New proofs of classical insertion theorems

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Abstract. We provide new proofs for the classical insertion theorems of Dowker and Michael. The proofs are geometric in nature and highlight the connection with the preservation of normality in products. Both proofs follow directly from the Katětov-Tong insertion theorem and we also discuss a proof of this.

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A function from a topological space X to $\mathbb{R}$ is said to be upper semicontinuous if, for every a in $\mathbb{R}$, the preimage of $[a,\infty)$ is closed and lower semicontinuous if the preimage of $(-\infty,a]$ is closed. Given a pair of semicontinuous functions $g \leq h$ one can ask whether there is a continuous function f, with $g \leq f \leq h$. Such insertion results form part of the classical theory of general topology, tracing back to Hahn [5], who proved Theorem 1 in the realm of metrizable spaces, and Dieudonné [2], who proved Theorems 1 and 2 for paracompact spaces.

Theorem 1 (Katětov [7], Tong [16]). *A space X is normal if and only if whenever $g,h : X \to \mathbb{R}$ are upper (resp. lower) semi-continuous and $g \leq h$, there is a continuous $f : X \to \mathbb{R}$ such that $g \leq f \leq h$.*

Theorem 2 (Dowker [3]). *A space X is normal and countably paracompact if and only if whenever $g,h : X \to \mathbb{R}$ are upper (resp. lower) semi-continuous and $g < h$, there is a continuous $f : X \to \mathbb{R}$ such that $g < f < h$.*

Theorem 3 (Michael [13]). *A space X is perfectly normal if and only if whenever $g,h : X \to \mathbb{R}$ are upper (resp. lower) semi-continuous and $g \leq h$, there is a continuous $f : X \to \mathbb{R}$ such that $g \leq f \leq h$ and $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$.*

There is an intimate connection between insertion theorems and the normality of product spaces, an area that has been the focus of much attention ([14]). Normality is significant as it is precisely the property allowing continuous, real-valued functions on closed subspaces to be extended to the whole space. However, it does not behave well on taking products: for example it is known that a normal space X need not be binormal (that is, $X \times [0,1]$ need not be normal) ([4]).

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Binormality was, for a long time, an hypothesis in Borsuk's homotopy extension theorem (Rudin and Starbird eventually proved that normality suffices [14]). The connection between binormality and insertions was made by Dowker who proved, again in [3], that X is binormal if and only if X is both normal and countably paracompact (that is, paracompact with respect to countable open covers), thus confirming Eilenberg's conjecture that the existence of an insertion as in Theorem 2 implies the normality of $X \times [0,1]$. Theorem 3 also links insertions to products of normal spaces since $X \times [0,1]$ is perfectly normal if and only if it is hereditarily normal if and only if X, itself, is perfectly normal ([6], [12]).

In this short note, we present new proofs of necessity in Theorems 2 and 3. The proofs are direct and geometric in nature and highlight the connection between insertion theory and the normality of products. (In each case the converse is straight forward.) The proof for Theorem 2 relies on Theorem 1 and on the binormality of X and enables us to complete the circle of implications in Dowker's original result [3, Theorem 4]. The proof for Theorem 3 is essentially the same but relies on the hereditary normality of $X \times [0,1]$.

Lane [9], [10] also gives a direct proof that normality of $X \times [0,1]$ implies Dowker's insertion property and Katětov proves Theorem 2 using Theorem 1. In fact, Lane proves a much more abstract version of Theorem 2, but in both cases our proof is different.

Proof of Theorem 2: Let X be a normal, countably paracompact space, g an upper and h a lower semicontinuous function such that $g < h$. It suffices to find and upper semicontinuous u and lower semicontinuous l such that $g < u \leq l < h$, since Theorem 1 then implies the existence of a continuous f such that $g < u \leq f \leq l < h$.

It is clear that we may assume that both g and h map X to $(0,1)$. Let $A = \{(x,r) : r \geq h(x)\}$ and $B = \{(x,r) : r \leq g(x)\}$. The semicontinuity of g and h imply that both A and B are closed subsets of $X \times [0,1]$. Moreover, they are disjoint since $g < h$ and, as $X \times [0,1]$ is normal, there are open sets U and V such that $A \subseteq V$, $B \subseteq U$ and $\overline{U} \cap \overline{V} = \emptyset$.

