] & & \\
& & & \text{2-star-Lindelöf} \arrow[d] & & \\
& & & 2\tfrac{1}{2}\text{-star-Lindelöf} \arrow[rr, shift left] & & \text{DCCC} \arrow[ll, shift left, "T_3"]
\end{tikzcd}
$$

Note that neither of the two properties, star-Lindelöfness and weak separability, implying centered-Lindelöfness in the diagram, follows from the other. A non-separable compact (hence star-Lindelöf) space is not weakly separable; a variety of examples of weakly separable spaces which are not star-Lindelöf is given in the examples section. Therefore the arrows from star-Lindelöfness and weak separability to centered-Lindelöfness cannot be reversed.

The "trunk" of the diagram, the line connecting n-star-Lindelöf properties for different n, has been studied in details in [11] and other papers. It is now reasonable to give further structure to the interval between star-Lindelöf and $1\frac{1}{2}$-star-Lindelöf by introducing centered-, linked- and CCC-Lindelöfness. In this paper we consider the first of these three properties as compared with star-Lindelöfness. To show the delicacy of the problem, let us compare it with the situation around compactness-type properties. A space is centered-compact ([8]) if every open cover has a subcover that can be represented as the union of finitely many centered subfamilies.

Theorem 2 ([8])**.** *A Hausdorff space is centered-compact iff it is starcompact* (*equivalently, countably compact*)*.*

(However, a T_1, centered-compact, non-starcompact space does exist ([8])). Therefore, in the class of Hausdorff spaces one can expect to distinguish between "centered" and "star" only in the infinite level. The problem to distinguish centered-Lindelöf and star-Lindelöf spaces was stated in [8] where a Hausdorff example was given. In this paper we give several Tychonoff examples, but first we consider some conditions under which centered-Lindelöfness does imply star-Lindelöfness. First of all, as we noted before, a collectionwise Hausdorff DCCC space has countable extent. Therefore all collectionwise Hausdorff (in particular, all monotonically normal, all GO, etc.) centered-Lindelöf spaces are star-Lindelöf.

Further, it is easy to see that a locally separable $1\frac{1}{2}$-star-Lindelöf space is star-Lindelöf. Therefore, every locally separable centered-Lindelöf space is star-Lindelöf. The next observation seems less trivial.

Proposition 3. *Every Hausdorff locally compact centered-Lindelöf space is star-Lindelöf.*

Proof: Let X be a Hausdorff locally compact centered-Lindelöf space and $\mathcal{U}$ an open cover of X. Define the family $\mathcal{V} = \{V : V \text{ is open in } X, \overline{V} \text{ is compact and } \overline{V} \subset U \text{ for some } U \in \mathcal{U}\}$. Then $\mathcal{V}$ is an open refinement of $\mathcal{U}$. As X is centered-Lindelöf, there exist countably many centered subfamilies $\mathcal{V}_n$ $(n \in \omega)$ of $\mathcal{V}$ such that $\bigcup\{\bigcup\mathcal{V}_n : n \in \omega\} = X$. For every $V \in \mathcal{V}$, choose an $U_V \in \mathcal{U}$ such that $\overline{V} \subset U_V$. Then for every $n \in \omega$, $\mathcal{U}_n = \{U_V \in \mathcal{U} : V \in \mathcal{V}_n\}$ is a subfamily of $\mathcal{U}$ such that $\bigcap\mathcal{U}_n \neq \emptyset$ and $\bigcup\{\bigcup\mathcal{U}_n : n \in \omega\} = X$. For every $n \in \omega$, fix a point $x_n \in \bigcap\mathcal{U}_n$. Then $F = \{x_n : n \in \omega\}$ is a countable subset of X such that $St(F,\mathcal{U}) = X$, and so X is star-Lindelöf. □

Question 1. *Must every Hausdorff locally compact linked-Lindelöf* (*or CCC-Lindelöf*) *space be star-Lindelöf?*

A Hausdorff locally compact $1\frac{1}{2}$-star-Lindelöf space need not be star-Lindelöf (even CCC-Lindelöf). Indeed, let Z be the discrete space of cardinality ω_1. Then $X = (\beta Z \times (\omega_2 + 1)) \setminus ((\beta Z \setminus Z) \times \{\omega_2\})$ is a counterexample.

