Induced near-homeomorphisms

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Abstract. We construct examples of mappings f and g between locally connected continua such that 2^f and $C(f)$ are near-homeomorphisms while f is not, and 2^g is a near-homeomorphism, while g and $C(g)$ are not. Similar examples for refinable mappings are constructed.

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For a metric continuum X we denote by 2^X and $C(X)$ the hyperspaces of all nonempty closed and of all nonempty closed connected subsets of X, respectively. Given a mapping $f : X \to Y$ between continua X and Y, we let $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ denote the corresponding induced mappings. The following theorem is known ([7, Lemma 2.1, p. 750]).

1. Theorem. *For any continua X and Y and a mapping $f : X \to Y$ the following three statements are equivalent:*

(a) $f : X \to Y$ *is monotone;*
(b) $2^f : 2^X \to 2^Y$ *is cell-like;*
(c) $C(f) : C(X) \to C(Y)$ *is cell-like.*

As applications of these results we show that if f is a monotone mapping between locally connected continua, then 2^f is a near-homeomorphism between Hilbert cubes. Moreover, if the continua X and Y contain no free arcs, then $C(f)$ is a near-homeomorphism, too. We show appropriate examples of mappings f and g such that 2^f and $C(f)$ are near-homeomorphisms while f is not, and 2^g is a near-homeomorphism, while g and $C(g)$ are not. Finally, we present examples of non-refinable mappings whose induced mappings are near-homeomorphisms, in particular are refinable. Several questions are asked.

All spaces considered in this paper are assumed to be metric. A *mapping* means a continuous function. A *continuum* means a compact connected space. Given a continuum X with a metric d, we denote by 2^X the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A,B) = \max\{\sup\{d(a,B) : a \in A\}, \sup\{d(b,A) : b \in B\}\}$$

(equivalently: with the Vietoris topology: see e.g. [6, (0.1), p. 1 and (0.12), p. 10]. Furthermore, we denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X. The reader is referred to Nadler's book [6] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between continua X and Y, we consider mappings (called the *induced* ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \ \text{ for every } \ A \in 2^X \ \text{ and } \ C(f)(A) = f(A) \ \text{ for every } \ A \in C(X).$$

A continuous mapping $\omega : 2^X \to \mathbb{R}$ is called a *Whitney map* provided that $\omega(\{x\}) = 0$ for each point $x \in X$, and that if A and B are nonempty closed subsets of X with $A \subset B$ and $A \neq B$, then $\omega(A) < \omega(B)$.

A continuum is said to have *trivial shape* if it is the intersection of a decreasing sequence of compact absolute retracts. A mapping $f : X \to Y$ between continua X and Y is called *cell-like* if, for each point $y \in Y$ the preimage $f^{-1}(y)$ is a continuum of trivial shape. In particular, cell-like mappings are monotone, i.e. the preimages of points are connected.

A mapping $f : X \to Y$ between continua X and Y is called a *near-homeomorphism* if f is the uniform limit of homeomorphisms from X onto Y. A proof of the following proposition is straightforward.

2. Proposition. *If a surjective mapping $f : X \to Y$ between continua X and Y is a near-homeomorphism, then the two induced mappings $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ are near-homeomorphisms, too.*

We will show that the converse implications do not hold.

An arc ab in a space X is said to be *free* provided that $ab \setminus \{a, b\}$ is an open subset of X.

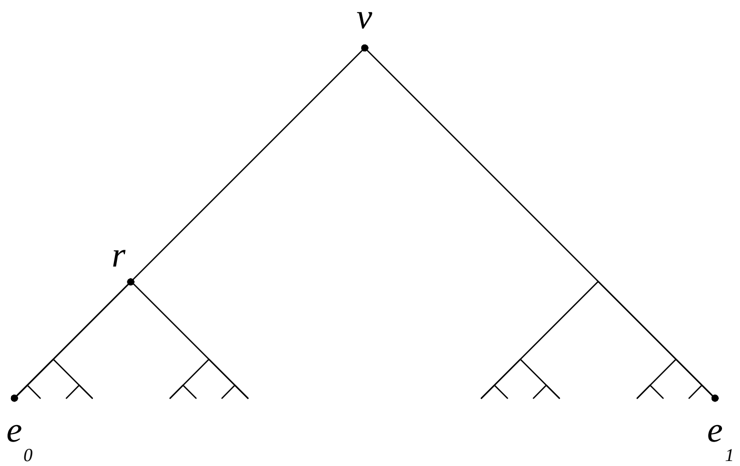
3. Theorem. *Let continua X and Y be locally connected, and let a mapping $f : X \to Y$ be monotone. Then 2^X and 2^Y are homeomorphic to the Hilbert cube, and the induced mapping 2^f is a near-homeomorphism. If, moreover, X and Y do not contain free arcs, then $C(X)$ and $C(Y)$ are homeomorphic to the Hilbert cube, and $C(f)$ is a near-homeomorphism.*

PROOF: The hyperspaces 2^X and 2^Y are homeomorphic to the Hilbert cubes by [6, (1.97), p. 137]. Similarly, if X and Y do not contain free arcs, then $C(X)$ and $C(Y)$ are homeomorphic to the Hilbert cube by [6, (1.98), p. 138]. Then by Theorem 1 the two induced mappings 2^f and $C(f)$ are cell-like mappings between Hilbert cubes, so they are near-homeomorphisms by [5, Theorem 7.5.7, p. 357 and Corollary 7.8.4, p. 372]. □

The next example shows that even if continua X and Y are homeomorphic, the conditions that both induced mappings are near-homeomorphisms do not imply that f is a near-homeomorphism.

