Limit points of arithmetic means of sequences in Banach spaces

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Abstract. We shall prove the following statements: Given a sequence $\{a_n\}_{n=1}^{\infty}$ in a Banach space $\mathbf{X}$ enjoying the weak Banach-Saks property, there is a subsequence (or a permutation) $\{b_n\}_{n=1}^{\infty}$ of the sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} b_j = a$$

whenever a belongs to the closed convex hull of the set of weak limit points of $\{a_n\}_{n=1}^{\infty}$. In case $\mathbf{X}$ has the Banach-Saks property and $\{a_n\}_{n=1}^{\infty}$ is bounded the converse assertion holds too. A characterization of reflexive spaces in terms of limit points and cores of bounded sequences is also given. The motivation for the problems investigated goes back to Lévy laplacian from potential theory in Hilbert spaces.

Keywords: Banach-Saks property, arithmetic means, limit points, subsequences, permutations of sequences

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1. Introduction

The motivation for the problems investigated goes back to Lévy laplacian from potential theory in Hilbert spaces, see [8], [9] and [10]. In discussing to what extent the definition of Lévy laplacian depends on the choice of an orthonormal basis, P. Lévy used the following result (without giving any proof of it), see [10, p. 63].

Proposition 1. *Given a bounded sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers and a real number a satisfying $\liminf a_n \le a \le \limsup a_n$, there is a permutation $\{b_n\}_{n=1}^{\infty}$ of the sequence $\{a_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} b_j = a.$$

The main goal of this note is to generalize this result for sequences in Banach spaces. First of all, we introduce some notation. Let $\mathbf{R}^m$ be the Euclidean space

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of dimension m. Denote by $\mathcal{P}$ the set of all permutations of the set $\mathbf{N}$ of natural numbers (i.e., the set of all one-to-one mappings ρ of $\mathbf{N}$ onto $\mathbf{N}$) and by $\mathcal{I}$ the set of all strictly increasing mappings $r : \mathbf{N} \to \mathbf{N}$. In what follows we shall assume that $\mathbf{X}$ is a real Banach space unless otherwise specified. A sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbf{X}$ is said to be $(C,1)$-*convergent* provided that there is $a \in \mathbf{X}$ such that

$$a = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} a_j, \tag{1}$$

i.e., $a = (C,1)\text{-}\lim_{n\to\infty} a_n$.

Let $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbf{X}$. Then $w\text{-}\lim_{n\to\infty} a_n$ stands for the weak limit of $\mathbf{a}$ if it exists. Denote

$$\mathcal{L}(\mathbf{a}) = \{a \in \mathbf{X};\ \exists\, r \in \mathcal{I} : w\text{-}\lim_{n\to\infty} a_{r(n)} = a\},$$

i.e., $\mathcal{L}(\mathbf{a})$ is the set of weak limit points of $\mathbf{a}$ in $\mathbf{X}$,

$$\mathcal{K}(\mathbf{a}) = \{a \in \mathbf{X};\ \exists\, r \in \mathcal{I} : (C,1)\text{-}\lim_{n\to\infty} a_{r(n)} = a\}$$

and

$$\mathcal{N}(\mathbf{a}) = \{a \in \mathbf{X};\ \exists\, \rho \in \mathcal{P} : (C,1)\text{-}\lim_{n\to\infty} a_{\rho(n)} = a\}.$$

For $M \subset \mathbf{X}$ denote by $\operatorname{co}(M)$ the convex hull of M, by $\overline{M}$ the closure of M in the norm topology and $\overline{\operatorname{co}}(M) = \overline{\operatorname{co}(M)}$. Finally, define the so-called *core* of $\mathbf{a}$ by

$$\mathcal{C}(\mathbf{a}) = \bigcap_{N=1}^{\infty} \overline{\operatorname{co}}(\{a_n\}_{n\geq N}).$$

Recall that a Banach space $\mathbf{X}$ is said to have *the weak Banach-Saks property* (WBS) provided that each weakly convergent sequence in $\mathbf{X}$ possesses a $(C,1)$-convergent subsequence.