Recall that a space is paraLindelöf provided every open cover has a locally countable open refinement. Every paraLindelöf centered-Lindelöf space is star-Lindelöf. Moreover, we have the following:

Proposition 4. *Every paraLindelöf $1\frac{1}{2}$-star-Lindelöf space is star-Lindelöf.*

Proof: Let X be a paraLindelöf $1\frac{1}{2}$-star-Lindelöf space and let $\mathcal{U}$ be an open cover of X. Then there is a locally countable open refinement $\mathcal{V}$ of $\mathcal{U}$. For every $x \in X$ fix an open set O_x such that $x \in O_x \subset V$ for some $V \in \mathcal{V}$ and O_x meets at most countably many elements of $\mathcal{V}$. Then $\mathcal{O} = \{O_x : x \in X\}$ is an open refinement of $\mathcal{V}$. Since X is $1\frac{1}{2}$-star-Lindelöf, there is a countable subfamily $\mathcal{O}_0 \subset \mathcal{O}$ such that $St(\bigcup\mathcal{O}_0, \mathcal{O}) = X$. Then we have $St(\bigcup\mathcal{O}_0, \mathcal{V}) = X$. Denote

$\mathcal{P} = \{(O, V) : O \in \mathcal{O}_0, V \in \mathcal{V}$ and $O \cap V \neq \emptyset\}$. Then $\mathcal{P}$ is a countable family. For every $(O, V) \in \mathcal{P}$, choose $z_{O,V} \in O \cap V$ and put $F = \{z_{O,V} : (O, V) \in \mathcal{P}\}$. Then $St(F, \mathcal{V}) = X$, and hence $St(F, \mathcal{U}) = X$. □

As we will see in the next section, in the previous proposition, paraLindelöfness cannot be replaced by metaLindelöfness (even by metacompactness).

As a natural generalization of star-Lindelöfness and centered-Lindelöfness one can consider the following cardinal functions:

Definition 3 ([18])**.** *The star-Lindelöf number of the space X is*

$$st\text{-}l(X) = \min\{\kappa : \forall \text{ open cover } \mathcal{U} \text{ of } X,\ \exists\, F \subset X \text{ such that } |F| \leq \kappa \text{ and } St(F, \mathcal{U}) = X\}.$$

Definition 4 ([8])**.** *The centered-Lindelöf number of the space X is*

$$ct\text{-}l(X) = \min\{\kappa : \text{ every open cover of } X \text{ has a } \kappa\text{-centered subcover}\}.$$

(κ-centered means representable as the union of κ many centered subfamilies.)

In all Hausdorff examples of centered-Lindelöf, non-star-Lindelöf spaces known to the authors (that is all examples described below and the example in [8]) the star-Lindelöf number is not greater than $\mathbf{c}$. Therefore, it is natural to ask

Question 2. *Is it true that the star-Lindelöf number of a centered-Lindelöf Hausdorff* (*regular, normal*) *space cannot be greater than* $\mathbf{c}$*?*

More general,

Question 3. *Is the following inequality true in the class of Hausdorff spaces?*

$$st\text{-}l(X) \leq 2^{ct\text{-}l(X)}.$$

It is true that some of our examples are weakly separable and for a Tychonoff weakly separable space X it follows from Theorem 1 that $st\text{-}l(X) \leq w(X) \leq 2^{d(\beta X)} = \mathbf{c}$. However, other examples in this paper and in [8] are far from being weakly separable which makes Questions 2 and 3 reasonable.

Note that for T_1, centered-Lindelöf spaces the star-Lindelöf number can be arbitrarily big: for every cardinal κ there exists a T_1, centered-Lindelöf space X with $st\text{-}l(X) > \kappa$. Indeed, let $\tau > \kappa$ be a cardinal such that $\tau^\kappa = \tau$. On a set $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$ and $|X_1| = |X_2| = \tau$ we define a T_1 topology, $\mathcal{T}$, all nonempty open sets in which take the form $A \cup (X_2 \setminus K)$ where $A \subset X_1$ is an arbitrary subset and $|K| \leq \kappa$. It is easy to see that $(X, \mathcal{T})$ is centered-Lindelöf (in fact every open cover is a centered family) and $st\text{-}l((X, \mathcal{T})) = \kappa^+$.

