Generalized n–coherence

J. Jirásko

Abstract. In this paper necessary and sufficient conditions for large subdirect products of n-flat modules from the category $Gen(Q)$ to be n-flat are given.

Keywords: relative finiteness conditions, relative coherence, large subdirect products of n-flat modules

Classification: 16D40

In what follows, R stands for an associative ring with a unit element and R-Mod (Mod-R) denotes the category of all unitary left (right) R-modules.

Let $\mathcal{F}$ be a filter on a set I and $\{M_i; i \in I\}$ be a family of left R-modules. We define an equivalence relation $\sim$ on $\prod_{i\in I} M_i$ as follows: For $(m_i), (n_i) \in \prod_{i\in I} M_i$, $(m_i) \sim (n_i)$ if $\{i \in I; m_i = n_i\} \in \mathcal{F}$. The equivalence class of $(0, 0, \dots)$ is called the $\mathcal{F}$-product and it is denoted by $\prod^{\mathcal{F}}_{i\in I} M_i$. Clearly, $\prod^{\mathcal{F}}_{i\in I} M_i$ is a submodule of $\prod_{i\in I} M_i$. For a set X let $|X|$ denotes the cardinality of X and for $m = (m_i)_{i\in I} \in \prod_{i\in I} M_i$ let $supp(m) = \{i \in I; m_i \neq 0\}$. For an infinite cardinal number $\aleph$ the $\aleph$-product is defined as $\prod^{\aleph}_{i\in I} M_i = \{m \in \prod_{i\in I} M_i; |supp(m)| < \aleph\}$. For an infinite cardinal number $\aleph$ let $\aleph^+$ be its immediate successor. Let $\mathcal{F}$ be a filter on an index set I and let $\aleph$ be $\sup\{|I \setminus X|; X \in \mathcal{F}\}$. According to [9] we define $\sup(\mathcal{F})$ to be $\aleph$ if the supremum is not attained and $\aleph^+$ if the supremum is attained. If $\aleph$ is an infinite cardinal number and $|I| \geq \aleph$ then $\mathcal{F} = \{X \subseteq I; |I \setminus X| < \aleph\}$ is a filter on I with $\sup(\mathcal{F}) = \aleph$ and $\prod^{\aleph}_{i\in I} M_i = \prod^{\mathcal{F}}_{i\in I} M_i$. If $|I| < \aleph$ then obviously $\prod^{\aleph}_{i\in I} M_i = \prod_{i\in I} M_i$. If $|I| = \aleph_s$ then we have $\sum^{\oplus}_{i\in I} M_i = \prod^{\aleph_0}_{i\in I} M_i \subseteq \prod^{\aleph_1}_{i\in I} M_i \subseteq \dots \subseteq \prod^{\aleph_s}_{i\in I} M_i \subseteq \prod^{\aleph_{s+1}}_{i\in I} M_i = \prod_{i\in I} M_i$. The $\mathcal{F}$-products ($\aleph$-products) of flat and projective modules were investigated in [9] and [10] by P. Loustaunau.

Let n be a nonnegative integer. A module $M \in$ Mod-R is called n-presented if there is a finite n-presentation of M i.e. an exact sequence

$$F_n \to F_{n-1} \to \dots \to F_1 \to F_0 \to M \to 0$$

in which every F_i is free of finite rank. A ring R is said to be right n-coherent if every n-presented right module is $(n+1)$-presented. The following definition of n-flat and n-FP-injective module is due to J. Chen and N. Ding. Let n be

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a positive integer. A left R-module Q is called n-flat if $\mathrm{Tor}_n^R(N,Q)=0$ for all n-presented right R-modules N. A right R-module M is said to be n-FP-injective if $\mathrm{Ext}_R^n(N,M)=0$ for all n-presented right R-modules N.

In [3] J. Chen and N. Ding characterize right n-coherent rings as rings for which direct products of n-flat left R-modules are n-flat. In [5] $(\aleph,Q)$-coherent rings were introduced and they were characterized as rings for which $\aleph$-products of flat modules from the category $Gen(Q)$ are flat. These rings were also studied in [11]. The aim of this paper is to generalize results of J. Chen and N. Ding and the results in [5] to $\aleph$-products of n-flat modules from the category $Gen(Q)$ for a fixed flat module Q.

