Characterizations of spreading models of l^1

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Abstract. Rosenthal in [11] proved that if (f_k) is a uniformly bounded sequence of real-valued functions which has no pointwise converging subsequence then (f_k) has a subsequence which is equivalent to the unit basis of l^1 in the supremum norm.

Kechris and Louveau in [6] classified the pointwise convergent sequences of continuous real-valued functions, which are defined on a compact metric space, by the aid of a countable ordinal index "γ". In this paper we prove some local analogues of the above Rosenthal 's theorem (spreading models of l^1) for a uniformly bounded and pointwise convergent sequence (f_k) of continuous real-valued functions on a compact metric space for which there exists a countable ordinal ξ such that $\gamma((f_{n_k})) > \omega^\xi$ for every strictly increasing sequence (n_k) of natural numbers. Also we obtain a characterization of some subclasses of Baire-1 functions by the aid of spreading models of l^1.

Keywords: uniformly bounded sequences of continuous real-valued functions, convergence index, spreading models of l^1, Baire-1 functions

Classification: 46B20, 46E99

1. Introduction

By $\mathbb{N}$ we mean the set of all natural numbers (i.e., $\mathbb{N} = \{1, 2, \dots\}$), by ω we mean the first infinite ordinal (i.e., $\omega = \{0, 1, 2, \dots\}$) and by ω_1 we mean the first uncountable ordinal. If X is a set then: $|X|$ denotes the cardinal number of X, $[X]^{<\omega}$ the set of all finite subsets of X and $[X]$ the set of all infinite subsets of X. Let $\mathcal{S}$ be the Schreier family (i.e., $\mathcal{S} = \{\emptyset\} \cup \{A \subset \mathbb{N} : A \neq \emptyset, |A| \leq \min A\}$). Alspach and Argyros in [1] defined the generalized Schreier families $\mathcal{F}_\xi$, $\xi < \omega_1$, where $\mathcal{F}_0 = \{\emptyset\} \cup \{\{n\} : n \in \mathbb{N}\}$ and $\mathcal{F}_1 = \mathcal{S}$.

A real-valued function f defined on a set X is bounded if $\|f\|_\infty := \sup_{x \in X} |f(x)| < +\infty$. A sequence (f_k) of real-valued functions defined on a set X is uniformly bounded if $\sup_k \|f_k\|_\infty < +\infty$.

Rosenthal in [11] proved that if (f_k) is a uniformly bounded sequence of real-valued functions which has no pointwise converging subsequence then (f_k) has a subsequence which is equivalent to the unit basis of l^1 in the supremum norm.

If (f_k) is a sequence of real-valued functions and $1 \leq \xi < \omega_1$ an ordinal we say that (f_k) is l^1_ξ-spreading model (or spreading model of l^1 of order ξ) if there are positive real numbers C and M such that

$$C \sum_{i=1}^m |c_i| \leq \| \sum_{i=1}^m c_i f_{k_i} \|_\infty \leq M \sum_{i=1}^m |c_i|$$

for every $F = \{k_1 < \ldots < k_m\} \in \mathcal{F}_\xi$ and for every real numbers $c_1, \ldots, c_m$.

Kechris and Louveau in [6] defined the convergence index "γ" of a sequence of continuous real-valued functions defined on a compact metric space and proved that $\gamma((f_k)) < \omega_1$ iff (f_k) is pointwise converging.

This paper is a continuation of the paper [8]. By using some results of [1], [3] and [8] and using few combinatorial lemmas we prove the following basic results:

If K is a compact metric space, (f_k) a uniformly bounded and pointwise converging sequence of continuous real-valued functions on K and $1 \leq \xi < \omega_1$ then the following hold: (a) If $\gamma((f_{n_k})) > \omega^\xi$ for every strictly increasing sequence (n_k) of natural numbers then there exists a strictly increasing sequence (n_k) of natural numbers such that the sequence (f_{n_k}) is l^1_ξ-spreading model (cf. Theorem 3.1). (b) If (n_k) is a strictly increasing sequence of natural numbers and (n'_k) a subsequence of (n_k) such that the sequence $(f_{n'_{2k+1}} - f_{n'_{2k}})$ is l^1_ξ-spreading model then $\gamma((f_{n_k})) > \omega^\xi$ (cf. Theorem 3.2).

By using (b) we prove that: If the sequence (f_k) is l^1_ξ-spreading model then $\gamma((f_{n_k})) > \omega^\xi$ for every strictly increasing sequence (n_k) of natural numbers (cf. Theorem 3.3). Combining these results and [8] we obtain some criteria (characterizations) for l^1_ξ-spreading models (cf. Theorem 3.4).

Also Kechris and Louveau in [6] classified the bounded Baire-1 functions, which are defined on a compact metric space K, to the subclasses $\mathcal{B}_1^\xi(K)$, $\xi < \omega_1$. Professor S. Negrepontis and the author ([7] or [10; Theorem 3.8]) proved the following: If K is compact metric space, $1 \leq \xi < \omega_1$, f a Baire-1 function on K with $f \notin \mathcal{B}_1^\xi(K)$ and (f_k) a uniformly bounded sequence of continuous real-valued functions on K pointwise converging to f, then (f_k) has a subsequence which is l^1_ξ-spreading model (cf. Theorem 3.5 (i)). In this paper we obtain this result as consequence of Theorem 3.1. Also using Theorem 3.3 we obtain the following result:

If K is a compact metric space, $1 \leq \xi < \omega_1$, f a bounded real-valued function on K and (f_k) a uniformly bounded sequence of continuous real-valued functions defined on K and pointwise converging to f such that for every sequence (g_k) of convex blocks of (f_k) (i.e., $g_k \in conv((f_p)_{p\geq k})$ for all k) there exists a subsequence of (g_k) which is l^1_ξ-spreading model then $f \notin \mathcal{B}_1^\xi(K)$ (cf. Theorem 3.5 (ii)). (Here $conv((h_k))$ denotes the set of convex combinations of the h_k's.) For $\xi = 1$, the above result has been proved by Haydon, Odell and Rosenthal in [5].

By using the above results we prove the following: (i) If every uniformly bounded and pointwise converging to zero sequence (f_k) of continuous real-valued functions on a compact metric space K with $\inf_k \|f_k\|_\infty > 0$ has a subsequence which is l^1_ξ-spreading model then all bounded and non-continuous Baire-1 functions on K do not belong to $\mathcal{B}_1^\xi(K)$. (ii) If every uniformly bounded and pointwise converging to zero sequence of continuous real-valued functions on a compact metric space K does not have a subsequence which is l^1_ξ-spreading model, then all bounded Baire-1 functions on K belong to $\mathcal{B}_1^\xi(K)$ (cf. Theorem 3.6).

2. Preliminaries

Let K be a compact metric space and $C(K)$ the set of continuous real-valued functions on K. By $\mathbb{R}$ we mean the set of all real numbers. A function $f : K \to \mathbb{R}$ is Baire-1 if there exists a sequence (f_k) in C(K) that converges pointwise to f. Let $\mathcal{B}_1(K)$ be the set of all bounded Baire-1 real-valued functions on K. Haydon, Odell and Rosenthal in [5], Kechris and Louveau in [6] defined the oscillation index $\beta(f)$ of a general function $f : K \to \mathbb{R}$ and proved that f is Baire-1 iff $\beta(f) < \omega_1$.

Definition 2.1 (cf. [5], [6]). Let K be a compact metric space, $f : K \to \mathbb{R}$, $P \subseteq K$ and $\epsilon > 0$. Let $P^0_{\epsilon,f} = P$ and for any ordinal α let $P^{\alpha+1}_{\epsilon,f}$ be the set of those $x \in P^\alpha_{\epsilon,f}$ such that for every open set U around x there are two points x_1 and x_2 in $P^\alpha_{\epsilon,f} \cap U$ such that $|f(x_1) - f(x_2)| \geq \epsilon$.

At a limit ordinal α we set $P^\alpha_{\epsilon,f} = \bigcap_{\beta<\alpha} P^\beta_{\epsilon,f}$.