The examples described below have different sets of additional properties from the following list: pseudocompact, c.c.c., connected, zero-dimensional, topological linear space, scattered, metaLindelöf, $< \mathbf{c}$-discrete (i.e. with all subsets of cardinality $< \mathbf{c}$ closed and discrete; see Section 2.3), etc. However the following question remains open:

Question 4. *Is there a ZFC example of a centered-Lindelöf normal space which is not star-Lindelöf?*

(A consistent example is given in Section 2.4.)

We conclude this section with one more theorem showing that the difference between the two classes of spaces we consider in this paper is not so big: every space from one class is representable as a closed G_δ-set in another class.

Theorem 5. *Every Tychonoff centered-Lindelöf space is a closed G_δ-set in some Tychonoff star-Lindelöf space. Specifically, if X is Tychonoff and centered-Lindelöf, then $\mathcal{R}(X) = \beta X \times (\omega + 1) \setminus ((\beta X \setminus X) \times \{\omega\})$ is star-Lindelöf.*

Proof: Let $\mathcal{U}$ be an open cover of $\mathcal{R}(X)$. For every $x \in X$, choose $U(x) \in \mathcal{U}$, $n(x) \in \omega$, and $V(x)$, an open neighbourhood of x in X so that $\overline{V(x) \times [n(x), \omega]} \subset U(x)$. Then $\mathcal{V} = \{V(x) : x \in X\}$ is an open cover of X. Since X is centered-Lindelöf, we have $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \omega\}$ where each $\mathcal{V}_n$ is a centered family. For $n, m \in \omega$, denote $\mathcal{V}_{n,m} = \{V = V(x) \in \mathcal{V}_n : n(x) = m\}$. Every $\mathcal{V}_{n,m}$ also is a centered family. Since βX is compact, there is a common adherence point $z_{n,m}$ for $\mathcal{V}_{n,m}$. Put $p_{n,m} = (z_{n,m}, m)$ and $P = \{p_{n,m} : n, m \in \omega\}$. We claim that $St(P,\mathcal{U}) \supset X \times \{\omega\}$. Indeed, let $x \in X$. Then $p_{n(x),n(x)} \in \overline{V(x) \times [n(x), \omega]} \subset U(x)$, hence $(x, \omega) \in St(p_{n(x),n(x)},\mathcal{U}) \subset St(P,\mathcal{U})$. Next, since $\beta X \times \{n\}$ is compact, there is a finite set $Q_n \subset \beta X \times \{n\}$ such that $St(Q_n,\mathcal{U}) \supset \beta X \times \{n\}$. Put $F = P \cup \bigcup\{Q_n : n \in \omega\}$. Then $St(F,\mathcal{U}) = \mathcal{R}(X)$. □

2. Examples

A variety of examples of weakly separable, non separable spaces was constructed in [9], [10], [17]. However, most of these examples are either Lindelöf or countably compact (hence they are star-Lindelöf).

Below we present a collection of examples of centered-Lindelöf, non star-Lindelöf spaces. Most of these examples have been known before. Many of them are weakly separable, but not all: for the example X from Section 2.1 we have $d(bX) = \omega_1$ for every compactification bX; moreover, for every cardinal $\kappa > \omega_1$ this example can be easily modified so that $d(bX) \geq \kappa$ for every compactification.

2.1 A "fat-Ψ" example

The idea of constructing a "fat-Ψ" space is to replace isolated points in the well-known Isbell-Mrówka space Ψ (see [13, 3.6.1.(a)]) with some infinite "building blocks" (see [5], [8], [11], [20], [26] for particular "fat-Ψ" spaces constructions).