4. Example. *There are a locally connected continuum X and a mapping $f : X \to X$ such that the induced mappings 2^f and $C(f)$ are near-homeomorphisms, while f is not.*

PROOF: To describe the example recall that a *Gehman dendrite* is a dendrite G having the Cantor ternary set in $[0,1]$ as the set $E(G)$ of its end points, such that all ramification points of G (the set of which is denoted by $R(G)$) are of order 3 and are situated in G in such a way that $E(G) = \operatorname{cl} R(G) \setminus R(G)$ (see the figure).



Figure

Let e_0 and e_1 denote two end points of G being of the maximal distance apart, i.e., these end point of G correspond to points 0 and 1 of the Cantor set when it is embedded into $[0,1]$ in the natural way. Let r be a ramification point of G lying in the left half of G and having the maximal distance from e_0. Let K be the component of $G \setminus \{r\}$ containing the end point e_1, and let D be the closure of the union of two other components of $G \setminus \{r\}$. Note that D is a copy of G diminished thrice with respect to the size of G. Thus there is a homothety $h : D \to G$ with the center e_0 and the ratio 3, which maps homeomorphically D onto G. Therefore, if $g : G \to D$ is a monotone retraction of G onto D which shrinks K to the singleton $\{r\}$ and which is the identity on D, then the composition $h \circ g : G \to G$ is a monotone mapping which is not a near-homeomorphism. The above construction is due to Dr. K. Omiljanowski, see [1, Example 5.3, p. 177].

Let

$$f = (h \circ g) \times \mathrm{id} : G \times [0,1] \to G \times [0,1],$$

and observe that the induced mappings 2^f and $C(f)$ are near-homeomorphisms, again by Theorem 3.

To see that f is not a near-homeomorphism note that $f((e_1, 0)) = (v, 0)$, where v is the highest point of G, and that each neighborhood of the point $(e_1, 0)$ contains the Cartesian product of a triod by an interval, while small neighborhoods of $(v, 0)$ do not contain such products. □

5. Example. *There are a locally connected continuum X and a mapping $f : X \to X$ such that the induced mapping 2^f is a near-homeomorphism, while f and $C(f)$ are not.*

Proof: Let $X = G$ be the Gehman dendrite, and let the mappings g and h have the same meaning as in the previous example. Put $f = h \circ g$, and observe that 2^f is a near-homeomorphism, again by Theorem 3. So, we need only to verify that $C(f)$ is not a near-homeomorphism. Denote, as previously, by v the top of G. Thus $f(e_1) = v$. Note that if N is a closed connected neighborhood of e_1, then $\dim C(N) = \infty$ by [6, (1.103), p. 142], while dimension of the hyperspace of subcontinua of a small closed connected neighborhood of v is two. Therefore there is no homeomorphism from $C(X)$ to $C(X)$ sending $\{e_1\}$ into a neighborhood of $\{v\}$. This shows that $C(f)$ is not a near-homeomorphism. The proof is complete. □

6. Questions. Let a mapping $f : X \to Y$ between continua X and Y be such that the induced mapping $C(f)$ is a near-homeomorphism (in particular, $C(X)$ and $C(Y)$ are homeomorphic). Does it imply that 2^f is a near-homeomorphism? The same question, if $X = Y$.

Now we are going to discuss relations between refinable induced mappings. Let us start with a definition. A surjective mapping $f : X \to Y$ is called *refinable* (see [2, p. 263]; see also a survey article [4] for more information) if for each $\varepsilon > 0$ there is a surjective ε-mapping $g : X \to Y$ (called *ε-refinement of f*) which is ε-close to f, that is, $\rho(f, g) < \varepsilon$ (where ρ means the supremum metric on the functional space Y^X) and $\operatorname{diam} g^{-1}(y) < \varepsilon$ for each $y \in Y$. In particular, every near-homeomorphism is refinable, while in general the continua X and Y do not have to be homeomorphic. However, if there exists a refinable mapping from X onto Y, then X has to be Y-like, in particular

$$\dim X \leq \dim Y. \tag{7}$$

It is known that if f is refinable, then 2^f is refinable, [3, Theorem 2.4 (i), p. 3]. Now we will investigate other possible relations between the three conditions:

(A) f is refinable;
(B) 2^f is refinable;
(C) $C(f)$ is refinable.

8. Example. *Let $f : [0, 1]^2 \to [0, 1]$ be the natural projection. Then 2^f is a near-homeomorphism (in particular it is refinable), while $C(f)$ and f are not refinable.*

Proof: 2^f is a near-homeomorphism by Theorem 3. $C(f)$ and f are not refinable because inequality (7) is not satisfied. □

9. Example. *Let* $f : [0,1]^3 \to [0,1]^2$ *be the natural projection. Then* 2^f *and* $C(f)$ *are near-homeomorphisms (in particular they are refinable), while* f *is not refinable.*

PROOF: The argument is exactly the same as for the previous example. $\square$

The following two questions remain open.

10. Question (Hosokawa, [3, p. 2])**.** Does f refinable imply $C(f)$ refinable?

11. Question. Does $C(f)$ refinable imply 2^f refinable?

A surjective mapping $f : X \to Y$ is called *monotonely refinable* if it is refinable, and each ε-refinement of f can be chosen to be a monotone mapping. In particular each near-homeomorphism is monotonely refinable. It is known that if the mapping f is monotonely refinable, then the two induced mappings, 2^f and $C(f)$ also are monotonely refinable, [3, Theorem 2.4 (ii), p. 3]. Example 9 shows that none of the two opposite implications is true. Furthermore, by Example 8, 2^f is monotonely refinable does not imply that $C(f)$ is monotonely refinable.

12. Question. Does $C(f)$ monotonely refinable imply 2^f monotonely refinable?

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