Comment.Math.Univ.Carolin. 41,1 (2000)53–59

Topological sequence entropy for maps of the circle

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Abstract. A continuous map f of the interval is chaotic iff there is an increasing sequence of nonnegative integers T such that the topological sequence entropy of f relative to T, $h_T(f)$, is positive ([FS]). On the other hand, for any increasing sequence of nonnegative integers T there is a chaotic map f of the interval such that $h_T(f) = 0$ ([H]). We prove that the same results hold for maps of the circle. We also prove some preliminary results concerning topological sequence entropy for maps of general compact metric spaces.

Keywords: chaotic map, circle map, topological sequence entropy

Classification: Primary 26A18, 54H20, 58F13

Introduction

Let (X, ρ) be a compact metric space; denote by $C(X)$ the space of all continuous maps of this space into itself. We will pay a special attention to the case when X is the circle $\mathbb{S} = \{z \in \mathbb{C}; |z| = 1\}$; the metric on $\mathbb{S}$ is given by $\|x, y\| = \operatorname{dist}(\Pi^{-1}x, \Pi^{-1}y)$ where Π denotes the natural projection of the real line $\mathbb{R}$ onto $\mathbb{S}$, i.e., $\Pi(x) = e^{2\pi i x}$. By $\mathbb{N}$ we denote the set of all positive integers. If $T = (t_i)_{i=1}^{\infty}$ is an arbitrary sequence of nonnegative integers then the (T, f, n)-*trajectory* of $x \in X$ is the sequence $(f^{t_i}x)_{i=1}^{n}$. The set of all periodic points of f is denoted by $\operatorname{Per}(f)$ and the set of periods of all periodic points of f by $P(f)$. A set $A \subseteq X$ is called a *retract* of X if there is a map $r : X \to A$ such that $r(a) = a$ for every $a \in A$.

Definition. Let (X, ρ) be a compact metric space. A map $f \in C(X)$ is said to be *chaotic* if there are points $x, y \in X$ such that

$$\limsup_{i\to\infty} \rho(f^i x, f^i y) > 0,$$
$$\liminf_{i\to\infty} \rho(f^i x, f^i y) = 0.$$

(The set $\{x, y\}$ is called a *scrambled* set.) A map is called *nonchaotic* if it is not chaotic.

Remark. This definition of a chaotic map is equivalent to the original one by Li and Yorke in [LY] for maps of the interval (see [KuS]) and for maps of the circle (see [Ku]).

The author has been supported by the Slovak grant agency, grant number 1/4015/97.

Definition. Let (X, ρ) be a compact metric space, $f \in C(X)$ and $T = (t_i)_{i=1}^{\infty}$ be an increasing sequence of nonnegative integers. We say that a set $A \subseteq X$ (T, f, ε, n)-*spans* a set $B \subseteq X$ if for any $x \in B$ there is $y \in A$ such that $\rho(f^{t_i}x, f^{t_i}y) < \varepsilon$ for all $1 \leq i \leq n$. (We also say that the point y spans the point x.)

Definition ([G]). Let (X, ρ) be a compact metric space, $f \in C(X)$ and $T = (t_i)_{i=1}^{\infty}$ be an increasing sequence of nonnegative integers.

A set $A \subseteq X$ is said to be (T, f, ε, n)-*separated* if for any $x, y \in A$, $x \neq y$ there is an index i, $1 \leq i \leq n$, such that $\rho(f^{t_i}x, f^{t_i}y) > \varepsilon$. Let $\operatorname{Sep}(T, f, \varepsilon, n)$ denote the largest of cardinalities of all (T, f, ε, n)-separated sets. Put

$$\operatorname{Sep}(T, f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{Sep}(T, f, \varepsilon, n).$$

A subset of X is said to be a (T, f, ε, n)-*span* if it (T, f, ε, n)-spans X. Let $\operatorname{Span}(T, f, \varepsilon, n)$ denote the smallest of cardinalities of all (T, f, ε, n)-spans. Put

$$\operatorname{Span}(T, f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{Span}(T, f, \varepsilon, n).$$

Then $\operatorname{Sep}(T, f) = \operatorname{Span}(T, f)$ (see [G]) and we define the *topological sequence entropy* of f relative to T, $h_T(f)$, to be $\operatorname{Sep}(T, f)$.