In our case the building block will be the ordinal space ω_1 with order topology; put $J = \bigcup\{J_n : n \in \omega\}$, the discrete sum of subspaces J_n each of which is homeomorphic to ω_1: $J_n = \{p_{n\alpha} : \alpha \in \omega_1\}$. Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω, $|\mathcal{R}| = \mathbf{c}$. Put $X = J \cup \mathcal{R}$. We declare J to be open in X. A basic neighbourhood of the point $r \in \mathcal{R}$ takes the form $O_{Kf}(r) =$

$\{r\} \cup \{p_{n\alpha} : n \in r \setminus K, \alpha > f(n)\}$, where K is an arbitrary finite subset of ω and f is an arbitrary function from ω to ω_1.

First we prove that X is centered-Lindelöf (note that X is not weakly separable because $\beta\omega_1$ is not separable). Let $\mathcal{U}$ be an open cover of X. Since each J_n is countably compact, hence centered-compact, it is covered by finitely many centered subfamilies of $\mathcal{U}$. Therefore, the whole J is covered by countably many centered subfamilies of $\mathcal{U}$, say $\mathcal{V}_n : n \in \omega$. For each point $r \in \mathcal{R}$ fix an element $U_r \in \mathcal{U}$ containing this point. Then for some $n = n(r)$, U_r contains a final interval of J_n. Then $\mathcal{W}_n = \{U_r : n(r) = n\}$ is a centered subfamily of $\mathcal{U}$ and $\mathcal{R} \subset \cup\bigcup\{\mathcal{W}_n : n \in \omega\}$. Therefore $X \subset \cup\bigcup\{\mathcal{W}_n \cup \mathcal{V}_n : n \in \omega\}$, that is X is centered-Lindelöf.

Now we prove that X is not star-Lindelöf. Enumerate all countable subsets of J on type $\mathbf{c}$: $[J]^\omega = \{C_\alpha : \alpha < \mathbf{c}\}$ and enumerate $\mathcal{R}$ on type $\mathbf{c} \times \mathbf{c}$: $\mathcal{R} = \{r_{\alpha\beta} : \alpha, \beta < \mathbf{c}\}$. For $r = r_{\alpha\beta} \in \mathcal{R}$ put $O_r = \{r\} \cup (\bigcup\{J_n : n \in r\} \setminus C_\alpha)$. Then O_r is open in X and $O_r \cap \mathcal{R} = \{r\}$ $\quad(\dagger)$. Put $\mathcal{U} = \{O_r : r \in \mathcal{R}\} \cup \{J\}$. Then $\mathcal{U}$ is an open cover of X and for every countable subset $F \subset X$, $St(F,\mathcal{U}) \neq X$. Indeed, $F = F_{\mathcal{R}} \cup F_J$, where $F_{\mathcal{R}} = F \cap \mathcal{R}$ and $F_J = F \cap J$. Then $F_J = C_\alpha$ for some α and for every $\beta < \mathbf{c}$ we have that $r_{\alpha\beta} \notin St(F_J,\mathcal{U})$. For all these points, by $(\dagger)$, the only possibility to be in $St(F_{\mathcal{R}},\mathcal{U})$ is that $r_{\alpha\beta} \in F_{\mathcal{R}}$. But $|F_{\mathcal{R}}| \leq \omega$ and we have $\mathbf{c}$-many points $r_{\alpha\beta}$.

Note that $st\text{-}l(X) = d(X) = \omega_1$ and that the space X is zero-dimensional and scattered (recall that a space Y is *scattered* provided every closed subset $A \subset Y$ contains a point which is isolated in A).

2.2 $C_p(X)$ examples

In this section we consider only Tychonoff spaces.

As we noted above, $C_p(X)$ is weakly separable (hence centered-Lindelöf) whenever $|X| \leq \mathbf{c}$. Also, it is easy to see that a space is Lindelöf iff it is both star-Lindelöf and metaLindelöf. Therefore, we will obtain the desired example (of a centered-Lindelöf, non star-Lindelöf space of the form $C_p(X)$) if we find X such that $|X| \leq \mathbf{c}$ and $C_p(X)$ is metaLindelöf but not Lindelöf. We will use the following

Theorem 6 ([12]). *If X is a compact Hausdorff space and $w(X) \leq \omega_1$, then $C_p(X)$ is hereditarily metaLindelöf.*

So if X is compact, $w(X) \leq \omega_1$ and $C_p(X)$ is not Lindelöf, then $C_p(X)$ is (weakly separable but) not star-Lindelöf.

