Strongly sequential spaces

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Abstract. The problem of Y. Tanaka [10] of characterizing the topologies whose products with each first-countable space are sequential, is solved. The spaces that answer the problem are called strongly sequential spaces in analogy to strongly Fréchet spaces.

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Introduction

In 1976 Y. Tanaka investigated in [10] the problem of characterizing the topologies whose product with every first-countable topology is sequential ([1]). He obtained some necessary conditions and some sufficient conditions, but he established a characterization only for Fréchet topologies. His result reads as follows.

Theorem 0.1 ([10, Theorem 1.1]). *Let X be a Fréchet topology, or a sequential topology each of whose points is a G_δ set. Let Y be first-countable. Then $X \times Y$ is sequential if and only if X is strongly Fréchet or Y is locally countably compact.*

I present here an analogous result with neither the assumptions of Fréchetness nor of separation (Theorem 5.1). The solution is based on the extension of the problem to the setting of convergences in which I get a characterization. The solution of the problem of Y. Tanaka appears as a particularly eloquent application of general methods of continuous duality developed in [8].

The problem of Tanaka can be decomposed into two parts:

Problem 0.2. Characterize topologies (or convergences) ξ such that $\xi \times \tau$ is sequential for every first-countable topology (convergence) τ.

Problem 0.3. Characterize couples of topologies (convergences) (ξ, τ) such that τ is first-countable and $\xi \times \tau$ is sequential.

Problem 0.2 is related to the classical theorem [7, Theorem 4.2] of Michael that states that a Hausdorff regular topology ξ is locally countably compact if and only if its product with every sequential topology is sequential. Indeed, regular Hausdorff topologies whose products with each Fréchet topology are sequential

[1] In [10], all topologies are T_1 and regular.

are also exactly locally countably compact ones (see [4, Theorem 12.2]), but no answer of similar type to Problem 0.2 was known.

In this paper I introduce the class of *strongly sequential* convergences, that answers Problem 0.2 in the setting of convergences. I give several characterizations of this class of convergences. Strongly sequential convergences also provide a full answer to Problem 0.3 in the case of first-countable regular T_1 topologies τ. Their relationship to sequential spaces is analogous to that of strongly Fréchet spaces with respect to general Fréchet spaces.

1. Convergences

A *convergence* ξ on a set X is a relation between X and the filters on X, denoted by $x \in \lim_\xi \mathcal{F}$ whenever x and $\mathcal{F}$ are in relation, such that $x \in \lim_\xi (x)$ for each fixed ultrafilter (x) and such that $\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}$ if $\mathcal{F} \subset \mathcal{G}$.

I denote by $|\xi|$ the underlying set of the convergence ξ. A convergence ξ is *finer* than a convergence ϑ ($\xi \geq \vartheta$) whenever $\lim_\xi \mathcal{F} \subset \lim_\vartheta \mathcal{F}$ for every filter $\mathcal{F}$. A map $f : |\xi| \to |\tau|$ is *continuous* if $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$; this implies the definitions of initial and final convergences, hence of product, sum, subspace and so on. If $f : |\xi| \to |\tau|$, then I will denote by f^- the inverse relation of f and by $f^-\tau$ the initial convergence with respect to f and τ.

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X are said to *mesh*, in symbol $\mathcal{A}\#\mathcal{B}$, whenever $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A subset A of X is ξ-*closed* whenever $\lim_\xi \mathcal{F} \subset A$ for every filter $\mathcal{F}$ with $A\#\mathcal{F}$. The set of all ξ-closed sets gives rise to a topology, called *topological modification of* ξ and denoted $T\xi$. The map T is a concrete reflector called the *topologizer*. Let $\mathcal{F}$ be a filter on a convergence space X. The *adherence* of $\mathcal{F}$ is the union of the limits of all filters that are finer than $\mathcal{F}$:

$$\operatorname{adh}_\xi \mathcal{F} = \bigcup_{\mathcal{G} \supset \mathcal{F}} \lim_\xi \mathcal{G}.$$

In particular, the *adherence* $\operatorname{adh}_\xi A$ of a set A is the adherence of its principal filter, while the *closure* $\operatorname{cl}_\xi A$ of A is the (idempotent) adherence of A for $T\xi$. There are various ways to characterize the topologizer. For example,

$$\lim_{T\xi} \mathcal{F} = \bigcap_{C \# \mathcal{F}} \operatorname{cl}_\xi C. \tag{1.1}$$

For each point x, the neighborhood filter for $T\xi$ is denoted by $\mathcal{N}_\xi(x)$. Continuous maps from a convergence ξ to the Sierpiński topology \$ ([2]) are precisely the indicator functions of ξ-closed sets ([3]). Therefore

$$T\xi = \bigvee_{f \in C(\xi,\$)} f^-\$, \tag{1.2}$$

[2]that is, the two point set $\{0, 1\}$ in which 1 is isolated while 0 is not.
[3]The indicator function of A takes the value 0 on A and 1 on A^c.

where $C(\xi, \$)$ denotes the set of continuous maps from ξ to $\$$.