Throughout all the paper ${}_RQ$ denotes a fixed flat left R-module and $\aleph$ denotes an infinite cardinal number.

The notions of $(\aleph,Q)$-finitely generated, $(\aleph,Q)$-finitely presented and $(\aleph,Q)$-coherent modules were introduced in [5]. In the following lemmas we summarize basic properties of these modules.

Lemma 1.1. *Let $\{Q_i; i\in I\}$ be a set of left R-modules. Then*

(i) *if $\mathcal{F}$ is a filter on I with $\sup(\mathcal{F})\le\aleph$ then $\prod_{i\in I}^{\mathcal{F}} Q_i\subseteq\prod_{i\in I}^{\aleph} Q_i$;*

(ii) *let $\mathcal{F}$ be a filter on I with $\sup(\mathcal{F})=\aleph$ and $q\in\prod_{i\in I}^{\aleph} Q_i$. If $S=supp(q)$ then there is $X\in\mathcal{F}$ and an injective map $f\colon S\to I\backslash X$. Since $X\subseteq I\backslash f(S)$ the element $\overline{q}$ defined by $\overline{q}_i=q_{f^{-1}(i)}$ for $i\in f(S)$ and $\overline{q}_i=0$ for $i\in I\backslash f(S)$ belongs to $\prod_{i\in I}^{\mathcal{F}} Q_i$.*

Proof: (i). If $q\in\prod_{i\in I}^{\mathcal{F}} Q_i$ then $|supp(q)|<\sup(\mathcal{F})\le\aleph$ and consequently $q\in\prod_{i\in I}^{\aleph} Q_i$.
(ii). If $\sup(\mathcal{F})=\aleph$ and $|S|<\aleph$ then there is $X\in\mathcal{F}$ with $|S|\le|I\backslash X|$. The rest is clear. □

Lemma 1.2. *Let $\mathcal{F}$ be a filter on I with $\sup(\mathcal{F})=\aleph$, $\{Q_i; i\in I\}$ be a family of left R-modules and M be a right R-module. Then the following conditions are equivalent:*

(i) *the natural homomorphism $\varphi_{\mathcal{F}}\colon M\otimes_R\prod_{i\in I}^{\mathcal{F}} Q_i\to\prod_{i\in I}^{\mathcal{F}}(M\otimes_R Q_i)$ defined via $\varphi_{\mathcal{F}}(m\otimes(q_i)_{i\in I})=(m\otimes q_i)_{i\in I}$ is an epimorphism;*

(ii) *the natural homomorphism $\varphi_{\aleph}\colon M\otimes_R\prod_{i\in I}^{\aleph} Q_i\to\prod_{i\in I}^{\aleph}(M\otimes_R Q_i)$ defined via $\varphi_{\aleph}(m\otimes(q_i)_{i\in I})=(m\otimes q_i)_{i\in I}$ is an epimorphism.*

Proof: (i) implies (ii). Let $\varphi_{\mathcal{F}}$ be an epimorphism, $q\in\prod_{i\in I}^{\aleph}(M\otimes Q_i)$, $S=supp(q)$ and $\overline{q}\in\prod_{i\in I}^{\mathcal{F}}(M\otimes Q_i)$ be the element defined in Lemma 1.1(ii). Then there is an element $m_1\otimes q_1+\dots+m_r\otimes q_r\in M\otimes\prod_{i\in I}^{\mathcal{F}} Q_i$ with $(m_1\otimes q_{1i}+\dots+m_r\otimes q_{ri})_{i\in I}=\overline{q}$. We can assume without loss of generality that $q_{ij}=0$ for $j\in I\backslash f(S)$ and $i=1,\dots,r$. Let $p_j\in\prod_{i\in I}^{\aleph} Q_i$ such that $p_{j_t}=0$ for $t\in I\backslash S$ and $p_{j_s}=q_{j\,f(s)}$ for $s\in S$, $j=1,\dots,r$. Hence $q_s=\overline{q}_{f(s)}=m_1\otimes q_{1\,f(s)}+\dots+m_r\otimes q_{r\,f(s)}=m_1\otimes p_{1_s}+\dots+m_r\otimes p_{r_s}$ for $s\in S$ and consequently $\varphi_{\aleph}$ is an epimorphism.