Theorem 7 ([4]). *If $C_p(X)$ is a Lindelöf space, then $t(X^n) = \omega$ for every finite n.*

Therefore for $X = \omega_1 + 1$ (with order topology) $C_p(X)$ is weakly separable but not star-Lindelöf.

Using another known fact about $C_p(X)$ spaces we obtain a separable compact space X with the same properties.

Theorem 8 ([23], see also [1])**.** *If X is an uncountable, compact, separable, scattered space the ω_1-th derivative of which is empty, then $C_p(X)$ is not normal (and hence not Lindelöf).*

(Recall that in a scattered space the *α-th derivative* of the set K is defined by induction: $K^0 = K$; $K^{\alpha+1} = K^\alpha \setminus \{p \in K^\alpha : p \text{ is isolated}\}$, $K^\alpha = \bigcap\{K^\gamma : \gamma < \alpha\}$ if α is a limit ordinal).

Now we give an example of a compact space F which satisfies all conditions of Theorems 6 and 8 and hence $C_p(F)$ is weakly separable but not star-Lindelöf.

Let $\mathcal{N}$ be a countable set and $\mathcal{R}$ be an almost disjoint family of infinite subsets of $\mathcal{N}$, $|\mathcal{R}| = \omega_1$. We define the topology on $F_0 = \mathcal{N} \cup \mathcal{R}$ just like that of the usual Isbell-Mrówka space (the only difference is that the family $\mathcal{R}$ is not necessarily *maximal* almost disjoint): the points of $\mathcal{N}$ are isolated in F_0 while a basic neighbourhood of a point $r \in \mathcal{R}$ takes the form $\{r\} \cup (r \setminus K)$, where K is an arbitrary finite subset of $\mathcal{N}$. Then F_0 is locally compact. Denote F the one-point compactification of F_0. Then F satisfies all conditions of Theorems 6 and 8; in particular $F^3 = \emptyset$.

Question 5 (A.V. Arhangel'skii)**.** *Does there exist a compact space X for which $C_p(X)$ is star-Lindelöf but not Lindelöf?*

The motivation for this question is that it is difficult to construct compact spaces X such that $C_p(X)$ is not metaLindelöf: the existence of such spaces remained an open problem for some time ([2, Problem 76]) until it was solved in [12].

2.3 A pseudocompact example

E.A. Reznichenko constructed [25] (see also [1]) an example of a dense subspace $X \subset I^{\mathbf{c}}$ with the following properties:

(1) X is pseudocompact,
(2) X is connected,
(3) $\forall\, H \subset X$, $|H| < \mathbf{c} \Rightarrow H$ is closed and discrete in X,
(4) $|X| = \mathbf{c}$.

Since X is dense in $I^{\mathbf{c}}$, it is weakly separable and hence centered-Lindelöf. To prove that X is not star-Lindelöf we need also the following property

(5) $\exists\, Z \subset X$ such that $|Z| = \mathbf{c}$ and Z is closed and discrete in X.

Lemma 9. $(3), (4), (5) \Rightarrow st\text{-}l(X) = \mathbf{c}$.

Proof: Enumerate the points of Z on type $\mathbf{c}$: $Z = \{z_\alpha : \alpha < \mathbf{c}\}$ and the points of $X \setminus Z$ on type $\kappa = |X \setminus Z| \le \mathbf{c}$: $X \setminus Z = \{x_\gamma : \gamma < \kappa\}$. For $\alpha < \mathbf{c}$, put $U_\alpha = \{z_\alpha\} \cup \{x_\gamma : \gamma \ge \alpha\}$. Then $\mathcal{U} = \{U_\alpha : \alpha < \mathbf{c}\}$ is an open cover of X such that for every $A \subset X$ with $|A| < \mathbf{c}$, we have $St(A, \mathcal{U}) \ne X$. It remains to note that $st\text{-}l(X) \le |X| = \mathbf{c}$. □

Now we adjust the construction of Reznichenko's example to fulfil (5).