Let $\beta(f,\epsilon)$ be the least α with $K^\alpha_{\epsilon,f} = \emptyset$ if such an α exists, and $\beta(f,\epsilon) = \omega_1$, otherwise. Define the oscillation index $\beta(f)$ of f by

$$\beta(f) = \sup\{\beta(f,\epsilon) : \epsilon > 0\}.$$

For every $\xi < \omega_1$ we define $\mathcal{B}_1^\xi(K) = \{f \in \mathcal{B}_1(K) : \ \beta(f) \leq \omega^\xi\}$.

The complexity of pointwise convergent sequences of continuous real-valued functions defined on a compact metric space is described by a countable ordinal index "γ" which is defined in the following way.

Definition 2.2 (cf. [6]). Let K be a compact metric space, (f_k) a sequence of continuous real-valued functions defined on K, $P \subseteq K$ and $\epsilon > 0$. Let $P^0_{\epsilon,(f_k)} = P$ and for any ordinal α let $P^{\alpha+1}_{\epsilon,(f_k)}$ be the set of those $x \in P^\alpha_{\epsilon,(f_k)}$ such that for every open set U around x and for every $p \in \mathbb{N}$ there are $m, n \in \mathbb{N}$ with $m > n > p$ and a point x' in $P^\alpha_{\epsilon,(f_k)} \cap U$ such that $|f_m(x') - f_n(x')| \geq \epsilon$.

At a limit ordinal α we set $P^\alpha_{\epsilon,(f_k)} = \bigcap_{\beta<\alpha} P^\beta_{\epsilon,(f_k)}$. (It can be noticed that $P^\alpha_{\epsilon,(f_k)}$ is a closed subset of P in the relative topology of P.) Let $\gamma((f_k),\epsilon)$ be the least α with $K^\alpha_{\epsilon,(f_k)} = \emptyset$ if such an α exists, and $\gamma((f_k),\epsilon) = \omega_1$, otherwise. (Notice that if $\gamma((f_k),\epsilon) < \omega_1$ then it is a successor ordinal.) Define the convergence index $\gamma((f_k))$ of (f_k) by

$$\gamma((f_k)) = \sup\{\gamma((f_k),\epsilon) : \epsilon > 0\}.$$

Also in [6] it is proved that, $\gamma((f_k)) < \omega_1$ iff (f_k) is pointwise converging.

Generalized Schreier families.

Definition 2.3 (cf. [1]). If F and H are finite non-empty subsets of $\mathbb{N}$ and $n \in \mathbb{N}$, then we define $F < H$ iff $\max F < \min H$, $n \leq F$ iff $n \leq \min F$. Let $\mathcal{F}_0 = \mathcal{F}_0^{'} = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ and $\mathcal{F}_1 = \mathcal{F}_1^{'}$ be the usual Schreier family, i.e., $\mathcal{F}_1 = \mathcal{F}_1^{'} = \{\emptyset\} \cup \{A \subset \mathbb{N} : A \neq \emptyset, |A| \leq \min A\}$. If $\mathcal{F}_\xi$, $\mathcal{F}_\xi^{'}$ have been defined then we set

$$\mathcal{F}_{\xi+1} = \bigcup_{k=1}^{\infty} \Big\{ \bigcup_{i=1}^{k} F_i : F_1, \dots, F_k \in \mathcal{F}_\xi \ \text{ with } \ k \leq F_1 < \dots < F_k \Big\}$$

and

$$\mathcal{F}_{\xi+1}^{'} = \bigcup_{k=1}^{\infty} \Big\{ \bigcup_{i=1}^{k} F_i : F_1, \dots, F_k \in \mathcal{F}_\xi^{'} \ \text{ with } \ k \leq F_1 < \dots < F_k \Big\}.$$

If ξ is a limit ordinal with $\mathcal{F}_\zeta$, $\mathcal{F}_\zeta^{'}$ defined for each $\zeta < \xi$, choose and fix a strictly increasing sequence of ordinals (ξ_k) and a strictly increasing sequence of successor ordinals $(\xi_k^{'})$ with $\xi = \sup_k \xi_k = \sup_k \xi_k^{'}$ and let

$$\mathcal{F}_\xi = \bigcup_{k=1}^{\infty} \{F \in \mathcal{F}_{\xi_k} : \min F \geq k\}, \quad \mathcal{F}_\xi^{'} = \bigcup_{k=1}^{\infty} \{F \in \mathcal{F}_{\xi_k^{'}}^{'} : \min F \geq k\}.$$

It can be noticed that the families $\mathcal{F}_m$, $1 \leq m < \omega$, appeared for the first time in an example constructed by Alspach and Odell [2]. (Also it is obvious that $\mathcal{F}_m = \mathcal{F}_m^{'}$ for every $m < \omega$.)

Lemma 2.4. (a) *For every $\zeta < \xi < \omega_1$ there exists $n \equiv n(\zeta, \xi) \in \mathbb{N}$ such that if $n \leq F \in \mathcal{F}_\zeta$ then $F \in \mathcal{F}_\xi$ and, if $n \leq F \in \mathcal{F}_\zeta^{'}$ then $F \in \mathcal{F}_\xi^{'}$* (see also [3; Lemma 2.1.8 (a)]).

(b) *For every $\xi < \omega_1$, whenever $F = \{n_1 < \dots < n_k\} \in \mathcal{F}_\xi$ (resp. $F = \{n_1 < \dots < n_k\} \in \mathcal{F}_\xi^{'}$) and $m_i \geq n_i$ for $1 \leq i \leq k$ then we have $\{m_1, \dots, m_k\} \in \mathcal{F}_\xi$ (resp. $\{m_1, \dots, m_k\} \in \mathcal{F}_\xi^{'}$)* (see also [3; Lemma 2.1.8 (b)]).

(c) *If $\zeta \leq \xi < \omega_1$ then there exists a strictly increasing sequence (λ_k) of natural numbers such that if $F \in \mathcal{F}_\zeta^{'}$ then $\{\lambda_j : j \in F\} \in \mathcal{F}_\xi$.*

(d) *If $\zeta \leq \xi < \omega_1$ then there exists a strictly increasing sequence (μ_k) of natural numbers such that if $F \in \mathcal{F}_\zeta$ then $\{\mu_j : j \in F\} \in \mathcal{F}_\xi^{'}$.*

Proof: (a) and (b) are proved easily by induction on $\xi < \omega_1$. We shall prove (c) by induction on $\xi < \omega_1$. For $\xi = 0$ it is obvious by Definition 2.3. Suppose that $\xi \geq 1$ and that the conclusion holds for every $\eta < \xi$. Assume that $\xi = \eta + 1$, where $\eta < \omega_1$. If $\zeta \leq \eta$ then there exists a strictly increasing sequence (λ_k) of natural numbers such that if $F \in \mathcal{F}^{'}_{\zeta}$ then $\{\lambda_j : j \in F\} \in \mathcal{F}_{\eta} \subseteq \mathcal{F}_{\eta+1} = \mathcal{F}_{\xi}$. Let $\zeta = \xi = \eta + 1$. By the induction assumption, there exists a strictly increasing sequence (λ_k) of natural numbers such that if $F \in \mathcal{F}^{'}_{\eta}$ then $\{\lambda_j : j \in F\} \in \mathcal{F}_{\eta}$. Then we easily see that if $F \in \mathcal{F}^{'}_{\zeta} = \mathcal{F}^{'}_{\eta+1}$ then $\{\lambda_j : j \in F\} \in \mathcal{F}_{\eta+1} = \mathcal{F}_{\xi}$.

Assume ξ is a limit ordinal and let (ξ_k) be the strictly increasing sequence of ordinals with $\sup_k \xi_k = \xi$ that defines the family $\mathcal{F}_{\xi}$. If $\zeta < \xi$ then there exists $n_0 \in \mathbb{N}$ with $\zeta < \xi_n$ for all $n \geq n_0$. We set $\lambda_k^{n_0} = k$ for all $k \in \mathbb{N}$. By induction on $n > n_0$, there exists a subsequence (λ_k^n) of (λ_k^{n-1}) such that if $F \in \mathcal{F}^{'}_{\zeta}$ then $\{\lambda_j^n : j \in F\} \in \mathcal{F}_{\xi_n}$. Consider the sequence $(\lambda_{n_0+k}^{n_0+k})$. By using the assumption and (b) we have that if $F \in \mathcal{F}^{'}_{\zeta}$ and $k = \min F$ then $F^{'} = \{\lambda_{n_0+j}^{n_0+j} : j \in F\} \in \mathcal{F}_{\xi_{n_0+k}}$ and $F^{'} \geq \lambda_{n_0+k}^{n_0+k} \geq n_0 + k$. Therefore $F^{'} \in \mathcal{F}_{\xi}$.