Define $u : X \to [0,1]$ and $l : X \to [0,1]$ by,

$$u(x) = \sup\{r : (x,s) \in \overline{U} \ \text{ for all } \ s < r\}$$
$$l(x) = \inf\{r : (x,s) \in \overline{V} \ \text{ for all } \ s > r\}.$$

Since $(x,s) \in U$ for all $s < g(x)$ and since $(x,h(x)) \notin \overline{U}$, u is well defined and $u(x) \geq g(x)$ for all x. Indeed, $u(x) > g(x)$, since U is open and $(x,g(x)) \in U$. Similarly $l(x) < h(x)$. Since $\mathbb{R}$ is connected, for each x there is an s_0 such that $(x,s_0) \notin \overline{U} \cup \overline{V}$, so $u(x) \leq s_0 \leq l(x)$. We claim that u is upper semi-continuous. If x is not in $u^{-1}[t,\infty)$, there is some $s_0 \in [u(x),t)$ such that $(x,s_0) \notin \overline{U}$. Hence there is some open subset of W containing x such that $(y,s_0) \notin \overline{U}$ for all $y \in W$ and $u(y) \leq s_0 < t$ for all $y \in W$. This shows that $u^{-1}[t,\infty)$ is closed. In a similar fashion l is lower semi-continuous and we are done. $\square$

Proof of Theorem 3: Again, by Theorem 1, it is enough to find an upper semicontinuous u and lower semicontinuous l such that $g \leq u \leq l \leq h$ and $g(x) < u(x) \leq l(x) < h(x)$, whenever $g(x) < h(x)$. We may assume that g and h map X to $(0,1)$. Since X is perfectly normal, $X \times [0,1]$ is hereditarily normal. Let $A = \{(x,r) : h(x) < r\}$ and $B = \{(x,r) : r < g(x)\}$. Note that any open set in $X \times [0,1]$ containing $(x, g(x))$ meets B and that, as above, $\{(x,r) : r \leq g(x)\}$ is closed. Thus $\overline{B} = \{(x,r) : r \leq g(x)\}$. Similarly $\overline{A} = \{(x,r) : h(x) \leq r\}$. Let $Y = (X \times [0,1]) \setminus (\overline{A} \cap \overline{B}) = (X \times [0,1]) \setminus \{(x,r) : g(x) = h(x) = r\}$. Y is an open normal subspace of $X \times [0,1]$ and, since $g \leq h$, $\overline{A} \cap Y$ and $\overline{B} \cap Y$ are disjoint closed subsets of Y. By normality, choose open subsets S, T, U and V of Y such that S and T are disjoint and $B \subseteq \overline{B} \cap Y \subseteq U \subseteq \overline{U}^Y \subseteq S$ and $A \subseteq \overline{A} \cap Y \subseteq V \subseteq \overline{V}^Y \subseteq T$.

Now, taking closures in X for the remainder of the proof, define functions u and l from X to $[0,1]$ as above. If $r > h(x)$ then there is some $s \in (h(x), r)$ and, as (x,s) is in $\overline{A} \setminus \overline{B}$, (x,s) is in Y but not in $\overline{U}$. Thus, if $(x,s) \in \overline{U}$ for all $s < r$, then $r \leq h(x)$. Also, (x,s) is in U for all $s < g(x)$, so u is well defined and $g(x) \leq u(x) \leq h(x)$. Similarly l is well defined and $g(x) \leq l(x) \leq h(x)$. As above, u and l are upper and lower semi-continuous functions respectively. When $g(x) = h(x)$ then clearly $g(x) = u(x) = l(x) = h(x)$. When $g(x) < h(x)$ then $(x, g(x)) \in \overline{B} \cap Y \subseteq U$ which is open and so $u(x) > g(x)$. Similarly $l(x) < h(x)$. Now if $l(x) < u(x)$ then there is an r such that $(x,r) \in \overline{U} \cap \overline{V} \cap Y \subseteq S \cap T$ which is a contradiction. So $u \leq l$ and we are done. $\square$

For completeness, we give another proof of Theorem 1. Katětov's proof mimics the classic onion skin proof of Urysohn's Lemma. Our proof is slightly more direct than Katětov's and that discussed in [4, 2.7.2] in that it does not rely on Katětov's lemma on partial orders ([8]) or the fact that separated F_σ-sets may be separated by open sets in a normal space. Instead we make the appropriate modifications to Mandelkern's proof [11] of the Tietze-Urysohn Theorem.