Let A be a set of cardinality $\mathbf{c}$. It can be represented as $A = \bigcup\{A_\alpha : \alpha < \mathbf{c}\}$ where $A_\alpha \cap A_\beta = \emptyset$ whenever $\alpha \neq \beta$ and $|A_\alpha| = \mathbf{c}$ for each α. Let b be a bijection of $\mathbf{c}$ onto $\mathbf{c} \times \mathbf{c}$; for each $\alpha < \mathbf{c}$ we denote $C_\alpha = \bigcup\{A_\gamma : \gamma \in b^{-1}\{(\alpha, \lambda) : \lambda < \mathbf{c}\}\}$. In other words, we define $\mathbf{c}$-many "big blocks" C_α as the unions of $\mathbf{c}$-many "small blocks" A_γ each. Then $A = \bigcup\{C_\alpha : \alpha < \mathbf{c}\}$, $C_\alpha \cap C_\beta = \emptyset$ whenever $\alpha \neq \beta$ and $|C_\alpha| = \mathbf{c}$ for each α. Put $Q = \bigcup\{I^B : B \subset A, |B| \leq \omega\}$. For each $q \in Q$ denote $B(q)$ the (unique) subset $B \subset A$ for which $q \in I^B$. Since $|Q| = \mathbf{c}^\omega = \mathbf{c}$, Q can be enumerated on type $\mathbf{c}$: $Q = \{q_\alpha : \alpha < \mathbf{c}\}$. For each $\alpha < \mathbf{c}$ we define the points $x_\alpha, z_\alpha \in I^A$ such that

$$x_\alpha(a) = \begin{cases} q_\alpha(a) & \text{if } a \in B(q_\alpha) \\ 1 & \text{if } a \in A_\alpha \setminus B(q_\alpha) \\ 0 & \text{if } a \in A \setminus (A_\alpha \cup B(q_\alpha)), \end{cases}$$

$$z_\alpha(a) = \begin{cases} 1 & \text{if } a \in C_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Put $X = \{x_\alpha : \alpha < \mathbf{c}\}$, $Z = \{z_\alpha : \alpha < \mathbf{c}\}$ and $X' = X \cup Z$. Note that X is exactly the result of the original Reznichenko's construction and thus it has properties (1)–(4). Also, it is easy to see that $Z \cap X = \emptyset$ and the only limit point for Z in I^A is constant zero which is outside X', hence Z is closed and discrete in X'. Further, X' is connected and pseudocompact since it contains a connected and pseudocompact dense subspace X. So it remains to check that

(3′) $\forall\, H \subset X, |H| < \mathbf{c} \Rightarrow \overline{H} \cap Z = \emptyset$.

Then X' will have properties (1)–(4) of Reznichenko's example and also property (5). Let $H \subset X$, $|H| < \mathbf{c}$ and $z = z_\alpha \in Z$. Then there exists a subset $M \subset \mathbf{c}$, $|M| < \mathbf{c}$ such that $H = \{x_\beta : \beta \in M\}$. Since $|M| < \mathbf{c}$ and $|B(q_\beta)| \leq \omega$ for each $\beta \in M$, there exists $a \in C_\alpha \setminus (\bigcup\{B(q_\beta) \cup A_\beta : \beta \in M\})$; then $z_\alpha(a) = 1$ and $x_\beta(a) = 0$ for every $\beta \in M$. Hence $z_\alpha \notin \overline{H}$.

2.4 Pixley-Roy examples

C. Pixley and P. Roy introduced a topology for certain spaces of subsets of a space X ([22]) which was later systematically studied in [10] where it was called the Pixley-Roy topology. We follow the notation of [10] (see also [27]).

For a space X, $\mathcal{A}(X)$ and $\mathcal{F}(X)$ denote the families of all nonempty subsets of X and all nonempty finite subsets of X respectively. If A is a subset of X and U is a neighbourhood of A in X, then we put

$$[A, U] = \{S \subset X : A \subset S \subset U\}.$$

The Pixley-Roy topology on $\mathcal{A}(X)$ is the topology generated by the base consisting of all sets of the form $[A, U]$. The subspace topologies generated on the subspaces of $\mathcal{A}(X)$ (for example on $\mathcal{F}(X)$) are also called Pixley-Roy topologies. Considering $S \subset \mathcal{A}(X)$ we write $[A, U]$ instead of $[A, U] \cap S$. The families $\mathcal{A}(X)$ and $\mathcal{F}(X)$ equipped with the Pixley-Roy topology are denoted $\mathcal{A}[X]$ and $\mathcal{F}[X]$.