Now suppose that $\zeta = \xi$ and let $(\zeta_k^{'})$ be the strictly increasing sequence of successor ordinals with $\sup_k \zeta_k^{'} = \zeta$ that defines the family $\mathcal{F}^{'}_{\zeta}$. For every $n \in \mathbb{N}$ there exists $j_n \in \mathbb{N}$ such that $j_n \geq n$ and $\zeta_n^{'} < \xi_{j_n}$. We set $\lambda_k^0 = k$ for all $k \in \mathbb{N}$. By induction on $n \geq 1$, there exists a subsequence (λ_k^n) of (λ_k^{n-1}) such that if $F \in \mathcal{F}^{'}_{\zeta_n^{'}}$ then $\{\lambda_j^n : j \in F\} \in \mathcal{F}_{\xi_{j_n}}$. The proof can be finished by taking the sequence $(\lambda_{j_k}^k)$ and using (b) and Definition 2.3. Similarly, we prove the condition (d). □

Repeated Averages.

S. Argyros, S. Mercourakis and A. Tsarpalias [3] defined a family $\{(M,\xi) : M \in [\mathbb{N}], \xi < \omega_1\}$ called Repeated Averages Hierarchy. The definition of this family follows.

Definition 2.5 (cf. [3])**.** Let $S_{l^1}^+$ be the positive part of the unit sphere of l^1. For $A = (a_k)$ in $S_{l^1}^+$ and $F = (x_k)$ bounded sequence in a Banach space X we denote by $A \cdot F$ the usual matrices product, that is:

$$A \cdot F = \sum_{k=1}^{\infty} a_k x_k.$$

For an $A = (a_k)$ in $S_{l^1}^+$ we set $\operatorname{supp} A = \{k \in \mathbb{N} : a_k \neq 0\}$. A sequence $(A_k) \subseteq S_{l^1}^+$ is said to be *block sequence* if $\operatorname{supp} A_k < \operatorname{supp} A_{k+1}$ for every $k = 1, 2, \ldots$.

For an $M \in [\mathbb{N}]$ an *M-summability method* is a block sequence (A_k) with $A_k \in S_{l^1}^+$ and $M = \bigcup_{k=1}^{\infty} \operatorname{supp} A_k$.

For every $M \in [\mathbb{N}]$ and $\xi < \omega_1$, an M-summability method (ξ_k^M) is defined inductively in the following way. (The notation (M, ξ) is also used for the same method.)

(i) For $\xi = 0$, $M = (m_k)$ we set $\xi_k^M = e_{m_k}$, where (e_k) is the unit basis of l^1 (i.e., $e_k = (0, 0, \dots, 1, 0, \dots)$, the 1 occurring in the k^{th} place).

(ii) If $\xi = \zeta + 1$, $M \in [\mathbb{N}]$ and (ζ_k^M) has been defined then we define (ξ_k^M) inductively as follows. We set $k_1 = 0$, $s_1 = \min \operatorname{supp} \zeta_1^M$, and

$$\xi_1^M = \frac{\zeta_1^M + \dots + \zeta_{s_1}^M}{s_1}.$$

Suppose that for $j = 1, 2, \dots, n-1$, k_j, s_j have been defined and

$$\xi_j^M = \frac{\zeta_{k_j+1}^M + \dots + \zeta_{k_j+s_j}^M}{s_j}.$$

Then we set $k_n = k_{n-1} + s_{n-1}$, $s_n = \min \operatorname{supp} \zeta_{k_n}^M$ and

$$\xi_n^M = \frac{\zeta_{k_n+1}^M + \dots + \zeta_{k_n+s_n}^M}{s_n}.$$

This completes the definition for successor ordinals.

(iii) If ξ is a limit ordinal and if we suppose that for every $\zeta < \xi$, $M \in [\mathbb{N}]$ the sequence (ζ_k^M) has been defined, then we define (ξ_k^M) as follows: We denote by (ζ_k) the strictly increasing sequence of successor ordinals with $\sup_k \zeta_k = \xi$ that defines the family $\mathcal{F}'_\xi$.

For $M = (m_j)$ we define inductively $M_1 = M$, $n_1 = m_1$, $M_2 = \{m_j : m_j \notin \operatorname{supp}[\zeta_{n_1}]_1^{M_1}\}$, $n_2 = \min M_2$, $M_3 = \{m_j : m_j \notin \operatorname{supp}[\zeta_{n_2}]_1^{M_2}\}$ and $n_3 = \min M_3$, and so on.

We set $\xi_1^M = [\zeta_{n_1}]_1^{M_1}, \xi_2^M = [\zeta_{n_2}]_1^{M_2}, \dots, \xi_k^M = [\zeta_{n_k}]_1^{M_k}, \dots$. Hence (ξ_k^M) has been defined. This completes the definition of Repeated Averages Hierarchy.

Remark 2.6 (cf. [3])**.** By induction on $\xi < \omega_1$ it is easy to show that for every $M \in [\mathbb{N}]$ and $\xi < \omega_1$ we have $\{\operatorname{supp} \xi_k^L : L \in [M], k = 1, 2, \dots\} \subseteq \mathcal{F}'_\xi$.

Notation 2.7 (cf. [3])**.** For $F \in [\mathbb{N}]^{<\omega}$ and $A = (a_k)$ in l^1 we denote by $\langle A, F \rangle$ the quantity $\sum_{k \in F} a_k$.

Definition 2.8. A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is said to be *hereditary* if $F \in \mathcal{F}$ and $G \subseteq F$ implies $G \in \mathcal{F}$. A family $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ is said to be *compact* if the set of all characteristic functions χ_F, where $F \in \mathcal{F}$, is a compact subspace of $\{0, 1\}^{\mathbb{N}}$ with the product topology. The family $\mathcal{F}$ is said to be *adequate* if $\mathcal{F}$ is hereditary and compact.

By Proposition 2.3.2 of [3], Theorem 2.2.6 of [3] and Lemma 2.4(d) we have the following theorem:

Theorem 2.9. *Let $\xi < \omega_1$ be an ordinal, $\mathcal{F}$ an adequate family of finite subsets of $\mathbb{N}$, $M \in [\mathbb{N}]$ and δ a positive real number such that for every $N \in [M]$ and for every $n \in \mathbb{N}$ we have that $\sup_{F\in\mathcal{F}}\langle \xi_n^N, F\rangle > \delta$.*

Then there exists a strictly increasing sequence (m_k) of members of M such that $\{m_j : j \in E\} \in \mathcal{F}$ for all $E \in \mathcal{F}_\xi$.

Trees.

Definition 2.10 (cf. [4])**.** Let X be a set. For every $n \in \mathbb{N}$ we set $X^n := \{(x_1, \dots, x_n) : x_1, \dots, x_n \in X\}$.

(i) A *tree* T on X will be a subset of $\bigcup_{n=1}^{\infty} X^n$ with the property that $(x_1, \dots, x_n) \in T$ whenever $n \in \mathbb{N}$ and $(x_1, \dots, x_n, x_{n+1}) \in T$.

(ii) Proceeding by induction we associate to each ordinal α a new tree T^α as follows: We set $T^0 = T$. If T^α is obtained, let

$$T^{\alpha+1} = \bigcup_{n=1}^{\infty} \{(x_1, \dots, x_n) \in T^\alpha : (x_1, \dots, x_n, x) \in T^\alpha \ \text{ for some } \ x \in X\}.$$

If β is a limit ordinal we set $T^\beta = \bigcap_{\alpha<\beta} T^\alpha$.

Notation 2.11. If T is a tree on a set X and $Y \subseteq X$ then we set:

$$T_{|Y} := T \cap \bigcup_{n=1}^{\infty} Y^n.$$

In the proofs of the main results (Theorems 3.1, 3.2, 3.3 and 3.4) we shall use some results from [8] which are contained in the following theorem.