Remark 1. It was already known to Cauchy that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $L \in [-\infty, +\infty]$, then $\lim_{n\to\infty} a_n = L$ implies $(C,1)\text{-}\lim_{n\to\infty} a_n = L$. Thus it is easily seen that the weak and the $(C,1)$-limit of a sequence in a Banach space are the same if both exist. Obviously, a Banach space $\mathbf{X}$ is (WBS) if and only if $\mathcal{L}(\mathbf{a}) \subset \mathcal{K}(\mathbf{a})$ for each sequence $\mathbf{a}$ in $\mathbf{X}$.

Finally, a Banach space $\mathbf{X}$ is said to have *the Banach-Saks property* (BS) provided that even each bounded sequence in $\mathbf{X}$ admits a $(C,1)$-convergent subsequence. It is well-known that every uniformly convex space has the Banach-Saks property and that every Banach space enjoying the Banach-Saks property is reflexive, see [3, p. 78]. Thus a Banach space $\mathbf{X}$ is (BS) if and only if $\mathbf{X}$ is (WBS) and reflexive. The main result is the following theorem.

Theorem. *Let* $\mathbf{a}$ *be a sequence in a Banach space* $\mathbf{X}$*,* $\mathcal{L} = \mathcal{L}(\mathbf{a})$*,* $\mathcal{K} = \mathcal{K}(\mathbf{a})$*,* $\mathcal{N} = \mathcal{N}(\mathbf{a})$ *and* $\mathcal{C} = \mathcal{C}(\mathbf{a})$*. The following statements hold:*

(i) *if* $\mathbf{X}$ *is (WBS), then* $\overline{\mathrm{co}}(\mathcal{L}) \subset \mathcal{K} \subset \mathcal{N} \subset \mathcal{C}$*;*
(ii) *if* $\mathbf{X}$ *is (BS) and* $\mathbf{a}$ *is bounded, then* $\overline{\mathrm{co}}(\mathcal{L}) = \mathcal{K} = \mathcal{N} = \mathcal{C}$*.*

Remark 2. Examples at the end of the note show that equalities in (ii) fail if $\mathbf{X}$ is supposed to be (WBS) only or if unbounded sequences are considered.

Remark 3. Obviously, in case $\mathbf{X} = \mathbf{R}$ Theorem includes Proposition 1 stated above. (Indeed, in this case for a bounded sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ of real numbers $\overline{\mathrm{co}}(\mathcal{L}(\mathbf{a}))$ is the closed interval with endpoints $\liminf a_n$ and $\limsup a_n$.)

2. Proof of Theorem

Our proof of Theorem is based on several auxiliary results. Recall that a Banach space $\mathbf{X}$ is (BS) if and only if $\mathbf{X}$ is (WBS) and reflexive. Then, by the following fact, it is sufficient to prove only the first part of Theorem.

Fact. *If* $\mathbf{X}$ *is a Banach space, then the following statements are equivalent to each other:*

(i) *the space* $\mathbf{X}$ *is reflexive;*
(ii) *each bounded sequence* $\mathbf{a}$ *in* $\mathbf{X}$ *satisfies* $\mathcal{L}(\mathbf{a}) \neq \emptyset$*;*
(iii) *each bounded sequence* $\mathbf{a}$ *in* $\mathbf{X}$ *satisfies* $\mathcal{C}(\mathbf{a}) \neq \emptyset$*;*
(iv) *each bounded sequence* $\mathbf{a}$ *in* $\mathbf{X}$ *satisfies* $\overline{\mathrm{co}}(\mathcal{L}(\mathbf{a})) = \mathcal{C}(\mathbf{a})$*.*

Proof: By the Eberlein-Šmulian theorem [4, p. 18], (i) is equivalent to (ii). Since $\mathcal{L}(\mathbf{a}) \subset \mathcal{C}(\mathbf{a})$ for each sequence $\mathbf{a}$ in $\mathbf{X}$, (ii) implies (iii).

Now we show that (i) follows from (iii) by James' theorem ([7]) according to which a Banach space $\mathbf{X}$ is reflexive if and only if each element of the dual $\mathbf{X}^*$ attains its norm on the closed unit ball B in $\mathbf{X}$. Indeed, for a given $f \in \mathbf{X}^*$ with $\|f\| = 1$ there is a sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ in B such that $\lim_{n\to\infty} f(a_n) = 1$. Putting

$$c_N = \inf\{f(x);\ x \in \overline{\mathrm{co}}(\{a_n\}_{n\geq N}),\ N \in \mathbf{N},$$

we get $\lim_{N\to\infty} c_N = 1$. Denoting $\mathcal{C} = \mathcal{C}(\mathbf{a})$, we have

$$c_N \leq \inf f(\mathcal{C}) \leq \sup f(\mathcal{C}) \leq 1,\ N \in \mathbf{N}$$

and, as a result, $f = 1$ on $\mathcal{C}$. By (iii), $\mathcal{C} \neq \emptyset$ so that f attains its norm on B.