Proposition 10 ([10], see also [27]). *Let X be a Hausdorff space. Then*

(i) *$\mathcal{A}[X]$ is a zero-dimensional Hausdorff space,*
(ii) *$\mathcal{F}[X]$ is hereditarily metacompact,*
(iii) *if X is first-countable, then $\mathcal{F}[X]$ is a Moore space,*
(iv) *if X is infinite, then $d(\mathcal{F}[X]) = |X|$,*
(v) *if X is infinite, then $d(\beta\mathcal{F}[X]) = nw(X)$.*

Modifying (iv) a little we obtain

Proposition 11. *If X is infinite and Hausdorff, then $st\text{-}l(\mathcal{F}[X]) = |X|$.*

Proof: Let $\mathcal{U}$ be any cover of $\mathcal{F}[X]$ by basic open sets. Suppose $F \subset \mathcal{F}[X]$ and $|F| = \tau < |X|$. Since F consists of finite sets, then for the set $G = \bigcup F$ we have $|G| = \tau < |X|$. Pick $x \in X \setminus G$. Every element O of the cover $\mathcal{U}$ that contains x is a basic open set, so it takes the form $O = [\{x\}, U]$ for some neighbourhood U of x in X. Every element of O must contain x but no element of F contains x. So $O \cap F = \emptyset$ and $\{x\} \notin St(F, \mathcal{U})$.

It remains to note that $st\text{-}l(\mathcal{F}[X]) \leq |\mathcal{F}[X]| = |X|$. $\square$

Therefore if X is a Hausdorff, first-countable space, $|X| = \mathbf{c}$ and $nw(X) = \omega$ (e.g. $X = \mathbb{R}$ like in the original example of Pixley and Roy), then $\mathcal{F}[X]$ is Hausdorff, zero-dimensional (hence Tychonoff), Moore, hereditarily metacompact, weakly separable (hence centered-Lindelöf) and $st\text{-}l(\mathcal{F}[X]) = \mathbf{c}$. Or, just a little bit more general, if $X \subset \mathbb{R}$ and $|X| = \kappa$, then $\mathcal{F}[X]$ is Hausdorff, zero-dimensional, Moore, hereditary metacompact, weakly separable (hence centered-Lindelöf) and $st\text{-}l(\mathcal{F}[X]) = \kappa$.

Now we recall the following definition:

Definition 5 ([27], [24], see [21] for the details). *A Q-set is an uncountable set of reals such that in the subspace topology, every subset of it is an F_σ-set. A Q-set S is strong if for all $n \in \omega$, S^n is a Q-set in $\mathbb{R}^n$ (i.e. has the same property in $\mathbb{R}^n$: every subset is an F_σ-set in the relative topology).*

It is a known fact that the existence of a Q-set is consistent with and independent from ZFC and that the existence of a Q-set implies the existence of a strong Q-set.

Proposition 12 ([27]). *If $X \subset \mathbb{R}$ is a strong Q-set, then $\mathcal{F}[X]$ is perfectly normal.*

Assuming the existence of a Q-set, hence of a strong Q-set, X we have that the space $\mathcal{F}[X]$ for such X is a *perfectly normal* space that, by Propositions 10 and 11, has all properties of previous example.

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References

[1] Arhangel'skii A.V., *Topological function spaces*, Moskov. Gos. Univ. Publ., Moscow, 1989 (in Russian); English translation in: Mathematics and its Applications (Soviet Series) **78**, Kluwer Academic Publishers Group, Dordrecht, 1992.

[2] Arhangel'skii A.V., *C_p-theory*, Recent Progress in General Topology, M. Hušek and J. van Mill Eds., Elsevier Sci. Pub. B.V., 1992, pp. 1–56.