Theorem 2.12. *Let K be a compact metric space, $1 \le \xi < \omega_1$ and (f_k) a sequence of continuous real-valued functions on K. The following hold:*

(i) *If $\gamma((f_{n_k})) > \omega^\xi$ for every strictly increasing sequence (n_k) of natural numbers then there exist $\epsilon > 0$ and a strictly increasing sequence (n_k) of natural numbers such that for every subsequence (n_k') of (n_k) and for every $E = \{k_1 < \dots < k_\lambda\} \in \mathcal{F}_\xi$, $(\lambda \in \mathbb{N})$, there exists $x_E \in K$ with $|f_{n'_{2k_j+1}}(x_E) - f_{n'_{2k_j}}(x_E)| > \epsilon$ for all $1 \le j \le \lambda$.*

(ii) *If $\epsilon > 0$, (n_k) a strictly increasing sequence of natural numbers and (n_k') a subsequence of (n_k) such that for every $E = \{k_1 < \dots < k_\lambda\} \in \mathcal{F}_\xi$, $(\lambda \in \mathbb{N})$, there exists $x_E \in K$ with $|f_{n'_{2k_j+1}}(x_E) - f_{n'_{2k_j}}(x_E)| > \epsilon$ for all $1 \le j \le \lambda$, then $\gamma((f_{n_k}), \epsilon) > \omega^\xi$.*

Proof: (i) We start with the next claim.

Claim. There exist a strictly increasing sequence (n_k) of natural numbers and $\epsilon > 0$ such that $\gamma((f_{n_k'}), \epsilon) > \omega^\xi$ for every subsequence (n_k') of (n_k).

[*Proof of Claim.* Assume the contrary. Then for every $\epsilon > 0$ and (n_k) strictly increasing sequence of natural numbers there exists a subsequence $(n_k^{'})$ of (n_k) such that $\gamma((f_{n_k^{'}}), \epsilon) \leq \omega^{\xi}$. We set $n_k^0 = k$ for every $k \in \mathbb{N}$. By induction on $i \geq 1$, there exists a subsequence (n_k^i) of (n_k^{i-1}) such that $\gamma((f_{n_k^i}), \frac{1}{i}) \leq \omega^{\xi}$ for every $i \in \mathbb{N}$. Then $\gamma((f_{n_k^k})) \leq \omega^{\xi}$, a contradiction.]

Therefore, by Claim and [8; Theorem 3.3 (i) $\Rightarrow$ (iii)], there are $\epsilon > 0$ and a strictly increasing sequence (n_k) of natural numbers such that for every subsequence $(n_k^{'})$ of (n_k) and for every $E = \{k_1 < \ldots, k_{\lambda}\} \in \mathcal{F}_{\xi}$, ($\lambda \in \mathbb{N}$), there is $x_E \in K$ with $|f_{n'_{2k_j+1}}(x_E) - f_{n'_{2k_j}}(x_E)| > \epsilon$ for all $1 \leq j \leq \lambda$.

(ii) By [8; Lemma 3.1.3, Definition 3.1.1], $\gamma((f_{n_k^{'}}), \epsilon) > \omega^{\xi}$ and hence $\gamma((f_{n_k}), \epsilon) > \omega^{\xi}$. □

3. Main results

In this section the complexity of pointwise convergent sequences of continuous real-valued functions defined on a compact metric space as described by the convergence index "γ" produces some local analogues (spreading models) of Rosenthal' s theorem (cf. Theorems 3.1, 3.2 and 3.3). By using these results and [8] we obtain a characterization of l_{ξ}^1-spreading models (cf. Theorem 3.4) and a characterization of those bounded Baire-1 functions which have the oscillation index greater than ω^{ξ}, where $1 \leq \xi < \omega_1$ (cf. Theorem 3.5). We start with the following theorem.

Theorem 3.1. *Let K be a compact metric space, $1 \leq \xi < \omega_1$ and (f_k) a uniformly bounded and pointwise converging sequence of continuous real-valued functions on K such that $\gamma((f_{n_k})) > \omega^{\xi}$ for every strictly increasing sequence (n_k) of natural numbers.*

Then there exists a strictly increasing sequence (n_k) of natural numbers such that the sequence (f_{n_k}) is l_{ξ}^1-spreading model.

For the proof of this theorem we need Lemmas 3.1.4, 3.1.5, 3.1.7, 3.1.8 which are proved by using a method, developed by Professor S. Negrepontis and the author (cf. [7] or [10; Definition 3.6, Lemma 3.7]). We start the next definition.

Definition 3.1.1 (cf. [1]). Let K be a compact metric space and $(f_k) \subseteq C(K)$ pointwise converging on K. Fix $\epsilon > 0$ and let

$$A_{n,m}^{+} = \{x \in K : f_n(x) - f_m(x) > \epsilon\}, \quad A_{n,m}^{-} = \{x \in K : f_n(x) - f_m(x) < -\epsilon\}.$$

For each countable ordinal α we define inductively a subset of K by $O^0(\epsilon, (f_k), K) = K$,

$$O^{\alpha+1}(\epsilon, (f_k), K) = \{x \in O^{\alpha}(\epsilon, (f_k), K) : \text{ for every neighborhood } U \text{ of } x$$

there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that

$$\bigcap_{m \geq m_n} A^+_{n,m} \cap O^\alpha(\epsilon, (f_k), K) \cap U \neq \emptyset \text{ or } \bigcap_{m \geq m_n} A^-_{n,m} \cap O^\alpha(\epsilon, (f_k), K) \cap U \neq \emptyset\}.$$

If β is a limit ordinal, $O^\beta(\epsilon, (f_k), K) = \bigcap_{\alpha<\beta} O^\alpha(\epsilon, (f_k), K)$.

Remark 3.1.2. It is easy to show that if (n_k) is a strictly increasing sequence of natural numbers and $x \in O^\alpha(\epsilon, (f_{n_k}), K)$ for some $\alpha < \omega_1$, then for every strictly increasing sequence (m_k) of natural numbers and $l \in \mathbb{N}$ with $m_j \in \{n_k : k = 1, 2, \dots\}$ for all $j \geq l$ we have $x \in O^\alpha(\epsilon, (f_{m_k}), K)$.

Definition 3.1.3. For $n \in \mathbb{N}$ and $\xi_1, \dots, \xi_n < \omega_1$ we say that the n-tuple $(\xi_1, \dots, \xi_n)$ has **property** (A) if whenever K is a compact metric space, $(f_k) \subseteq C(K)$ pointwise converging to f, (n_k) a strictly increasing sequence of natural numbers, $m \in \mathbb{N}$ and $\epsilon > 0$ such that for every subsequence (n'_k) of (n_k) and for every $E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n}$ with $m \leq E_1 < \dots < E_n$ there exists $x \in K$ such that $|f_{n'_{2j+1}}(x) - f_{n'_{2j}}(x)| > \epsilon$ for all $j \in \bigcup_{i=1}^n E_i$, then there exists a subsequence (n'_k) of (n_k) such that $O^{\omega^{\xi_n}+\dots+\omega^{\xi_1}}(\frac{\epsilon}{4}, (f_{n'_k}), K) \neq \emptyset$.

Lemma 3.1.4. *For every $\xi < \omega_1$, whenever $(\xi_1, \dots, \xi_n)$ has property (A) then $(\xi, \xi_1, \dots, \xi_n)$ has property (A).*

Proof: We proceed by induction on $\xi < \omega_1$.

Case 1. $(\xi = 0)$. Assume that $(\xi_1, \dots, \xi_n)$ have property (A) and we shall show that $(0, \xi_1, \dots, \xi_n)$ has property (A). Indeed, let K be a compact metric space, $(f_k) \subseteq C(K)$ pointwise converging to f, $\epsilon > 0$, (n_k) a strictly increasing sequence of natural numbers and $m \in \mathbb{N}$ such that for every subsequence (n'_k) of (n_k) and $k \in \mathbb{N}$, $E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n}$ with $m \leq k < E_1 < \dots < E_n$ there exists $x \in K$ with $|f_{n'_{2j+1}}(x) - f_{n'_{2j}}(x)| > \epsilon$ for all $j \in \{k\} \cup \bigcup_{i=1}^n E_i$. We shall prove that there exists a subsequence (n'_k) of (n_k) such that $O^{\omega^{\xi_n}+\dots+\omega^{\xi_1}+1}(\frac{\epsilon}{4}, (f_{n'_k}), K) \neq \emptyset$.