Now we prove that (i) implies (iv). Let $\mathbf{X}$ be reflexive and $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbf{X}$. Denote $\mathcal{L} = \mathcal{L}(\mathbf{a})$ and $\mathcal{C} = \mathcal{C}(\mathbf{a})$. Assume on the contrary that there is $a \in \mathcal{C} \setminus \overline{\mathrm{co}}(\mathcal{L})$. By a geometric version of the Hahn-Banach theorem ([2, p. 111]), there is $f \in \mathbf{X}^*$ and $\alpha \in \mathbf{R}$ such that

$$f(a) > \alpha > \sup\{f(x);\ x \in \overline{\mathrm{co}}(\mathcal{L})\}.$$

Now we are going to show that the half-space $H = \{x \in \mathbf{X};\ f(x) \geq \alpha\}$ intersects $\mathcal{L}$, which is impossible. Because of the boundedness of $\mathbf{a}$ and the reflexivity of $\mathbf{X}$ it is sufficient to show that $\{n \in \mathbf{N};\ a_n \in H\}$ is infinite. For each $N \in \mathbf{N}$ we have $a \in \mathcal{C}_N = \overline{\mathrm{co}}(\{a_n\}_{n \geq N})$ and, as a result, there is $n \geq N$ such that $a_n \in H$. Otherwise, we would get $f(a) \leq \alpha$, which is a contradiction; cf. [1, p. 139].

Finally, let $\mathbf{X}$ be a non-reflexive Banach space. Of course, there is a bounded sequence $\mathbf{b} = \{b_n\}_{n=1}^{\infty}$ in $\mathbf{X}$ having no weakly convergent subsequences, i.e., with $\mathcal{L}(\mathbf{b}) = \emptyset$. Then the bounded sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ in $\mathbf{X}$, defined by $a_{2n-1} = b_n$ and $a_{2n} = -b_n$, $n \in \mathbf{N}$, satisfies $\mathcal{L}(\mathbf{a}) = \emptyset$ and $(C,1)\text{-}\lim_{n\to\infty} a_n = 0$, which shows that (iv) implies (i). $\square$

Remark 4. Sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in a Banach space $\mathbf{X}$ are said to be *tail equivalent* provided that there are integers k and l such that $a_{k+n} = b_{l+n}$, $n \in \mathbf{N}$. For such sequences it is easy to check that $(C,1)\text{-}\lim_{n\to\infty} a_n = L$ if and only if $(C,1)\text{-}\lim_{n\to\infty} b_n = L$.

Lemma 1. *If $\mathbf{a}$ is a sequence in a Banach space $\mathbf{X}$, then $\mathcal{N}(\mathbf{a}) \subset \mathcal{C}(\mathbf{a})$.*

Proof: Let $\rho \in \mathcal{P}$, $\{a_n\}_{n=1}^{\infty} \subset \mathbf{X}$ and $a \in \mathbf{X}$ be such that $(C,1)\text{-}\lim_{n\to\infty} a_{\rho(n)} = a$. Given $N \in \mathbf{N}$, we have to show that $a \in \mathcal{C}_N = \overline{\mathrm{co}}(\{a_n\}_{n\geq N})$. Choose an $m \in \mathbf{N}$ such that $\{1,\dots,N-1\} \subset \rho(\{1,\dots,m\})$. Then, by Remark 4, $a = (C,1)\text{-}\lim_{n\to\infty} a_{\rho(n+m)} \in \mathcal{C}_N$. $\square$

Lemma 2. *If $\mathbf{a}$ is a sequence in $\mathbf{X}$, then $\mathcal{K}(\mathbf{a}) \subset \mathcal{N}(\mathbf{a})$.*