[3] Arhangel'skii A.V., Ponomarev V.I., *Foundations of General Topology: Problems and Exercises*, Moscow, Nauka Publ., 1974 (in Russian); English translation in: Mathematics and its applications, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster **13** (1984).

[4] Asanov M.O., *On cardinal invariants of the spaces of all continuous functions* (in Russian), Contemporary Topology and Set Theory, Izhevsk **2** (1979), 8–12.

[5] Berner A.J., *Spaces with dense conditionally compact subsets*, Proc. Amer. Mat. Soc. **81** (1979), 137–142.

[6] Beshimov R.B., *Weakly separable spaces and mappings*, Thesis, Moscow State University, 1994.

[7] Beshimov R.B., *Some properties of weakly separable spaces Uzbek. Mat. Zh.* **1** (1994), 7–11.

[8] Bonanzinga M., *Star-Lindelöf and absolutely star-Lindelöf spaces*, Questions Answers Gen. Topology **16** (1998), 79–104.

[9] van Douwen E.K., *Density of compactifications*, Set-Theoretic Topology, G.M. Reed Ed., N.Y., 1977, pp. 97–110.

[10] van Douwen E.K., *The Pixley-Roy topology on spaces of subsets*, Set-Theoretic Topology, G.M. Reed Ed., N.Y., 1977, pp. 111–134.

[11] van Douwen E.K., Reed G.M., Roscoe A.W., Tree I.J., *Star covering properties*, Topology Appl. **39** (1991), 71–103.

[12] Dow A., Junnila H., Pelant J., *Weak covering properties of weak topologies*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 349–368.

[13] Engelking R., *General Topology*, Warszawa, 1977.

[14] Fleischman W.M., *A new extension of countable compactness*, Fund. Math. **67** (1970), 1–9.

[15] Ikenaga S., *A class which contains Lindelöf spaces, separable spaces and countably compact spaces*, Mem. of Numazu College of Tech. **18** (1983), 105–108.

[16] Ikenaga S., *Topological concept between Lindelöf and Pseudo-Lindelöf*, Res. Rep. of Nara Nat. College of Technology **26** (1990), 103–108.

[17] Levy R., McDowell R.H., *Dense subsets of βX*, Proc. Amer. Mat. Soc. **507** (1975), 426–429.

[18] Matveev M.V., *Pseudocompact and related spaces*, Thesis, Moscow State University, 1985.

[19] Matveev M.V., *A survey on star covering properties Topology Atlas, preprint No 330 (1998) http://www.unipissing.ca/topology/v/a/a/a/19.htm.*

[20] Matveev M.V., Uspenskii V.V., *On star-compact spaces with a G_δ-diagonal*, Zbornik Radova Filosofskogo Fakulteta v Nisu, Ser. Mat. **6:2** (1992), 281–290.

[21] Miller A.W., *Special subsets of the real line*, Handbook of Set-Theoretic Topology, K. Kunen and J.E. Vaughan Eds., Elsevier Sci. Publ., 1984, pp. 201–233.

[22] Pixley C., Roy P., *Uncompletable Moore spaces*, Proceedings of the Auburn Topology Conference, Auburn Univ., Auburn, Alabama, 1969, pp. 75–85.

[23] Pol R., *A theorem on weak topology on $C(X)$ for compact scattered X*, Fund. Math. **106** (1980), no. 2, 135–140.

[24] Przymusiński T., Tall F.D., *The undecidability of the existence of a non-separable normal Moore space satisfying the countable chain condition*, Fund. Math **85** (1974), 291–297.

[25] Reznichenko E.A., *A pseudocompact space in which only the sets of full cardinality are not closed and not discrete*, Vestnik MGU, Ser. I: Matem., Mekh., 1989, No. 6, pp. 69–70 (in Russian); English translation in: Moscow Univ. Math. Bull. **44** (1989) 70–71.

[26] Scott B.M., *Pseudocompact, metacompact spaces are compact*, Topology Proc. **4** (1979), 577–587.

[27] Tall F.D., *Normality versus collectionwise normality*, Handbook of Set-Theoretic Topology, K. Kunen and J.E. Vaughan Eds., Elsevier Sci. Publ., 1984, pp. 685–732.

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