We set $P_1 := \{x \in K : |f_{n_{2m+1}}(x) - f_{n_{2m}}(x)| \geq \epsilon\}$. By the continuity of $f_{n_{2m}}, f_{n_{2m+1}}$, P_1 is a closed subset of K and hence it is a compact subspace of K. Also we set $n^0_k = n_{2m+k+1}$ for all $k = 1, 2, \dots$. Then for every subsequence (n'_k) of (n^0_k) we consider the subsequence (n''_k) of (n_k) with $n''_k = n_k$ for $1 \leq k \leq 2m+1$ and $n''_k = n'_k$ for $k \geq 2m+2$. By applying the assumption we have that for every $E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n}$ with $m+1 \leq E_1 < \dots < E_n$ there exists $x \in P_1$ such that $|f_{n'_{2j+1}}(x) - f_{n'_{2j}}(x)| > \epsilon$ for all $j \in \bigcup_{i=1}^n E_i$. Since $(\xi_1, \dots, \xi_n)$ has property (A), there exists a subsequence (n^1_k) of (n^0_k) and $x_1 \in O^{\omega^{\xi_n}+\dots+\omega^{\xi_1}}(\frac{\epsilon}{4}, (f_{n^1_k}), P_1)$. Then clearly $|f_{n_{2m+1}}(x_1) - f_{n_{2m}}(x_1)| \geq \epsilon$.

By induction on $j \geq 1$ and using that $(\xi_1, \dots, \xi_n)$ has property (A), there exists a strictly increasing sequence (n^{j+1}_k) of elements of $\{n^j_{2m+k+1} : k = 1, 2, \dots\}$ and

$x_{j+1} \in K$ with $x_{j+1} \in O^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}}(\frac{\epsilon}{4}, (f_{n_k^{j+1}}), P_{j+1})$, where $P_{j+1} := \{x \in K : |f_{n^j_{2m+1}}(x) - f_{n^j_{2m}}(x)| \geq \epsilon\}$.

Since K is compact metric space, there exists a subsequence (x_{λ_j}) of (x_j) and $x \in K$ such that $\lim_{j\to\infty} x_{\lambda_j} = x$. Then $|f_{n^{\lambda_j-1}_{2m+1}}(x_{\lambda_j}) - f_{n^{\lambda_j-1}_{2m}}(x_{\lambda_j})| \geq \epsilon$ for all $j = 1, 2, \ldots$. Then it is easy to choose a subsequence (λ_{μ_j}) of (λ_j) and $n_j^{'} \in \{n^{\lambda_{\mu_j}-1}_{2m}, n^{\lambda_{\mu_j}-1}_{2m+1}\}$ for $j = 1, 2, \ldots$, such that one of the following holds:

(1) $f_{n_j^{'}}(x_{\lambda_{\mu_j}}) - f(x_{\lambda_{\mu_j}}) > \frac{\epsilon}{3}$ for all $j = 1, 2, \ldots$,

(2) $f_{n_j^{'}}(x_{\lambda_{\mu_j}}) - f(x_{\lambda_{\mu_j}}) < -\frac{\epsilon}{3}$ for all $j = 1, 2, \ldots$.

We shall prove that $x \in O^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+1}(\frac{\epsilon}{4}, (f_{n_k^{'}}), K)$. Indeed, let U be a neighborhood of x. Since $\lim_{j\to\infty} x_{\lambda_{\mu_j}} = x$, there exists $j_0 \in \mathbb{N}$ such that $x_{\lambda_{\mu_j}} \in U$ for all $j \geq j_0$.

Suppose that (1) holds. Since (f_k) converges pointwise to f for every $j \geq j_0$ there exists $m_j \in \mathbb{N}$ such that

$$f_{n_j^{'}}(x_{\lambda_{\mu_j}}) - f_{n_m^{'}}(x_{\lambda_{\mu_j}}) \geq \frac{\epsilon}{3} > \frac{\epsilon}{4} \text{ for all } m \geq m_j.$$

So, by using Remark 3.1.2, $x_{\lambda_{\mu_j}} \in \bigcap_{m\geq m_j} A^+_{j,m} \cap O^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}}(\frac{\epsilon}{4}, (f_{n_k^{'}}), K) \cap U$ for all $j \geq j_0$. Therefore $x \in O^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+1}(\frac{\epsilon}{4}, (f_{n_k^{'}}), K)$. A similar argument shows that $x \in O^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+1}(\frac{\epsilon}{4}, (f_{n_k^{'}}), K)$ if (2) holds.

Case 2. $(\xi \geq 1)$. Suppose that the conclusion holds for every $\zeta < \xi$ and we shall show it for ξ. Assume that $(\xi_1, \ldots, \xi_n)$ has property (A) and we shall show that $(\xi, \xi_1, \ldots, \xi_n)$ has property (A). Indeed, let K be a compact metric space, $(f_k) \subseteq C(K)$ pointwise converging to f, $\epsilon > 0$, (n_k) a strictly increasing sequence of natural numbers and $m \in \mathbb{N}$ such that for every subsequence $(n_k^{'})$ of (n_k) and $E \in \mathcal{F}_\xi, E_1 \in \mathcal{F}_{\xi_1}, \ldots, E_n \in \mathcal{F}_{\xi_n}$ with $m \leq E < E_1 < \ldots < E_n$ there exists $x \in K$ with $|f_{n^{'}_{2j+1}}(x) - f_{n^{'}_{2j}}(x)| > \epsilon$ for all $j \in E \cup \bigcup_{i=1}^n E_i$. We set $n_k^m = n_k$ for all $k \in \mathbb{N}$. Consider these two subcases:

(a) $\xi = \zeta + 1$. Then for every subsequence $(n_k^{'})$ of (n_k), $j \in \mathbb{N}$ with $j \geq m$ and $F_1, \ldots, F_j \in \mathcal{F}_\zeta, E_1 \in \mathcal{F}_{\xi_1}, \ldots, E_n \in \mathcal{F}_{\xi_n}$ with $j \leq F_1 < \ldots < F_j < E_1 < \ldots < E_n$ there exists $x \in K$ such that $|f_{n^{'}_{2k+1}}(x) - f_{n^{'}_{2k}}(x)| > \epsilon$ for all $k \in \bigcup_{l=1}^j F_l \bigcup_{i=1}^n E_i$. By the induction hypothesis, $(\underbrace{\zeta, \ldots, \zeta}_{j-times}, \xi_1, \ldots, \xi_n)$ has property (A) for all $j \in \mathbb{N}$. So, by induction on $j > m$, there exists a subsequence (n_k^j) of (n_k^{j-1}) such that $O^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+j\omega^\zeta}(\frac{\epsilon}{4}, (f_{n_k^j}), K) \neq \emptyset$. We set $n_k^{'} = n^{m+k}_{m+k}$ for all $k \in \mathbb{N}$. Therefore, by the compactness of K and using Definition 3.1.1 and Remark 3.1.2, we get that the set $O^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+\omega^\xi}(\frac{\epsilon}{4}, (f_{n_k^{'}}), K)$ is non-empty.

(b) ξ is a limit ordinal. Let (ζ_k) be the strictly increasing sequence of ordinals with $\sup_k \zeta_k = \xi$ that defines the family $\mathcal{F}_\xi$. Then for every subsequence $(n_k^{'})$ of (n_k), $j \in \mathbb{N}$ with $j \geq m$ and $E \in \mathcal{F}_{\zeta_j}, E_1 \in \mathcal{F}_{\xi_1}, \ldots, E_n \in \mathcal{F}_{\xi_n}$ with $j \leq E < E_1 < \ldots < E_n$ there exists $x \in K$ such that $|f_{n_{2k+1}^{'}}(x) - f_{n_{2k}^{'}}(x)| > \epsilon$ for all $k \in E \cup \bigcup_{i=1}^n E_i$. By the induction hypothesis, $(\zeta_j, \xi_1, \ldots, \xi_n)$ has property (A) for every $j \in \mathbb{N}$. So, by induction on $j > m$, there exists a subsequence (n_k^j) of (n_k^{j-1}) such that $O^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+\omega^{\zeta_j}}(\frac{\epsilon}{4}, (f_{n_k^j}), K)$ is non-empty. We set $n_k^{'} = n_{m+k}^{m+k}$ for all $k \in \mathbb{N}$. By the compactness of K and using Definition 3.1.1 and 3.1.2, we get $O^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}+\omega^{\xi}}(\frac{\epsilon}{4}, (f_{n_k^{'}}), K) \neq \emptyset$. □