Remark. If $t_i = i - 1$, $i = 1, 2, \ldots$ then $h_T(f)$ is the topological entropy $h(f)$ of f. Topological sequence entropy can be viewed as the topological entropy of the nonautonomous dynamical system given on the space X by the sequence of maps $f^{t_1}, f^{t_2 - t_1}, f^{t_3 - t_2}, \ldots$ (see [KS]).

In [FS] Franzová and Smítal proved that a continuous map f of the interval is chaotic if and only if there is an increasing sequence of nonnegative integers T such that $h_T(f) > 0$. A natural question arose whether there is some universal sequence which characterizes chaos. This is not the case as it was proved in [H] — for any increasing sequence of nonnegative integers T there is a chaotic map f with $h_T(f) = 0$. The main aim of this paper is to prove the same results for maps of the circle.

Theorem 1. *A map $f \in C(\mathbb{S})$ is chaotic if and only if there is an increasing sequence of nonnegative integers T such that $h_T(f) > 0$.*

Remark. Theorem 1 does not hold in general, even for triangular maps of the square. There is a nonchaotic triangular map with positive topological sequence entropy relative to a suitable sequence (see [FPS, Theorem 2]) and, on the other hand, there is a chaotic triangular map with zero topological sequence entropy relative to any sequence (see [FPS, Theorem 3]).

Theorem 2. *Let X be a compact metric space containing a homeomorphic image of an interval and let T be an increasing sequence of nonnegative integers. Then there is a chaotic map $f \in C(X)$ such that $h_T(f) = 0$.*

Remark. The analysis of the proof of Theorem 6 in [H] shows that Theorem 2 holds also when X is a Cantor set.

Corollary 3. *Let T be an increasing sequence of nonnegative integers. Then there is a chaotic map $f \in C(\mathbb{S})$ such that $h_T(f) = 0$.*

Preliminary results

Let (X, ρ) and (Y, σ) be compact metric spaces, $f \in C(X)$, $g \in C(Y)$, and let $\pi : X \to Y$ be a continuous map such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{\ g\ } & Y \end{array}$$

commutes. In this situation we have the following

Lemma 4. *Let T be an increasing sequence of nonnegative integers. Then*

(i) *if π is injective then $h_T(f) \le h_T(g)$;*
(ii) *if π is surjective then $h_T(f) \ge h_T(g)$;*
(iii) *if π is bijective then $h_T(f) = h_T(g)$.*

Proof:

(ii) and (iii). See [G, p. 332].

(i). We have that π is a homeomorphism between X and πX. Thus, by (iii), $h_T(f) = h_T(g|_{\pi X})$. Now let $E \subseteq \pi X$ be $(T, g|_{\pi X}, \varepsilon, n)$-separated. Trivially, it is also (T, g, ε, n)-separated which gives $h_T(g|_{\pi X}) \le h_T(g)$. □

It is known that some of the properties of topological entropy are not satisfied by topological sequence entropy. For example, contrary to the formula $h(f^k) = k \cdot h(f)$, an analogous formula for topological sequence entropy does not hold — it is even possible that $h_S(f) < h_T(f)$ for a subsequence S of T ([L]). In this case the following result can be useful.

Theorem 5. *Let (X, ρ) be a compact metric space, $f \in C(X)$, T be an increasing sequence of nonnegative integers and k be a positive integer. Then there is an increasing sequence of nonnegative integers S such that $h_S(f^k) \ge h_T(f)$.*

Proof: Since X is compact, $f, f^2, \dots, f^{k-1}$ are equicontinuous, i.e., for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ and $\rho(x, y) \le \delta$ then $\rho(f^i x, f^i y) < \varepsilon$ for $i = 1, \dots, k-1$. We may assume that $\delta \le \varepsilon$.