A convergence ξ is a pretopology whenever $x \in \lim_\xi \mathcal{F}$ provided $x \in \mathrm{adh}_\xi A$ for each $A \# \mathcal{F}$. The map P assigning to each convergence ξ the finest pretopology coarser than ξ is a concrete reflector.

$$\lim_{P\xi} \mathcal{F} = \bigcap_{A \# \mathcal{F}} \mathrm{adh}_\xi A. \tag{1.3}$$

For each point x, the infimum of all filters that ξ-converge to x is a $P\xi$-convergent filter called *vicinity filter* of x and denoted $\mathcal{V}_\xi(x)$. The pretopologizer may be also characterized via initial images of a single pretopology Λ ([4]).

$$P\xi = \bigvee_{f \in C(\xi, \Lambda)} f^- \Lambda. \tag{1.4}$$

A class $\mathfrak{J}$ of filters is said to be *composable* if it contains the class of principal filters and if $\mathcal{H}\mathcal{F}$ ([5]) is a (possibly degenerate) $\mathfrak{J}$-filter on Y for each $\mathfrak{J}$-filter $\mathcal{F}$ on X, each set Y and each $\mathfrak{J}$-filter $\mathcal{H}$ on $X \times Y$. In particular, the image of a $\mathfrak{J}$-filter under a relation (identified with its principal filter) is a $\mathfrak{J}$-filter. For example the classes of countably based filters and of principal filters are composable while the class of filters generated by sequences is not.

If $\mathfrak{J}$ is a class of filters, the coreflector $\mathrm{Base}_\mathfrak{J}$ on $\mathfrak{J}$-based convergences is defined by

$$x \in \lim_{\mathrm{Base}_\mathfrak{J} \xi} \mathcal{F} \iff \underset{\mathcal{G} \le \mathcal{F}, \mathcal{G} \in \mathfrak{J}}{\exists} x \in \lim_\xi \mathcal{G}. \tag{1.5}$$

The coreflector on countably based convergences is denoted First, while the coreflector on convergences based in filters generated by sequences is denoted Seq. Extending the notion of sequential topology, a convergence is said to be *sequential* if every sequentially closed set is closed, that is, if $T\xi = T\,\mathrm{Seq}\,\xi$, or equivalently if

$$\xi \ge T\,\mathrm{First}\,\xi. \tag{1.6}$$

Hence, a convergence ξ solves Problem 0.2 if and only if $\xi \times \tau \ge T\,\mathrm{First}(\xi \times \tau)$ for every $\tau = \mathrm{First}\,\tau$, equivalently, if and only if

$$\xi \times \tau \ge T(\mathrm{First}\,\xi \times \tau) \tag{1.7}$$

for every convergence $\tau = \mathrm{First}\,\tau$.

In the next sections, I characterize such convergences both internally and in terms of product properties.

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[4]The underlying set of Λ is the three point set $\{0,1,2\}$ endowed with the following pretopology: $\mathcal{V}(0) = \{\Lambda\}$, $\mathcal{V}(1) = \{\Lambda\}$, $\mathcal{V}(2) = \big\{\{0,1,2\},\ \{1,2\}\big\}$. See [1, II.2] for details.

[5]$HF = \{y : \exists_{x \in F} (x,y) \in H\}$ and $\mathcal{H}\mathcal{F}$ is the filter generated by $\{HF : H \in \mathcal{H}, F \in \mathcal{F}\}$.