Lemma 3.1.5. *For every $n \in \mathbb{N}$ and $\xi_1, \ldots, \xi_n < \omega_1$ the n-tuple $(\xi_1, \ldots, \xi_n)$ has property (A).*

Proof: By Lemma 3.1.4, it is enough to show that the 1-tuple (ξ) has property (A) for every $\xi < \omega_1$. We shall prove it by induction on $\xi < \omega_1$. For $\xi = 0$, let K be a compact metric space, $(f_k) \subseteq C(K)$ pointwise convergent to f, (n_k) a strictly increasing sequence of natural numbers, $m \in \mathbb{N}$ and $\epsilon > 0$ such that for every subsequence $(n_k^{'})$ of (n_k) and for every $E = \{k\} \in \mathcal{F}_0$ there exists $x \in K$ such that $|f_{n_{2k+1}^{'}}(x) - f_{n_{2k}^{'}}(x)| > \epsilon$. Then working as in the proof of the case 1 of Lemma 3.1.4 we prove that there exists a subsequence $(n_k^{'})$ of (n_k) such that $O^1(\frac{\epsilon}{4}, (f_{n_k^{'}}), K) \neq \emptyset$.

Now suppose that $\xi \geq 1$, the 1-tuple (ζ) has property (A) for every $\zeta < \xi$ and we shall prove that (ξ) has property (A). If $\xi = \zeta + 1$, then for every $k \in \mathbb{N}$, the k-tuple $(\underbrace{\zeta, \ldots, \zeta}_{k-times})$ has property (A) by Lemma 3.1.4. If ξ is a limit ordinal and (ξ_k) the strictly increasing sequence of ordinals with $\sup_k \xi_k = \xi$ that defines $\mathcal{F}_\xi$ then for every $k \in \mathbb{N}$, the 1-tuple (ξ_k) has property (A) by the induction assumption. Therefore, by using the definition of the property (A) and using a diagonal argument we get the desired conclusion (as in the case 2 of Lemma 3.1.4). □

Definition 3.1.6. For any $n \in \mathbb{N}$ and $\xi_1, \ldots, \xi_n < \omega_1$ we say that the n-tuple $(\xi_1, \ldots, \xi_n)$ has **property** (B) if whenever T is a tree on ω such that $0 < m_1 < \ldots < m_k$ for every $(0, m_1, \ldots, m_k) \in T$ and $M \in [\mathbb{N}]$ such that $(0) \in (T_{|N \cup \{0\}})^{\omega^{\xi_n}+\ldots+\omega^{\xi_1}}$ for every $N \in [M]$, then there exists a strictly increasing sequence (m_k) of elements of M such that for every $E_1 \in \mathcal{F}_{\xi_1}, \ldots, E_n \in \mathcal{F}_{\xi_n}$ with $E_1 < \ldots < E_n$ and $\bigcup_{i=1}^n E_i = \{k_1 < \ldots < k_\lambda\}$, (where $\lambda \in \mathbb{N}$), we have $(0, m_{k_1}, \ldots, m_{k_\lambda}) \in T$.

Lemma 3.1.7. *For every $\xi < \omega_1$, whenever $(\xi_1, \ldots, \xi_n)$ has property (B) then $(\xi, \xi_1, \ldots, \xi_n)$ has property (B).*

Proof: We proceed by induction on $\xi < \omega_1$.

Case 1. ($\xi = 0$). Let $(\xi_1, \dots, \xi_n)$ have property (B), let T be a tree on ω such that $0 < m_1 < \dots < m_k$ for every $(0, m_1, \dots, m_k) \in T$ and $M \in [\mathbb{N}]$ such that $(0) \in (T_{|N \cup \{0\}})^{\omega^{\xi_n} + \dots + \omega^{\xi_1} + 1}$ for every $N \in [M]$.

Claim. There exists $M_0 \in [M]$ such that for every $M' \in [M_0]$ there is $m \in M'$ such that $(0, m) \in (T_{|L \cup \{0\}})^{\omega^{\xi_n} + \dots + \omega^{\xi_1}}$ for all $L \in [M']$ with $\min L > m$.

[*Proof of Claim.* Assume the contrary. Then there exists a decreasing sequence (M_λ) of infinite subsets of M such that if $m_\lambda = \min M_\lambda$ then $m_\lambda < m_{\lambda+1}$ and $(0, m_\lambda) \notin (T_{|\{0, m_\lambda\} \cup M_{\lambda+1}})^{\omega^{\xi_n} + \dots + \omega^{\xi_1}}$ for all $\lambda \in \mathbb{N}$. Consider the set $L = \{m_\lambda : \lambda = 1, 2, \dots\}$. Then from the assumption we have that $(0) \in (T_{|L \cup \{0\}})^{\omega^{\xi_n} + \dots + \omega^{\xi_1} + 1}$. Hence there exists $\lambda \in \mathbb{N}$ such that $(0, m_\lambda) \in (T_{|L \cup \{0\}})^{\omega^{\xi_n} + \dots + \omega^{\xi_1}}$. Then $(0, m_\lambda) \in (T_{|\{0, m_\lambda\} \cup M_{\lambda+1}})^{\omega^{\xi_n} + \dots + \omega^{\xi_1}}$, a contradiction. This completes the proof of the claim.]

For every $m \in M$ we define the tree

$T_m = \{(0)\} \cup \{(0, n_1, \dots, n_j) : j \in \mathbb{N}, (0, m, n_1, \dots, n_j) \in T\}$.

By induction on $\alpha < \omega_1$, it is easy to show that $(0, m, n_1, \dots, n_j) \in (T_{|N \cup \{0\}})^\alpha$ iff $(0, n_1, \dots, n_j) \in (T_{m|N \cup \{0\}})^\alpha$ and $(0, m) \in (T_{|N \cup \{0\}})^\alpha$ iff $(0) \in (T_{m|N \cup \{0\}})^\alpha$ for every $N \in [M]$.

By repeated application of Claim and using that $(\xi_1, \dots, \xi_n)$ has property (B), we find strictly increasing sequences $M_\lambda = (m_k^\lambda)$, $\lambda \in \mathbb{N}$ of elements of M and a strictly increasing sequence (m_λ) of elements of M such that for every $\lambda \in \mathbb{N}$ it holds $m_\lambda \in M_\lambda$, $m_\lambda^\lambda \le m_\lambda < \min M_{\lambda+1}$ and for every $E_1 \in \mathcal{F}_{\xi_1}, \dots, E_n \in \mathcal{F}_{\xi_n}$ with $E_1 < \dots < E_n$ and $\bigcup_{i=1}^n E_i = \{k_1 < \dots < k_\mu\}$, (where $\mu \in \mathbb{N}$), we have $(0, m_{k_1}^{\lambda+1}, \dots, m_{k_\mu}^{\lambda+1}) \in T_{m_\lambda}$. The proof can be finished by taking the sequence (m_λ) and using Lemma 2.4(b).

Case 2. ($\xi \ge 1$). Assume that the conclusion of our Lemma is true for every $\zeta < \xi$ and we shall show that it is true for ξ. Suppose that $(\xi_1, \dots, \xi_n)$ has property (B) and we shall show that $(\xi, \xi_1, \dots, \xi_n)$ has property (B). Let T be a tree on ω such that $0 < m_1 < \dots < m_k$ for every $(0, m_1, \dots, m_k) \in T$ and $M \in [\mathbb{N}]$ such that $(0) \in (T_{|N \cup \{0\}})^{\omega^{\xi_n} + \dots + \omega^{\xi_1} + \omega^\xi}$ for all $N \in [M]$. Consider these two subcases:

(a) $\xi = \zeta + 1$. Then $(0) \in (T_{|N \cup \{0\}})^{\omega^{\xi_n} + \dots + \omega^{\xi_1} + \lambda\omega^\zeta}$ for all $N \in [M]$, $\lambda \in \mathbb{N}$ and by the induction hypothesis, $(\underbrace{\zeta, \dots, \zeta}_{\lambda-times}, \xi_1, \dots, \xi_n)$ has property (B) for every $\lambda \in \mathbb{N}$.