(ii) implies (i). If $\varphi_{\aleph}$ is an epimorphism and $q \in \prod^{\mathcal{F}}_{i\in I}(M \otimes Q_i)$ then there is an element $m_1 \otimes q_1 + \dots + m_r \otimes q_r \in M \otimes \prod^{\aleph}_{i\in I} Q_i$ with $(m_1 \otimes q_{1i} + \dots m_r \otimes q_{ri})_{i\in I} = q$. If $S = supp(q)$ then $I \backslash S \in \mathcal{F}$. Without loss of generality we can take q_i such that $q_{ij} = 0$ for $j \in I\backslash S$ and $i = 1,\dots,r$. Thus $q_i \in \prod^{\mathcal{F}}_{i\in I} Q_i$ for $i = 1,\dots,r$ and consequently $\varphi_{\mathcal{F}}$ is an epimorphism. □

The following definition is motivated by the definition of R.R. Colby and E.A. Rutter of the Q-finitely generated module in [4] and the definition of P. Loustaunau of the $\aleph$-finitely generated module in [9].

Definition 1.3. *A right R-module M is said to be $(\aleph, Q)$-finitely generated if every subset T of $M \otimes_R Q$ with $|T| < \aleph$ is contained in $N \otimes_R Q$ for some finitely generated submodule N of a module M.*

Lemma 1.4. *Let M be a right R-module. Then the following conditions are equivalent:*

(i) *M is $(\aleph, Q)$-finitely generated;*
(ii) *if I is a set and $Q_i \in Gen(Q)$, $i \in I$ then the natural homomorphism $\varphi\colon M \otimes_R \prod^{\aleph}_{i\in I} Q_i \to \prod^{\aleph}_{i\in I}(M \otimes_R Q_i)$ defined via $\varphi(m \otimes (q_i)_{i\in I}) = (m \otimes q_i)_{i\in I}$ is an epimorphism;*
(iii) *if I is a set then the natural homomorphism $\varphi\colon M \otimes_R \prod^{\aleph}_{i\in I} Q \to \prod^{\aleph}_{i\in I}(M \otimes_R Q)$ defined via $\varphi(m \otimes (q_i)_{i\in I}) = (m \otimes q_i)_{i\in I}$ is an epimorphism.*