Let $T = (t_i)_{i=1}^{\infty}$. Define $S = (s_i)_{i=1}^{\infty}$ as follows. Put $s_1 = \left[\frac{t_1}{k}\right]$ (where $[\cdot]$ stands for the integer part) and for any m let s_{m+1} will be the first $\left[\frac{t_i}{k}\right]$ greater than s_m.

Let $E \subseteq X$ be an (T, f, ε, n)-separated set. We are going to show that E is a (S, f^k, δ, m)-separated set where m is such that $s_m = \left[\frac{t_n}{k}\right]$. To this end let $x, y \in E$, $x \ne y$. Then for some $i \in \{1, 2, \dots, n\}$, $\rho(f^{t_i} x, f^{t_i} y) > \varepsilon$. Take j with $s_j = \left[\frac{t_i}{k}\right]$. Then $j \le m$ and from the definition of δ we have $\rho(f^{k \cdot s_j} x, f^{k \cdot s_j} y) > \delta$. Thus E is an (S, f^k, δ, m)-separated set. From this we have $\mathrm{Sep}(T, f, \varepsilon, n) \le$

$\operatorname{Sep}(S, f^k, \delta, m)$. Now, $h_T(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Sep}(T, f, \varepsilon, n) \leq$
$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Sep}(S, f^k, \delta, m) \leq \lim_{\delta\to 0} \limsup_{m\to\infty} \frac{1}{m} \log \operatorname{Sep}(S, f^k, \delta, m) = h_S(f^k)$.

□

Corollary 6. *Let X be a compact metric space, $f \in C(X)$ and k be a positive integer. Then the following two conditions are equivalent:*

(i) *there is an increasing sequence T of nonnegative integers such that $h_T(f) > 0$;*

(ii) *there is an increasing sequence T of nonnegative integers such that $h_T(f^k) > 0$.*

In the sequel we will discuss the space of maps of the circle. The space $C(\mathbb{S})$ can be decomposed into the following classes (see [ALM, Chapter 3], cf. also [Ku, p. 384]):

$C_1 = \{f \in C(\mathbb{S});\ f \text{ has no periodic point}\}$;
$C_2 = \{f \in C(\mathbb{S});\ P(f^n) = \{1\} \text{ or } P(f^n) = \{1, 2, 2^2, \dots\} \text{ for some } n \in \mathbb{N}\}$;
$C_3 = \{f \in C(\mathbb{S});\ P(f^n) = \mathbb{N} \text{ for some } n \in \mathbb{N}\}$.

According to this we will distinguish three different cases.

Maps withou t periodic points

In all of this section we assume $f \in C(\mathbb{S})$ to have no periodic point. We are going to show that Theorem 1 holds for such maps. Since, by [Ku, Theorem B], f is not chaotic, we need only to show that $h_T(f) = 0$ for any increasing sequence T. So fix T. If f is a homeomorphism then $h_T(f) = 0$ by [KS, Theorem D]. Otherwise, by [AK, Theorem 1 and Theorem 2], there is a nowhere dense perfect set E which is the only ω-limit set of f, all (closed) contiguous intervals are wandering, the preimage of any contiguous interval is a contiguous interval, the image of any contiguous interval is either a contiguous interval or a point from E. Moreover, $f|_E$ is monotone. By linear extension of $f|_E$ we obtain a monotone map $g \in C(\mathbb{S})$. By [KS, Theorem D], $h_T(g) = 0$. By Lemma 4(i), $h_T(f|_E) \leq h_T(g)$. Hence, $\limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Span}(T, f|_E, \varepsilon, n) = 0$ for any $\varepsilon > 0$.