Proof: It is sufficient to prove that if $\{a_n\}_{n=1}^{\infty} \subset \mathbf{X}$ and $p \in \mathcal{I}$ are such that $(C,1)\text{-}\lim_{n\to\infty} a_{p(n)} = 0$, then there is $\rho \in \mathcal{P}$ with $(C,1)\text{-}\lim_{n\to\infty} a_{\rho(n)} = 0$. In addition, we may assume that $\mathbf{N} \setminus p(\mathbf{N})$ is infinite because $(C,1)$-limits of two tail equivalent sequences are the same. Thus there is a unique $q \in \mathcal{I}$ with $q(\mathbf{N}) = \mathbf{N} \setminus p(\mathbf{N})$. Denote by $\|\,.\,\|$ the norm on $\mathbf{X}$. We claim that there is $r \in \mathcal{I}$ such that, for each $m \in \mathbf{N}$,

$$\begin{aligned} &\left\| \frac{1}{n+m}\left(\sum_{j=1}^{m} a_{q(j)} + \sum_{j=1}^{n} a_{p(j)}\right)\right\| \\ &\leq \frac{1}{n+m}\left\|\sum_{j=1}^{m} a_{q(j)}\right\| + \frac{n}{n+m}\left\|\frac{1}{n}\sum_{j=1}^{n} a_{p(j)}\right\| < \frac{1}{m},\ n \geq r(m). \end{aligned} \tag{2}$$

Indeed, for a given $m \in \mathbf{N}$ the middle term of (2) tends to 0 as $n \to \infty$. Set $r(0) = 0$ and, for each $m \in \mathbf{N}$, $\rho(r(m)+m) = q(m)$ and

$$\rho(j) = p(j-m+1),\ \ r(m-1)+m-1 < j < r(m)+m.$$

Then $\rho \in \mathcal{P}$ and, by (2), $(C,1)\text{-}\lim_{n\to\infty} a_{\rho(n)} = 0$. $\square$

In order to prove Theorem we have to show only the first inclusion in (i). To do this, we use a deep result concerning Banach spaces with the weak Banach-Saks property.

Definition. *A sequence* $\{a_n\}_{n=1}^{\infty}$ *in* $\mathbf{X}$ *is said to be stable provided that there is* $a \in \mathbf{X}$ *such that*

$$\lim_{n\to\infty} \sup \left\{ \left\| \frac{1}{n} \sum_{j=1}^{n} a_{r(j)} - a \right\| ;\ r \in \mathcal{I} \right\} = 0,$$

i.e., $(C,1)\text{-}\lim_{n\to\infty} a_{r(n)} = a$ *uniformly with respect to* $r \in \mathcal{I}$.

Then it holds that $\mathbf{X}$ is (WBS) if and only if each weakly convergent sequence in $\mathbf{X}$ admits even a stable subsequence, see [6] and cf. also [5].

Let $\mathbf{Q}$ be the set of rational numbers. For $M \subset \mathbf{X}$ denote by $\mathrm{co}_{\mathbf{Q}}(M)$ the set of all convex combinations of points of M with rational coefficients.

Lemma 3. *Let* $\mathbf{X}$ *be (WBS),* $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathbf{X}$ *and* $a \in \mathrm{co}_{\mathbf{Q}}(\mathcal{L}(\mathbf{a}))$. *Then there is a sequence* $\{\omega_n\}_{n=1}^{\infty}$ *of positive numbers tending to* 0 *such that, for each* $r_0 \in \mathbf{N}$, *there is* $r \in \mathcal{I}$ *with* $r(1) > r_0$ *and*

$$\left\| \frac{1}{n} \sum_{j=1}^{n} a_{r(j)} - a \right\| \le \omega_n,\ n \in \mathbf{N}$$

(*in particular,* $(C,1)\text{-}\lim_{n\to\infty} a_{r(n)} = a$).