Proof of Theorem 1: Assume X is normal. For $t \in \mathbb{Q}$ define $H(t) = \{x \in X : h(x) \leq t\}$ and $G(t) = \{x \in X : g(x) < t\}$. Index the set $P = \{(r,s) : r, s \in \mathbb{Q}$ and $r < s\}$ as $\{(r_n, s_n) : n \in \mathbb{N}\}$. Clearly, for any $r < s$, $H(r)$ is closed, $G(s)$ is open and $H(r) \subseteq G(s)$.

By induction, we construct a family of closed subsets of X, $\{D(r,s) : (r,s) \in P\}$ such that $H(r) \subseteq D(r,s)^\circ \subseteq D(r,s) \subseteq G(s)$ for $(r,s) \in P$ and $D(r,s) \subseteq D(t,u)^\circ$ whenever $r < t$ and $s < u$. Given $D(r_k, s_k)$ for all $k < n$, let $J = \{j : j < n, r_j < r_n$ and $s_j < s_n\}$ and $K = \{k : k < n, r_n < r_k$ and $s_n < s_k\}$. By normality, choose a closed set $D(r_n, s_n)$ such that

$$H(r_n) \cup \bigcup_{j \in J} D(r_j, s_j) \subseteq D(r_n, s_n)^\circ \subseteq D(r_n, s_n) \subseteq G(s_n) \cap \bigcap_{k \in K} D(r_k, s_k)^\circ.$$

Now, for each t in $\mathbb{Q}$, let $F(t) = \bigcap_{s > t} D(t,s)$. $F(t)$ is closed and contains $H(t)$, and $F(t) \subseteq G(s)$ if $t < s$. Then $\bigcup_{t \in \mathbb{Q}} F(t) = X$ and $\bigcap_{t \in \mathbb{Q}} F(t) = \emptyset$ and also $F(t) \subseteq F(s)^\circ$ whenever $t < s$. Hence $f(x) = \inf\{t : x \in F(t)\}$ is a well defined continuous function (by the standard argument used in Urysohn's Lemma). If $f(x) = y$ then $x \in F(s)$ for all $s > y$ and hence $x \in G(s)$ for all $s > y$. Therefore, $g(x) \leq y$. If $h(x) = y$, then $x \in H(s) \subseteq F(s)$ for all $s \geq y$ and hence $f(x) \leq y$. Thus $g \leq f \leq h$. $\square$

References

[1] Bonan E., *Sur un lemme adapté au théorème de Tietze-Urysohn*, C.R. Acad. Paris Sér. A **270** (1970), 1226–1228.

[2] Dieudonné J., *Une généralisation des espaces compacts*, J. Math. Pures Appl. **23** (1944), 65–76.

[3] Dowker C.H., *On countably paracompact spaces*, Canad. J. Math. **3** (1951), 219–224.

[4] Engelking R., *General Topology*, Helderman Verlag, Berlin, 1989.

[5] Hahn H., *Über halbstetige und unstetige Funktionen*, Sitzungsberichte Akad. Wiss. Wien Abt. IIa **126** (1917), 91–110.

[6] Katětov M., *Complete normality of Cartesian products*, Fund. Math. **36** (1948), 271–274.

[7] Katětov M., *On real-valued functions in topological spaces*, Fund. Math. **38** (1951), 85–91.

[8] Katětov M., *Correction to "On real-valued functions in topological spaces"*, Fund. Math. **40** (1953), 203–205.

[9] Lane E.P., *Insertion of a continuous function and $X \times I$*, Topology Proc. **6** (1981), 329–334.

[10] Lane E.P., *A conjecture concerning $X \times I$*, Questions Answers Gen. Topology **8** (1990), 387–403.

[11] Mandelkern M., *A short proof of the Tietze-Urysohn extension theorem*, Arch. Math. (Basel) **60** (1993), 364–366.

[12] Michael E., *A note on paracompact spaces*, Proc. Amer. Math. Soc. **4** (1953), 831–838.

[13] Michael E., *Continuous selections I*, Annals of Math. **63** (1956), 361–382.

[14] Rudin M.E., *Dowker spaces*, in: Handbook of Set-Theoretic Topology, North Holland, Amsterdam, 1984, pp. 761–780.

[15] Scott B.M., *A "more topological" proof of the Tietze-Urysohn theorem*, Amer. Math. Monthly **85** (1988), 117–123.

[16] Tong H., *Some characterizations of normal and perfectly normal spaces*, Duke Math. J. **19** (1952), 289–292.

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