Now fix an arbitrary $\varepsilon > 0$. We are going to estimate $\operatorname{Span}(T, f, \varepsilon, n)$. Let $I_1, \dots, I_k$ be all contiguous intervals longer than $\frac{\varepsilon}{2}$. Let A be a $(T, f|_E, \frac{\varepsilon}{2}, n)$-span. Take any point x whose (T, f, n)-trajectory lies in $\mathbb{S} \setminus \bigcup_{i=1}^k I_i$. If $x \in E$ then x is (T, f, ε, n)-spanned by A. For $x \notin E$ put y to be an endpoint of the contiguous interval which contains x. Then $\|f^{t_i}x, f^{t_i}y\| \leq \frac{\varepsilon}{2}$ for all $1 \leq i \leq n$. Since $y \in E$ is $(T, f, \frac{\varepsilon}{2}, n)$-spanned by a point $z \in A$, the set A obviously (T, f, ε, n)-spans all such points x.

So it remains to consider those points whose (T, f, n)-trajectories meet $\bigcup_{i=1}^k I_i$. Fix $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. We are going to show that there is a set of cardinality at most $n \cdot k \cdot N^k$ which (T, f, ε, n)-spans all considered points. It is sufficient to show that there is a set with cardinality at most N^k which (T, f, ε, n)-spans the set $I(t_i, I_j) = \{x \in \mathbb{S};\ f^{t_i}x \in I_j\}$ (for fixed $1 \le i \le n$ and $1 \le j \le k$). First, it is obvious that $I(t_i, I_j)$ is a contiguous interval. Consider its (T, f, n)-trajectory $(f^{t_1}I(t_i, I_j), \dots, f^{t_n}I(t_i, I_j))$. Each element in this trajectory is either a contiguous interval or a point from E. At most k of them have lengths greater than or equal to ε — cut each of such elements to N segments shorter than ε. All the other elements of the trajectory will be considered to be segments themselves. To each $x \in I(t_i, I_j)$ assign its code — the sequence $(S_1(x), \dots, S_n(x))$ where $S_l(x)$ is the segment containing $f^{t_l}x$. We have at most N^k different codes and all points with the same code can be (T, f, ε, n)-spanned by one point.

From what has been said above we see that $\mathrm{Span}(T, f, \varepsilon, n) \le \mathrm{Span}(T, f|_E, \frac{\varepsilon}{2}, n) + n \cdot k \cdot N^k$ which finishes the proof of Theorem 1 for maps without periodic points.

Maps with periodic points

We will first deal with the case C_2. We know that for any $n \in \mathbb{N}$ f is chaotic if and only if f^n is chaotic. Taking into account Corollary 6 we can without loss of generality assume that $P(f) = \{1\}$ or $P(f) = \{1, 2, 2^2, \dots\}$. Since f has a fixed point, by [Ku, Lemma 2.5] there is a lifting F and an F-invariant compact interval J longer than 1. In the following discussion of the case C_2 we will write F and Π instead of $F|_J$ and $\Pi|_J$, respectively, as in the next commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{F} & J \\ \Pi\big\downarrow & & \big\downarrow\Pi \\ \mathbb{S} & \xrightarrow{f} & \mathbb{S} \end{array}$$

Note that if $x, y \in J$ then $\|\Pi x, \Pi y\| \le |x - y|$ with the equality whenever $|x - y| \le \frac{1}{2}$.

Lemma 7. *F is chaotic if and only if f is chaotic.*

Proof: Let F be chaotic. Then there are two points $u, v \in J$ such that

$$\limsup_{i\to\infty} |F^i u - F^i v| = \gamma > 0; \tag{1}$$

$$\liminf_{i\to\infty} |F^i u - F^i v| = 0. \tag{2}$$

We claim that $\Pi u, \Pi v$ form a scrambled set for f. From (2), $\liminf_{i\to\infty} \|f^i \Pi u, f^i \Pi v\| = 0$. Now put $\eta = \min\{\gamma, \frac{1}{2}\}$. Take $0 < \delta < \eta$ such that $|x - y| < \delta$ implies $|Fx - Fy| < \eta$. From this and (1) and (2) we have that $|F^i u - F^i v| \in [\delta, \eta]$ infinitely many times. Since $\eta \le \frac{1}{2}$, the same holds for $\|f^i \Pi u, f^i \Pi v\|$. Thus $\limsup_{i\to\infty} \|f^i \Pi u, f^i \Pi v\| \ge \delta > 0$.