Proof: Denote $\mathcal{L} = \mathcal{L}(\mathbf{a})$. Given $a \in \mathrm{co}_{\mathbf{Q}}(\mathcal{L})$, there are $m \in \mathbf{N}$, $\{\alpha_1, \dots, \alpha_m\} \subset \mathbf{Q} \cap (0,1]$ and $\{c_1, \dots, c_m\} \subset \mathcal{L}$ such that

$$a = \sum_{j=1}^{m} \alpha_j c_j \quad \text{and} \quad 1 = \sum_{j=1}^{m} \alpha_j.$$

Of course, there are $\{k_1, \dots, k_m\} \subset \mathbf{N}$ and $l \in \mathbf{N}$ such that $\alpha_j = k_j/l$, $j = 1, \dots, m$. Let us remark that $l = k_1 + \dots + k_m$. For each $j = 1, \dots, m$, there is $p_j \in \mathcal{I}$ such that $w\text{-}\lim_{n\to\infty} a_{p_j(n)} = c_j$ (and, in particular, $\{a_{p_j(n)}\}_{n=1}^{\infty}$ is bounded) and because $\mathbf{X}$ is (WBS) we may choose p_j such that $\{a_{p_j(n)}\}_{n=1}^{\infty}$ is in addition stable, i.e., the numbers

$$\omega_n^j = \sup \left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} a_{p_j(r(i))} - c_j \right\| ;\ r \in \mathcal{I} \right\} \tag{3}$$

tend to 0 as $n \to \infty$. Let

$$K = \max\{ \sup_{n \in \mathbf{N}} \|a_{p_j(n)}\|;\ j = 1, \dots, m\} < +\infty.$$

For each $h \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and $t \in \{0, \dots, l-1\}$, put $\omega_{hl+t}^0 = 2K(l-1)/(hl+1)$ and

$$\omega_{hl+t} = \sum_{j=1}^{m} \alpha_j \omega_{hk_j}^j + \omega_{hl+t}^0 \tag{4}$$

where $\omega_0^j = 0$, $j = 1, \dots, m$. Of course, $\lim_{n\to\infty} \omega_n = 0$.

Now, for a given $r_0 \in \mathbf{N}$, we construct a desired $r \in \mathcal{I}$ by induction on h: Put $r(0) = r_0$ and $r_j(0) = 0$, $j = 1, \dots, m$. Assume that, for some $h \in \mathbf{N}_0$, we have defined integers

$$r_0 = r(0) < r(1) < \cdots < r(hl) \quad \text{and} \quad 0 = r_j(0) < r_j(1) < \cdots < r_j(hk_j),\ j = 1, \dots, m.$$

Then there is $u \in \mathbf{N}$ such that $r_1(hk_1) < u$ and $r(hl) < p_1(u)$. For $t = 1, \dots, k_1$, put

$$r_1(hk_1 + t) = u + t - 1 \quad \text{and} \quad r(hl + t) = p_1(r_1(hk_1 + t)).$$

Next assume that, for $1 \le j \le m-1$, we have defined

$$r_0 = r(0) < \cdots < r(hl + \sum_{i=1}^{j} k_i) \quad \text{and} \quad 0 = r_i(0) < \cdots < r_i((h+1)k_i),\ i \le j.$$

Then there is $v \in \mathbf{N}$ such that $r_{j+1}(hk_{j+1}) < v$ and $r(hl + \sum_{i=1}^{j} k_i) < p_{j+1}(v)$. For $t = 1, \dots, k_{j+1}$, put

$$(5)\quad r_{j+1}(hk_{j+1} + t) = v + t - 1 \quad \text{and} \quad r(hl + \sum_{i=1}^{j} k_i + t) = p_{j+1}(r_{j+1}(hk_{j+1} + t)).$$

So we get

$$r_0 = r(0) < \cdots < r((h+1)l) \quad \text{and} \quad 0 = r_j(0) < \cdots < r_j((h+1)k_j),\ j = 1, \dots, m.$$

Thus we have constructed $r \in \mathcal{I}$ and $r_j \in \mathcal{I}$, $j = 1, \dots, m$ which satisfy (5). Denote for each $j = 1, \dots, m$

$$s_n^j = \frac{1}{n} \sum_{k=1}^{n} a_{p_j(r_j(k))} \quad \text{and} \quad s_n = \frac{1}{n} \sum_{k=1}^{n} a_{r(k)},\ n \in \mathbf{N}.$$

Given $n \in \mathbf{N}$, there are unique $h \in \mathbf{N}_0$ and $t \in \{0, \dots, l-1\}$ such that $n = hl + t$. Putting $s_0 = a$, it is easily seen that $\|s_m - s_{m-1}\| \le 2K/m$, $m \in \mathbf{N}$. Therefore

$$\|s_{hl+t} - s_{hl}\| \le \sum_{j=1}^{t} \|s_{hl+j} - s_{hl+j-1}\| \le \frac{2K}{hl+1}(l-1) = \omega_{hl+t}^0.$$