(b) ξ is a limit ordinal. Let (ζ_k) be the strictly increasing sequence of ordinals with $\sup_k \zeta_k = \xi$ that defines the family $\mathcal{F}_\xi$. Then $(0) \in (T_{|N \cup \{0\}})^{\omega^{\xi_n} + \dots + \omega^{\xi_1} + \omega^{\zeta_\lambda}}$ for every $N \in [M]$, $\lambda \in \mathbb{N}$ and by the induction assumption, $(\zeta_\lambda, \xi_1, \dots, \xi_n)$ has property (B) for every $\lambda \in \mathbb{N}$.

By using the definition of the property (B) and using a diagonal argument we get the desired conclusion in the two subcases. $\square$

Lemma 3.1.8. *For every $n \in \mathbb{N}$, $\xi_1, \dots, \xi_n < \omega_1$, the n-tuple $(\xi_1, \dots, \xi_n)$ has property (B).*

PROOF: By Lemma 3.1.7, it is enough to show that (ξ) has property (B) for every $\xi < \omega_1$. We shall use induction on ξ. Let $\xi = 0$, T be a tree on ω such that $0 < m_1 < \dots < m_k$ for every $(0, m_1, \dots, m_k) \in T$ and $M \in [\mathbb{N}]$ such that $(0) \in (T_{|N \cup \{0\}})^1$ for every $N \in [M]$. Then there exist a strictly decreasing sequence $M_1 \supset M_2 \supset \dots \supset M_k \supset \dots$ of infinite subsets of M and a strictly increasing sequence (m_k) such that $m_k \in M_k$ and $(0, m_k) \in T$ for all $k \in \mathbb{N}$. Therefore the sequence (m_k) is the desired sequence.

Now let $1 \leq \xi < \omega_1$ such that (ζ) has property (B) for every $\zeta < \xi$. If $\xi = \zeta + 1$ then for every $k \in \mathbb{N}$, $(\underbrace{\zeta, \dots, \zeta}_{k-times})$ has property (B) by Lemma 3.1.7. If ξ is a limit ordinal and (ζ_k) is the strictly increasing sequence with $\sup_k \zeta_k = \xi$ that defines the family $\mathcal{F}_\xi$ then the 1-tuple (ζ_k) has property (B) for all $k \in \mathbb{N}$.

By using the definition of the property (B) and using a diagonal argument we prove that (ξ) has property (B). □

PROOF OF THEOREM 3.1: By Lemma 3.1.5, the 1-tuple (ξ) has property (A). So, by Theorem 2.12(i) and by the definition of the property (A), there exist $\delta > 0$ and a subsequence (n'_k) of (n_k) such that $O^{\omega^\xi}(\delta, (f_{n'_k}), K) \neq \emptyset$. By Remark 3.1.2, $O^{\omega^\xi}(\delta, (f_{n''_k}), K) \neq \emptyset$ for every subsequence (n''_k) of (n'_k). Consider the next tree on ω:

$$T := \{(0)\} \cup \bigcup_{n=1}^\infty \{(0, m_1, \dots, m_n) \in \omega^{n+1} : m_1 < \dots < m_n \text{ and } \|\sum_{i=1}^n c_i f_{m_i}\|_\infty \geq \delta \sum_{i=1}^n |c_i| \text{ for all } c_1, \dots, c_n \in \mathbb{R}\}.$$

We set $M := \{n'_k : k = 1, 2, \dots\}$. By using a result of Alspach and Argyros ([1; Theorem 3.1]), it is easy to see that $(T_{|N \cup \{0\}})^{\omega^\xi} \neq \emptyset$ for every $N \in [M]$. By Lemma 3.1.8, (ξ) has property (B). Therefore, by the definition of the property (B) there exists a subsequence (n''_k) of (n'_k) such that for every $E = \{k_1 < \dots < k_\lambda\} \in \mathcal{F}_\xi$, (where $\lambda \in \mathbb{N}$), the finite sequence $(0, n''_{k_1}, \dots, n''_{k_\lambda})$ belongs to T and since (f_k) is uniformly bounded we get that the sequence $(f_{n''_k})$ is l^1_ξ-spreading model. □

Combining some results of [3] and [8] we obtain the following theorem.

Theorem 3.2. *Let K be a compact metric space, $1 \leq \xi < \omega_1$, (f_k) a uniformly bounded and pointwise converging sequence of continuous real-valued functions on K, (n_k) a strictly increasing sequence of natural numbers and (n'_k) a subsequence of (n_k) such that the sequence $(f_{n'_{2k+1}} - f_{n'_{2k}})$ is l^1_ξ-spreading model. Then $\gamma((f_{n_k})) > \omega^\xi$.*

PROOF: By using Lemma 2.4(c) (for $\zeta = \xi$) and the definition of l^1_ξ-spreading model for the sequence $(f_{n'_{2k+1}} - f_{n'_{2k}})$ there exist a strictly increasing sequence

(λ_k) of natural numbers and $\delta > 0$ such that

$$(*) \qquad \delta \sum_{i=1}^{m} |c_i| \leq \| \sum_{i=1}^{m} c_i (f_{n'_{2\lambda_{k_i}+1}} - f_{n'_{2\lambda_{k_i}}}) \|_\infty \leq 2(\sup_k \|f_k\|_\infty) \sum_{i=1}^{m} |c_i|$$

for every $\{k_1 < \ldots < k_m\} \in \mathcal{F}'_\xi$, $c_1, \ldots, c_m \in \mathbb{R}$. For every $x \in K$ let $F_x = \{l \in \mathbb{N} : |f_{n'_{2\lambda_l+1}}(x) - f_{n'_{2\lambda_l}}(x)| \geq \frac{\delta}{2}\}$. Since (f_k) is pointwise converging the sequence $(f_{n'_{2\lambda_k+1}} - f_{n'_{2\lambda_k}})$ converges pointwise to zero and so F_x is finite for every $x \in K$. We set $\mathcal{F} = \{F \in [\mathbb{N}]^{<\omega} : \; F \subseteq F_x \; \text{for some} \; x \in K\}$. We shall prove that $\mathcal{F}$ is adequate. By Definition 2.8 and the definition of $\mathcal{F}$ it is enough to show that the set $\{\chi_F : F \in \mathcal{F}\}$ is closed subspace of $\{0,1\}^{\mathbb{N}}$ with the product topology. Indeed, If $A \subseteq \mathbb{N}$, $A = (a_n)$, with $\chi_A \in cl_{\{0,1\}^{\mathbb{N}}}(\{\chi_F : F \in \mathcal{F}\})$ then for every $n \in \mathbb{N}$ there exists $x_n \in K$ such that $\{a_1, \ldots, a_n\} \subseteq F_{x_n}$. Then $a_k \in F_{x_n}$ for every $n \geq k$. Since K is a compact metric space there exist a subsequence (x_{k_n}) of (x_n) and $x \in K$ with $\lim_n x_{k_n} = x$. By the continuity of f_k's we have $A \subseteq F_x$ and so A is finite and $A \in \mathcal{F}$. Hence $\{\chi_F : F \in \mathcal{F}\}$ is closed.