Let F be nonchaotic. Then by [JS, Theorem 3] every trajectory of F is approximable by cycles, i.e. for any $\varepsilon > 0$ and any $x \in J$ there is some periodic point $p \in \mathrm{Per}(F)$ such that

$$|F^i x - F^i p| < \varepsilon \quad \text{for all} \quad i = 0, 1, 2, \dots. \tag{3}$$

Fix any $z \in \mathbb{S}$. Take any of its preimages $x \in \Pi^{-1}z$. Let $\varepsilon > 0$ be arbitrary, $p \in \mathrm{Per}(F)$ such that (3) is satisfied. Clearly, Πp is a periodic point for f and $\|f^i z, f^i \Pi p\| < \varepsilon$ for all $i = 0, 1, 2, \dots$. Thus f is not chaotic by [Ku, Theorem A]. □

Lemma 8. *Let F be chaotic. Then there is an increasing sequence T such that $h_T(f) > 0$.*

Proof: If F has a periodic point of period $k \cdot 2^m$ where $k > 1$ is odd then, by Sharkovsky theorem, it has also a periodic point of period $k' \cdot 2^m$ where $k' > \operatorname{diam} J + 1$ is odd. Since $\Pi|J$ is at most $[\operatorname{diam} J] + 1$ to one, f has a periodic point of period $k'' \cdot 2^{m'}$ where $k'' > 1$ is odd. This is a contradiction since $P(f)$ is $\{1\}$ or $\{1, 2, 2^2, \dots\}$. So F is of type 2^∞, chaotic. By [S] there is an orbit of periodic intervals of period $p > \operatorname{diam} J$ such that F^p is chaotic on each of them. At least one interval K in this orbit is shorter than 1. Then $\Pi|_K$ is injective and so $F^p|_K$ is topologically conjugate with $f^p|_{\Pi K}$. By [FS, Theorem] there is an increasing sequence of nonnegative integers S such that $h_S(F^p|_K) > 0$. Since $h_{p \cdot S}(f) = h_S(f^p)$ it is sufficient to use Lemma 4(iii) and (i) to get $h_{p \cdot S}(f) \geq h_S(f^p|_\pi K) = h_S(F^p|_K) > 0$. □

We are going to show that Theorem 1 holds for maps from the class C_2. Let $f \in C_2$ be chaotic. Then we obtain the required result using Lemma 7 and Lemma 8.

Now let $f \in C_2$ and let there be an increasing sequence of nonnegative integers T such that $h_T(f) > 0$. Theorem 4(ii) then implies that $h_T(F) > 0$ where F has the same meaning as above. By [FS, Theorem] F is chaotic. Lemma 7 finishes the proof.

Finally we will discuss the situation for maps from the remaining class C_3. So let $P(f^n) = \mathbb{N}$ for some n. By [BC, Theorem IX.28(i) and (ii)] we have that $h(f^n)$ is positive and so is $h(f)$. By the same theorem, conditions (ii) and (iii), we have that $f^{m \cdot n}$ is strictly turbulent for a suitable $m \in \mathbb{N}$ which implies that f is chaotic for the same reason as in the interval case. This finishes the proof of Theorem 1.

Proof of Theorem 2

The space X contains a homeomorphic image J of the interval $[0, 1]$. The set J is a retract of X by [HY, Theorem 2-34]. Let $r : X \to J$ be a corresponding retraction. By [H, Theorem 6] there is a chaotic onto map $g \in C([0, 1])$ such that $h_T(g) = 0$. Let $\tilde{g} \in C(J)$ be a map topologically conjugate with g. Define $f \in C(X)$ by $f = \tilde{g} \circ r$. Since $\bigcap_{i=0}^{\infty} f^i X = fX = J$, we have that $h_T(f) = h_T(f|_J) = 0$ by [BCJ, Proposition 1].

Acknowledgment. The author thanks Professor Ľubomír Snoha for contributing ideas useful for this paper and for his helpful comments during the last stage of its preparation.

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(*Received* March 31, 1999)