Proof: (i) implies (ii). Let $u \in \prod^{\aleph}_{i\in I}(M \otimes Q_i)$, $T = supp(u)$ and $f_i\colon Q^{(J_i)} \to Q_i$, $i \in I$ be epimorphisms. Then $|T| < \aleph$ and $id_M \otimes f_i\colon M \otimes Q^{(J_i)} \to M \otimes Q_i$, $i \in I$ are epimorphisms. Hence $u_i = \sum_{j=1}^{n_i} m_{ij} \otimes f_i(q_{ij})$, where $m_{ij} \in M$, $q_{ij} \in Q^{(J_i)}$, $i \in I$ and $j = 1,\dots,n_i$. Now $q_{ij} = \sum_{k=1}^{t_{ij}} q_{ijk}$, where $q_{ijk} \in Q$, $k = 1,\dots,t_{ij}$. Let $S = \{m_{ij} \otimes q_{ijk}; i \in T, j = 1,\dots,n_i, k = 1,\dots,t_{ij}\}$. Then $|S| < \aleph$ and $S \subseteq M \otimes Q$. Thus $S \subseteq N \otimes Q$ for some finitely generated submodule $N = \sum_{p=1}^{l} n_p R$ of M. Hence $m_{ij} \otimes q_{ijk} = \sum_{p=1}^{l} n_p \otimes q_{ijkp}$ for some $q_{ijkp} \in Q$. Put $v_{ip} = \sum_{j=1}^{n_i} \sum_{k=1}^{t_{ij}} q_{ijkp}$ for $i \in T$ and $v_{ip} = 0$ for $i \in I\backslash T$, $p = 1,\dots,l$. Then $w_p = (f_i(v_{ip}))_{i\in I} \in \prod^{\aleph}_{i\in I} Q_i$, $p = 1,\dots,l$ and $u_i = \sum_{p=1}^{l} n_p \otimes f_i(\sum_{j=1}^{n_i} \sum_{k=1}^{t_{ij}} q_{ijkp}) = \sum_{p=1}^{l} n_p \otimes f_i(v_{ip})$, $i \in I$. Thus $\varphi(\sum_{p=1}^{l} n_p \otimes w_p) = (\sum_{p=1}^{l} n_p \otimes f_i(v_{ip}))_{i\in I} = u$ and consequently φ is an epimorphism.
(ii) implies (iii). Obvious.
(iii) implies (i). Let $S \subseteq M \otimes Q$ with $|S| < \aleph$ and I be a set such that $|S| \le |I|$ (e.g. $I = M \otimes Q$ or I is a set with $|I| \ge \aleph$). Then there is an injective map $f\colon S \to I$. Let us consider $u \in \prod^{\aleph}_{i\in I}(M \otimes Q)$ defined by $u_i = f^{-1}(i)$ for $i \in f(S)$ and $u_i = 0$ for $i \in I\backslash f(S)$. Then by assumption there is $\sum_{j=1}^{r} m_j \otimes q_j \in M \otimes \prod^{\aleph}_{i\in I} Q$ such that $(\sum_{j=1}^{r} m_j \otimes q_{j_i})_{i\in I} = u$. Now if $s \in S$ then $s = f^{-1}(i) = \sum_{j=1}^{r} m_j \otimes q_{j_i}$ for some $i \in f(S)$ and therefore $S \subseteq N \otimes Q$, where $N = \sum_{j=1}^{r} m_j R$ is a finitely generated submodule of M. □

Corollary 1.5. *The class of all $(\aleph, Q)$-finitely generated modules is closed under extensions, homomorphic images, finite direct sums, direct summands and contains the class of all finitely generated modules.*

Proof: It follows immediately from Lemma 1.4(ii) and the definition of $(\aleph, Q)$-finitely generated module.

□

Definition 1.6. *A right R-module M is said to be $(\aleph, Q)$-finitely presented if there is a short exact sequence $0 \to K \to F \to M \to 0$ with F free of finite rank and K $(\aleph, Q)$-finitely generated.*

Lemma 1.7. *Let M be a finitely generated right R-module. Then the following conditions are equivalent:*

(i) *M is $(\aleph, Q)$-finitely presented;*

(ii) *if $0 \to K \to P \to M \to 0$ is a projective presentation with P finitely generated then K is $(\aleph, Q)$-finitely generated;*

(iii) *if I is a set and $Q_i \in Gen(Q)$, $i \in I$ then the natural homomorphism $\varphi\colon M \otimes_R \prod_{i\in I}^{\aleph} Q_i \to \prod_{i\in I}^{\aleph}(M \otimes_R Q_i)$ defined via $\varphi(m \otimes (q_i)_{i\in I}) = (m \otimes q_i)_{i\in I}$ is an isomorphism;*

(iv) *if I is a set then the natural homomorphism $\varphi\colon M \otimes_R \prod_{i\in I}^{\aleph} Q \to \prod_{i\in I}^{\aleph}(M \otimes_R Q)$ defined via $\varphi(m \otimes (q_i)_{i\in I}) = (m \otimes q_i)_{i\in I}$ is an isomorphism.*

Proof: (i) implies (ii). Let $0 \to K_i \to P_i \to M \to 0$, $i = 1, 2$ be two projective presentations of M. By Schanuel's Lemma we have $P_1 \oplus K_2 \simeq P_2 \oplus K_1$. Now if P_1, P_2 are finitely generated and K_1 is $(\aleph, Q)$-finitely generated then K_2 is $(\aleph, Q)$-finitely generated by Corollary 1.5.