If $h > 0$ then, by (5),

$$s_{hl} = \frac{1}{hl} \sum_{j=1}^{m} hk_j s_{hk_j}^j = \sum_{j=1}^{m} \alpha_j s_{hk_j}^j$$

and hence, by (3),

$$\|s_{hl} - a\| \le \sum_{j=1}^{m} \alpha_j \|s_{hk_j}^j - c_j\| \le \sum_{j=1}^{m} \alpha_j \omega_{hk_j}^j.$$

Finally, by (4), we get $\|s_n - a\| \le \omega_n$, which completes the proof. $\square$

Lemma 4. *If* $\mathbf{X}$ *is (WBS) and* $\mathbf{a}$ *is a sequence in* $\mathbf{X}$, *then* $\overline{\mathrm{co}}(\mathcal{L}(\mathbf{a})) \subset \mathcal{K}(\mathbf{a})$.

Proof: Let $\mathbf{a} = \{a_n\}_{n=1}^{\infty} \subset \mathbf{X}$ and $\mathcal{L} = \mathcal{L}(\mathbf{a})$. Given $a \in \overline{\mathrm{co}}(\mathcal{L})$, there are $a^N \in \mathrm{co}_{\mathbf{Q}}(\mathcal{L})$, $N \in \mathbf{N}$ such that $\lim_{N\to\infty} a^N = a$. We may assume that $a = 0$ and $\|a^N\| < \frac{1}{N}$, $N \in \mathbf{N}$. By induction we construct $r \in \mathcal{I}$ such that $(C,1)\text{-}\lim_{n\to\infty} a_{r(n)} = 0$. Of course, by Lemma 3, there is $n_1 \in \mathbf{N}$ and $r_1 \in \mathcal{I}$ such that

$$\Delta_1 = \sup_{n\geq n_1} \left\| \frac{1}{n} \sum_{j=1}^{n} a_{r_1(j)} \right\| < 1.$$

Put $l_1 = n_1$ and $r(j) = r_1(j)$ for each $j \leq n_1$. Assume that for some $N \in \mathbf{N}$ we have $n_N \in \mathbf{N}$, $l_N \in \mathbf{N}$, $r_N \in \mathcal{I}$ and $r(1) < r(2) < \cdots < r(n_N)$ such that $r(n_N) < r_N(l_N+1)$ and

$$\Delta_N = \sup_{n\in\mathbf{N}_0} \left\| \frac{1}{n_N+n} \left(\sum_{j=1}^{n_N} a_{r(j)} + \sum_{j=l_N+1}^{l_N+n} a_{r_N(j)} \right) \right\| < \frac{1}{N}. \tag{6}$$

By Lemma 3, for $a^{N+1} \in \mathrm{co}_{\mathbf{Q}}(\mathcal{L})$ there is a sequence $\{\omega_n^{N+1}\}_{n=1}^{\infty}$ of positive numbers tending to 0 such that for each $r_0 \in \mathbf{N}$ there is $p \in \mathcal{I}$ with $p(1) > r_0$ and

$$\left\| \frac{1}{n} \sum_{j=1}^{n} a_{p(j)} - a^{N+1} \right\| \leq \omega_n^{N+1}, \ n \in \mathbf{N}. \tag{7}$$

Of course, there is $n_0 \in \mathbf{N}$ such that $\omega_n^{N+1} + \|a^{N+1}\| \leq \frac{1}{N}$, $n > n_0$. Choose $m \in \mathbf{N}$ such that, denoting $K = \max\{\omega_n^{N+1} + \|a^{N+1}\|;\ n = 1, \dots, n_0\}$,

$$\Delta_N + \frac{n_0}{n_N+m+n_0} K \leq \frac{1}{N}. \tag{8}$$

Put $r(n_N+j) = r_N(l_N+j)$ for $j = 1, \dots, m$. Thus there is $r_{N+1} \in \mathcal{I}$ such that $r_{N+1}(1) > r(n_N+m)$ and

$$\left\| \frac{1}{n} \sum_{j=1}^{n} a_{r_{N+1}(j)} - a^{N+1} \right\| \leq \omega_n^{N+1}, \ n \in \mathbf{N}. \tag{9}$$