By $(*)$ it is easy to see that $\|\xi_n^L \cdot ((f_{n'_{2\lambda_k+1}} - f_{n'_{2\lambda_k}}))\|_\infty \geq \delta$ for every $L \in [\mathbb{N}]$, $n \in \mathbb{N}$. Then for every $L \in [\mathbb{N}]$ and $n \in \mathbb{N}$ there exists $x \in K$ such that $|(\xi_n^L \cdot ((f_{n'_{2\lambda_k+1}} - f_{n'_{2\lambda_k}})))(x)| \geq \delta$. Also

$$\delta \leq |(\xi_n^L \cdot ((f_{n'_{2\lambda_k+1}} - f_{n'_{2\lambda_k}})))(x)| \leq \langle \xi_n^L, F_x \rangle \cdot 2 \sup_k \|f_k\|_\infty + \frac{\delta}{2}\,.$$

Then $\langle \xi_n^L, F_x \rangle \geq \frac{\delta}{4 \sup_k \|f_k\|_\infty}$. So, by Theorem 2.9, there exists a strictly increasing sequence (j_k) of natural numbers such that $\{j_l : l \in E\} \in \mathcal{F}$ for all $E \in \mathcal{F}_\xi$. We set $n''_1 = n'_1$, $n''_{2k+1} = n'_{2\lambda_{j_k}+1}$ and $n''_{2k} = n'_{2\lambda_{j_k}}$ for every $k \in \mathbb{N}$. Then the sequence (n''_k) is a subsequence of (n_k) and for every $E = \{k_1 < \ldots < k_m\} \in \mathcal{F}_\xi$ there is $x_E \in K$ such that $|f_{n''_{2k_j+1}}(x_E) - f_{n''_{2k_j}}(x_E)| > \frac{\delta}{2}$ for all $1 \leq j \leq m$. Therefore, by Theorem 2.12 (ii), $\gamma((f_{n_k}), \frac{\delta}{2}) > \omega^\xi$. Hence $\gamma((f_{n_k})) > \omega^\xi$. □

Theorem 3.3. *Let K be a compact metric space, $1 \leq \xi < \omega_1$ and (f_k) a uniformly bounded and pointwise converging sequence of continuous real-valued functions on K which is l^1_ξ-spreading model. Then $\gamma((f_{n_k})) > \omega^\xi$ for every strictly increasing sequence (n_k) of natural numbers.*

Proof: By induction on $1 \leq \xi < \omega_1$, it is easy to show that if $E = \{k_1 < \ldots < k_\lambda\} \in \mathcal{F}_\xi$ then $F = \{2k_1 < 2k_1 + 1 < \ldots < 2k_\lambda < 2k_\lambda + 1\} \in \mathcal{F}_\xi$. By using this fact, it is easy to see that if (f_k) is l^1_ξ-spreading model then for every strictly increasing sequence (n_k) of natural numbers the sequence $(f_{n_{2k+1}} - f_{n_{2k}})$ is also l^1_ξ-spreading model and so, by Theorem 3.2, $\gamma((f_{n_k})) > \omega^\xi$. □

Combining Theorems 3.1, 3.3, 2.12 and Theorem 3.3 of [8] we get the following criteria (characterizations) for the l^1_ξ-spreading model.

Theorem 3.4. *Let K be a compact metric space, $1 \leq \xi < \omega_1$ and (f_k) a uniformly bounded and pointwise converging sequence of continuous real-valued functions on K. Then the following are equivalent:*

(i) *there exists a subsequence (f'_k) of (f_k) which is l^1_ξ-spreading model;*

(ii) *there are $\epsilon > 0$ and a strictly increasing sequence (n_k) of natural numbers such that for every subsequence (n'_k) of (n_k) and for every $E = \{k_1 < \ldots < k_\lambda\} \in \mathcal{F}_\xi$ (where $\lambda \in \mathbb{N}$) there is $x_E \in K$ with $|f_{n'_{2k_j+1}}(x_E) - f_{n'_{2k_j}}(x_E)| > \epsilon$ for all $1 \leq j \leq \lambda$;*

(iii) *there are $\epsilon > 0$ and a strictly increasing sequence (n_k) of natural numbers such that for every $E = \{k_1 < \ldots < k_\lambda\} \in \mathcal{F}_\xi$ (where $\lambda \geq 2$) there is $x_E \in K$ with $|f_{n_{k_{j+1}}}(x_E) - f_{n_{k_j}}(x_E)| > \epsilon$ for all $1 \leq j \leq \lambda - 1$.*

Theorem 3.5. *Let K be a compact metric space, f a bounded real-valued function on K and $1 \leq \xi < \omega_1$. Then the following hold:*

(i) *If $f \notin \mathcal{B}_1^\xi(K)$ and $(f_k) \subseteq C(K)$ a uniformly bounded sequence pointwise converging to f, then (f_k) has a subsequence which is l^1_ξ-spreading model* (cf. [7] or [10; Theorem 3.8]).

(ii) *If $(f_k) \subseteq C(K)$ is a uniformly bounded sequence pointwise converging to f such that for every sequence (g_k) of convex blocks of (f_k) (i.e., $g_k \in conv((f_p)_{p\geq k})$) there exists a subsequence of (g_k) which is l^1_ξ-spreading model, then $f \notin \mathcal{B}_1^\xi(K)$. (Here $conv((h_k))$ denotes the set of convex combinations of the $h'_k s$.)*

Proof: The condition (i) is obvious by Theorem 3.1 and using that $\beta(f) \leq \gamma((f_k))$ (cf. [6; Proposition 1.1]).

(ii) By [6; Theorem 1.3] there exists a sequence (g_k) of convex blocks of (f_k) such that $\beta(f) = \gamma((g_k))$. By the hypothesis, let (g'_k) a subsequence of (g_k) which is l^1_ξ-spreading model. By Theorem 3.3 we have $\gamma((g'_k)) > \omega^\xi$. Also $\gamma((g'_k)) \leq \gamma((g_k)) = \beta(f)$. Hence $\beta(f) > \omega^\xi$ i.e., $f \notin \mathcal{B}_1^\xi(K)$. □

It can be noticed that Theorems 3.3 and 3.5 have been proved for the first time in the preprint [9], but for completeness we gave new proofs. Also for $\xi = 1$, Theorem 3.5 has been proved by Haydon, Odell and Rosenthal in [5].

Theorem 3.6. *Let K be a compact metric space and $1 \leq \xi < \omega_1$. Then the following hold:*

(i) *If every uniformly bounded and pointwise converging to zero sequence $(f_k) \subseteq C(K)$ with $\inf_k \|f_k\|_\infty > 0$ has a subsequence which is l^1_ξ-spreading model, then $\mathcal{B}_1(K) \setminus C(K) \subseteq \mathcal{B}_1(K) \setminus \mathcal{B}_1^\xi(K)$.*

(ii) *If no uniformly bounded and pointwise converging to zero sequence* $(f_k) \subseteq C(K)$ *has a subsequence which is* l^1_ξ*-spreading model then* $\mathcal{B}_1(K) \subseteq \mathcal{B}^\xi_1(K)$.

PROOF: (i) Let $f \in \mathcal{B}_1(K) \setminus C(K)$. By [6; Theorem 1.3] there exists a uniformly bounded sequence $(g_k) \subseteq C(K)$ pointwise converging to f such that $\gamma((g_k)) = \beta(f)$. Then for every strictly increasing sequence (n_k) of natural numbers the sequence $(g_{n_{2k+1}} - g_{n_{2k}})$ is pointwise converging to zero and $\inf_k \|g_{n_{2k+1}} - g_{n_{2k}}\|_\infty > 0$ because f is not continuous. Hence there exists a subsequence (h_k) of $(g_{n_{2k+1}} - g_{n_{2k}})$ which is l^1_ξ-spreading model. Choose a strictly increasing sequence (j_k) of natural numbers such that $h_k = g_{n_{2j_k+1}} - g_{n_{2j_k}}$ for all $k \in \mathbb{N}$. We set $n'_1 = n_1$, $n'_{2k} = n_{2j_k}$ and $n'_{2k+1} = n_{2j_k+1}$ for every $k \in \mathbb{N}$. So, $h_k = g_{n'_{2k+1}} - g_{n'_{2k}}$ for all $k \in \mathbb{N}$. Therefore, by Theorem 3.2, $\gamma((g_k)) > \omega^\xi$. Hence $\beta(f) > \omega^\xi$, i.e., $f \notin \mathcal{B}^\xi_1(K)$. This completes the proof of (i).

(ii) Assume the contrary. Then there exists $f \in \mathcal{B}_1(K) \setminus \mathcal{B}^\xi_1(K)$. Let $(f_k) \subseteq C(K)$ be a uniformly bounded sequence which converges pointwise to f. By Theorem 3.5(i), there exists a subsequence (f'_k) of (f_k) which is l^ξ_1-spreading model. Then the sequence $(f'_{2k+1} - f'_{2k})$ converges pointwise to zero. Also, by using that if $F = \{k_1 < \ldots < k_\lambda\} \in \mathcal{F}_\xi$ then $F' = \{2k_1 < 2k_1 + 1 < \ldots < 2k_\lambda < 2k_\lambda + 1\} \in \mathcal{F}_\xi$, it is easy to show that the sequence $(f'_{2k+1} - f'_{2k})$ is l^1_ξ-spreading model, a contradiction. □

Acknowledgment. I am grateful to referee for his (her) useful corrections.

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(*Received* August 20, 1998)