(ii) implies (iii). Let $0 \to K \to F \to M \to 0$ be an exact sequence, where F is free of finite rank and $Q_i \in Gen(Q), i \in I$. Consider the following commutative diagram

$$\begin{array}{ccccccc}
K \otimes \prod_{i\in I}^{\aleph} Q_i & \longrightarrow & F \otimes \prod_{i\in I}^{\aleph} Q_i & \longrightarrow & M \otimes \prod_{i\in I}^{\aleph} Q_i & \longrightarrow & 0 \\
\big\downarrow \varphi_K & & \big\downarrow \varphi_F & & \big\downarrow \varphi_M & & \\
\prod_{i\in I}^{\aleph}(K \otimes Q_i) & \longrightarrow & \prod_{i\in I}^{\aleph}(F \otimes Q_i) & \longrightarrow & \prod_{i\in I}^{\aleph}(M \otimes Q_i) & \longrightarrow & 0
\end{array}.$$

Then φ_F is obviously an isomorphism since F is free of finite rank and φ_K is an epimorphism since K is $(\aleph, Q)$-finitely generated. Hence φ_M is an isomorphism.

(iii) implies (iv). Obvious.

(iv) implies (i). Let $0 \to K \to F \to M \to 0$ be an exact sequence with F free of finite rank. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & K \otimes \prod_{i\in I}^{\aleph} Q & \longrightarrow & F \otimes \prod_{i\in I}^{\aleph} Q & \longrightarrow & M \otimes \prod_{i\in I}^{\aleph} Q & \longrightarrow & 0 \\
 & & \big\downarrow \varphi_K & & \big\downarrow \varphi_F & & \big\downarrow \varphi_M & & \\
0 & \longrightarrow & \prod_{i\in I}^{\aleph}(K \otimes Q) & \longrightarrow & \prod_{i\in I}^{\aleph}(F \otimes Q) & \longrightarrow & \prod_{i\in I}^{\aleph}(M \otimes Q) & \longrightarrow & 0
\end{array}.$$

Now φ_F and φ_M are isomorphisms. Hence φ_K is an epimorphism and K is $(\aleph, Q)$-finitely generated by Lemma 1.4. $\square$

Remark 1.8. *As it follows from Lemma* 1.2 *and the proof of Lemma* 1.7 *every* $\aleph$*-product* $\prod_{i\in I}^{\aleph}$ *in Lemma* 1.4 *and Lemma* 1.7 *can be replaced by* $\mathcal{F}$*-product* $\prod_{i\in I}^{\mathcal{F}}$ *for a filter* $\mathcal{F}$ *on* I *with* $\sup(\mathcal{F}) = \aleph$.

Definition 1.9. *Let n be a nonnegative integer. A right R-module M is called n-$(\aleph, Q)$-presented if there is a finite n-$(\aleph, Q)$-presentation of M i.e. an exact sequence*

$$0 \to K_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

in which every F_i is free of finite rank and K_n is $(\aleph, Q)$-finitely generated.

Definition 1.10. *Let n be a nonnegative integer. A ring R is said to be right n-$(\aleph, Q)$-coherent if every n-presented right R-module is $(n+1)$-$(\aleph, Q)$-presented.*

Lemma 1.11. *Let n be a positive integer, N be an n-$(\aleph, Q)$-presented right R-module and $\{Q_i; i \in I\}$ be a family of left R-modules from $Gen(Q)$. Then:*

(i) *there is an epimorphism* $\operatorname{Tor}_n^R(N, \prod_{i\in I}^{\aleph} Q_i) \to \prod_{i\in I}^{\aleph} \operatorname{Tor}_n^R(N, Q_i)$;

(ii) *there is an isomorphism* $\operatorname{Tor}_{n-1}^R(N, \prod_{i\in I}^{\aleph} Q_i) \cong \prod_{i\in I}^{\aleph} \operatorname{Tor}_{n-1}^R(N, Q_i)$.