For all $n \in \mathbf{N}_0$,

$$\left\| \frac{1}{n_N+m+n} \left(\sum_{j=1}^{n_N+m} a_{r(j)} + \sum_{j=1}^{n} a_{r_{N+1}(j)} \right) \right\| \leq \frac{1}{N}. \tag{10}$$

Indeed, for all $n \in \mathbf{N}$,

$$\left\| \frac{1}{n} \sum_{j=1}^{n} a_{r_{N+1}(j)} \right\| \leq \omega_n^{N+1} + \|a^{N+1}\|, \tag{11}$$

which is not greater than $1/N$ if $n > n_0$ and than K if $n \le n_0$. Furthermore, for a given $n \in \mathbf{N}_0$ the term on the left-hand side of (10) is not greater than

$$\frac{n_N+m}{n_N+m+n}\left\|\frac{1}{n_N+m}\sum_{j=1}^{n_N+m} a_{r(j)}\right\| + \frac{n}{n_N+m+n}\left\|\frac{1}{n}\sum_{j=1}^{n} a_{r_{N+1}(j)}\right\|. \tag{12}$$

In case $n > n_0$, by (6) and (11), (12) is a convex combination of two numbers not greater than $\frac{1}{N}$ and hence (10) is true. This holds also in case $n \le n_0$ by (8). Finally, there is $l_{N+1} \in \mathbf{N}$ such that, denoting $n_{N+1} = n_N + m + l_{N+1}$ and $r(n_N + m + j) = r_{N+1}(j)$ for $j = 1, \dots, l_{N+1}$,

$$\Delta_{N+1} = \sup_{n\in\mathbf{N}_0}\left\|\frac{1}{n_{N+1}+n}\left(\sum_{j=1}^{n_{N+1}} a_{r(j)} + \sum_{j=l_{N+1}+1}^{l_{N+1}+n} a_{r_{N+1}(j)}\right)\right\| < \frac{1}{N+1}\,.$$

In particular, by (6) and (10),

$$\left\|\frac{1}{n}\sum_{j=1}^{n} a_{r(j)}\right\| \le \frac{1}{N}$$

for each $n_N \le n < n_{N+1}$. Thus $(C,1)\text{-}\lim_{n\to\infty} a_{r(n)} = 0$. □

The proof of Theorem follows now immediately from Lemmas 1, 2, 4 and Fact.

3. Counterexamples and remarks

We begin with several examples showing that some of assertions of Theorem cannot be generalized.

Example 1. There is a sequence $\mathbf{a}$ of real numbers such that $\overline{\mathrm{co}}(\mathcal{L}(\mathbf{a})) \ne \mathcal{K}(\mathbf{a})$. In fact, given $\{b_n\}_{n=1}^{\infty} \subset \mathbf{R}$ such that $\lim_{n\to\infty} b_n = +\infty$ and $\lim_{n\to\infty} b_n/n = 0$, it is easy to show that the sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, defined by

$$a_{2n-1} = b_n \quad\text{and}\quad a_{2n} = -b_n,\ n \in \mathbf{N},$$

satisfies that $\mathcal{L}(\mathbf{a}) = \emptyset$ and $(C,1)\text{-}\lim_{n\to\infty} a_n = 0$ and so $0 \in \mathcal{K}(\mathbf{a})$.

Remark 5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in a Banach space $\mathbf{X}$ and $s_n = \frac{1}{n}\sum_{j=1}^{n} a_j$, $n \in \mathbf{N}$. Then it is easy to see that $\lim_{n\to\infty}(s_n - s_{n-1}) = 0$ if and only if $\lim_{n\to\infty} a_n/n = 0$.

Example 2. There is a sequence $\mathbf{a}$ of real numbers such that $\mathcal{K}(\mathbf{a}) \ne \mathcal{N}(\mathbf{a})$. In fact, define $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ by $a_{2n-1} = 0$ and $a_{2n} = n$, $n \in \mathbf{N}$. Then $\mathcal{K}(\mathbf{a}) = \{0\}$. Indeed, if $p \in \mathcal{I}$ and $\{a_{p(n)}\}_{n=1}^{\infty}$ is $(C,1)$-convergent, then $a_{p(n)} = 0$ except for a finite number of values of n. Otherwise, $a_{p(n)}/n \geq a_{p(n)}/p(n) = 1/2$ for an infinite number of values of n and hence it does not hold that