Proof: Let

$$0 \to K_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to N \to 0$$

be the finite n-$(\aleph, Q)$-presentation of N and $K_i = Ker(F_{i-1} \to F_{i-2})$ for $i = 2, \ldots, n$. Then the short exact sequence $0 \to K_i \to F_{i-1} \to K_{i-1} \to 0$ induces the commutative diagram

$$\begin{array}{ccccc}
0 \to Tor_1^R(K_{i-1}, \prod_{i\in I}^{\aleph} Q_i) & \longrightarrow & K_i \otimes \prod_{i\in I}^{\aleph} Q_i & \longrightarrow & F_{i-1} \otimes \prod_{i\in I}^{\aleph} Q_i \\
\downarrow f_{i-1} & & \downarrow \varphi_{K_i} & & \downarrow \varphi_{F_{i-1}} \\
0 \to \prod_{i\in I}^{\aleph} \operatorname{Tor}_1^R(K_{i-1}, Q_i) & \longrightarrow & \prod_{i\in I}^{\aleph}(K_i \otimes Q_i) & \longrightarrow & \prod_{i\in I}^{\aleph}(F_{i-1} \otimes Q_i)
\end{array} .$$

Then f_{n-1} is an epimorphism since K_n is $(\aleph, Q)$-finitely generated and f_{n-2} is an isomorphism since K_{n-1} is $(\aleph, Q)$-finitely presented, K_i being finitely presented for $i < n-1$. Now our lemma follows from the fact that $\operatorname{Tor}_{n-1}^R(N, -) \cong \operatorname{Tor}_1^R(K_{n-2}, -)$ and $\operatorname{Tor}_n^R(N, -) \cong \operatorname{Tor}_1^R(K_{n-1}, -)$. $\square$

Theorem 1.12. *Let n be a nonnegative integer. Then the following conditions are equivalent:*

(i) $\prod_{i\in I}^{\aleph} Q$ *is n-flat for every index set I;*

(ii) $\prod_{i\in I}^{\aleph} Q_i$ *is n-flat for every index set I and any family of n-flat modules $Q_i \in Gen(Q)$;*

(iii) *R is right n-$(\aleph, Q)$-coherent.*

(iv)
$$\mathrm{Tor}_n^R(N, \prod_{i\in I}{}^{\aleph} Q_i) \cong \prod_{i\in I}{}^{\aleph} \mathrm{Tor}_n^R(N, Q_i)$$

for every n-presented right R-module N and any family of left R-modules $Q_i \in Gen(Q)$.

Proof: (ii) implies (i). Obvious.

(i) implies (iii). Suppose that N is an n-presented right R-module,

$$F_n \to F_{n-1} \to \dots \to F_1 \to F_0 \to N \to 0$$

is a finite n-presentation of N and $K_i = Ker(F_{i-1} \to F_{i-2})$ for $i = 2, \dots, n$. Then the exact sequence $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ induces the following commutative diagram

$$\begin{array}{ccccc}
0 \to K_n \otimes \prod_{i\in I}^{\aleph} Q & \longrightarrow & F_{n-1} \otimes \prod_{i\in I}^{\aleph} Q & \longrightarrow & K_{n-1} \otimes \prod_{i\in I}^{\aleph} Q \to 0 \\
\Big\downarrow \varphi_{K_n} & & \Big\downarrow \varphi_{F_{n-1}} & & \Big\downarrow \varphi_{K_{n-1}} \\
0 \to \prod_{i\in I}^{\aleph} (K_n \otimes Q) & \longrightarrow & \prod_{i\in I}^{\aleph} (F_{n-1} \otimes Q) & \longrightarrow & \prod_{i\in I}^{\aleph} (K_{n-1} \otimes Q) \to 0
\end{array} .$$

Then $\mathrm{Tor}_1^R(K_{n-1}, \prod_{i\in I}^{\aleph} Q) \cong \mathrm{Tor}_n^R(N, \prod_{i\in I}^{\aleph} Q) = 0$ by assumption and the upper row is exact. The lower row is exact since Q is flat. Now $\varphi_{F_{n-1}}, \varphi_{K_{n-1}}$ are isomorphisms and consequently φ_{K_n} is an isomorphism. Thus K_n is $(\aleph, Q)$-finitely presented by Lemma 1.7. Hence N is $(n+1)$-$(\aleph, Q)$-presented.

(iii) implies (iv). It follows immediately from Lemma 1.11 (ii).

(iv) implies (ii). Obvious. □

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Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 166 29 Praha 6, Czech Republic

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