$$\lim_{n\to\infty} \frac{a_{p(n)}}{n} = 0$$

which is impossible by Remark 5. Now we show $\mathcal{N}(\mathbf{a}) = [0, +\infty)$. Obviously, $\mathcal{N}(\mathbf{a}) \subset [0, +\infty)$. Let $a \in [0, +\infty)$. Then there is $r = r_a \in \mathcal{I}$ such that $\lim_{n\to\infty}(r(n) - n) = +\infty$, $\lim_{n\to\infty} n^2/2r(n) = a$ and $r(n) \geq n$, $n \in \mathbf{N}$. Indeed, denoting by $[x]$ the greatest integer less or equal to a real number x, put $r_0(n) = n^3$, $n \in \mathbf{N}$ and $r_a(n) = [n^2/2a] + n$, $n \in \mathbf{N}$ if $a \in (0, +\infty)$. Then the sequence $\{b_n\}_{n=1}^{\infty}$, defined by $b_{r(n)} = n$, $n \in \mathbf{N}$ and $b_n = 0$ otherwise, is a permutation of the sequence $\{a_n\}_{n=1}^{\infty}$. Denote

$$s_n = \frac{1}{n}\sum_{j=1}^{n} b_j, \ n \in \mathbf{N}.$$

For each $n \in \mathbf{N}$ and $k \in \mathbf{N}$, $r(n) \leq k < r(n+1)$, we have

$$\frac{n(n+1)}{2r(n+1)} \leq s_k = \frac{n(n+1)}{2k} \leq \frac{n(n+1)}{2r(n)}\,.$$

Thus $\lim_{n\to\infty} s_n = a$ and hence $a \in \mathcal{N}(\mathbf{a})$.

Example 3. There is a sequence $\mathbf{a}$ of real numbers such that

$$(\emptyset =)\mathcal{N}(\mathbf{a}) \neq \mathcal{C}(\mathbf{a})(= \mathbf{R}).$$

In fact, define $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ by $a_n = (-1)^n n$, $n \in \mathbf{N}$. By definition, $\mathcal{C}(\mathbf{a}) = \mathbf{R}$. Now we show that $\mathcal{N}(\mathbf{a}) = \emptyset$. Assume on the contrary that there is $\rho \in \mathcal{P}$ such that $\{a_{\rho(n)}\}_{n=1}^{\infty}$ is $(C,1)$-convergent. Obviously, for an infinite number of values of n, $\rho(n) \geq n$, and hence

$$\frac{|a_{\rho(n)}|}{n} \geq \frac{|a_{\rho(n)}|}{\rho(n)} = 1,$$

which is impossible by Remark 5.

Let us note that the Proposition 1 stated at the beginning of the note follows easily from

Proposition 2. *Let* $\{b_n\}_{n=1}^{\infty}$ *and* $\{c_n\}_{n=1}^{\infty}$ *be subsequences of a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers with*

$$b = \lim_{n\to\infty} b_n < \lim_{n\to\infty} c_n = c.$$

If $b = -\infty$ *or* $c = +\infty$, *then assume that* $\lim_{n\to\infty} b_n/n = 0$ *or* $\lim_{n\to\infty} c_n/n = 0$, *respectively. Then for each* $a \in [b, c]$ *there is a permutation* ρ *of* $\mathbf{N}$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} a_{\rho(j)} = a.$$

Let us remark that there is a natural proof of Proposition 2 which is quite similar to that of the Riemann derangement theorem which tells us that a conditionally convergent series of real numbers can be rearranged to sum to any real number.

We end with open problems. It would be interesting

(1) to find a bounded sequence $\mathbf{a}$ in a Banach space $\mathbf{X}$ enjoying (WBS) such that $\mathcal{K}(\mathbf{a}) \neq \mathcal{N}(\mathbf{a})$ or $\mathcal{N}(\mathbf{a}) \neq \mathcal{C}(\mathbf{a})$;

(2) to characterize Banach spaces $\mathbf{X}$ in which $\mathcal{K}(\mathbf{a})$ and $\mathcal{N}(\mathbf{a})$ are closed and convex for any bounded sequence $\mathbf{a}$ in $\mathbf{X}$. Let us remark that, by Theorem, this is satisfied in any $\mathbf{X}$ with (BS).

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