2. Countably Antoine convergences

As indicated in the introduction, continuous duality plays a crucial role in the characterization of strongly sequential spaces. Given two convergences ξ and σ, the *continuous convergence* $\sigma[\xi]$ on the set of continuous mappings from ξ to σ is the coarsest convergence on $|\sigma[\xi]|$ that makes the evaluation map $w : \xi \times \sigma[\xi] \to \sigma$ defined by $w(x, f) = f(x)$ continuous. The reason why continuous convergence appears naturally in many problems that involve products is the exponential law:

$$\sigma[\xi \times \tau] = \sigma[\xi][\tau], \tag{2.1}$$

for every convergences ξ, τ, σ. Here the equality means the homeomorphism via the *transposition map* ${}^t : |\sigma[\xi \times \tau]| \to |\sigma[\xi][\tau]|$ defined by ${}^t f(y)(x) = f(x, y)$. In this paper I am primarily concerned with the duality between ξ and $\$[\xi]$. Given a filter $\mathcal{G}$ on $|\$[\xi]|$, denote by

$$|\mathcal{G}| = \{ \bigcup_{A \in G} A : G \in \mathcal{G}\} \tag{2.2}$$

the *reduced filter* of $\mathcal{G}$. It follows from the definition that a filter $\mathcal{G}$ converges to A_0 for $\$[\xi]$ (in symbols, $A_0 \in \lim_{\$[\xi]} \mathcal{G}$) if and only if

$$\mathrm{adh}_{\xi} |\mathcal{G}| \subset A_0. \tag{2.3}$$

Recall that a convergence ξ is said to be *Antoine* ([1]) whenever $\xi = i^{-}(\$[\$[\xi]])$ where $i : |\xi| \to |\$[\$[\xi]]|$ is the natural injection from ξ to its bidual. More generally, if $\mathfrak{J}$ is a composable class of filters, I call a convergence $\mathfrak{J}$-*Antoine* whenever $\xi = A_{\mathfrak{J}}\xi$, where the reflector $A_{\mathfrak{J}}$ is defined by

$$A_{\mathfrak{J}}\xi = i^{-}(\$[\mathrm{Base}_{\mathfrak{J}}(\$[\xi])]). \tag{2.4}$$

In particular, if $\mathfrak{J}$ stands for the class of countably based filters, then the reflector on $\mathfrak{J}$-Antoine convergences is denoted A_{ω}, and it is denoted A if $\mathfrak{J}$ is the class of all filters.

A convergence is said to be *atomic* if its all but one point are isolated.

Theorem 2.1. *Let $\mathfrak{J}$ be a composable class of filters. The convergence $A_{\mathfrak{J}}\theta$ is the coarsest convergence on $|\theta|$ among the convergences α that fulfill*

$$\alpha \times \tau \geq T(\theta \times \tau), \tag{2.5}$$

for each $\mathfrak{J}$-based convergence τ (equivalently, for each atomic $\mathfrak{J}$-based topology).

Proof: Assume $\alpha \ngeq A_{\mathfrak{J}}\theta$. There exists a filter $\mathcal{F}$ such that $x_0 \in \lim_{\alpha} \mathcal{F}$ but $i(x_0) \notin \lim_{\$[\mathrm{Base}_{\mathfrak{J}}(\$[\theta])]} i(\mathcal{F})$. Consequently, there exists a continuous map f_0 from θ to $\$$ and a $\mathfrak{J}$-filter $\mathcal{G}_0$ such that $f_0 \in \lim_{\$[\theta]} \mathcal{G}_0$ but

$$f_0(x_0) \notin \lim_{\$} w(\mathcal{F} \times \mathcal{G}_0). \tag{2.6}$$

Let τ be the atomic $\mathfrak{J}$-based topology on $|\$[\theta]|$ defined by $\mathcal{N}_\tau(f_0) = \mathcal{G}_0 \wedge (f_0)$. Since τ is finer than $\$[\theta]$, the evaluation $w : \theta \times \tau \to \$$ is continuous. In view of (2.6) and of (1.2), I conclude that $\alpha \times \tau \not\geq T(\theta \times \tau)$, contrary to (2.5).

Conversely, if $(x, y) \in \lim_{A_{\mathfrak{J}}\theta\times\tau}(\mathcal{F} \times \mathcal{G})$ and $f \in C(\theta \times \tau, \$)$, then $f(x, y) \in \lim_\$ f(\mathcal{F}\times\mathcal{G})$. Indeed, $f(\mathcal{F}\times\mathcal{G}) =^t f(\mathcal{G})(\mathcal{F})$ and ${}^t f(\mathcal{G})$ is a $\mathfrak{J}$-filter by composability. Since f is continuous, ${}^t f$ is also continuous so that ${}^t f(y) \in \lim_{\mathrm{Base}_{\mathfrak{J}}\$[\theta]}({}^t f)(\mathcal{G})$. Consequently, $f(x, y) =^t f(y)(x) \in \lim_\$ w(\mathcal{F},{}^t f(\mathcal{G})) = \lim_\$ f(\mathcal{F} \times \mathcal{G})$, by definition of $A_{\mathfrak{J}}$. Hence, (2.5) holds, in view of (1.2).

□

In particular, if $\mathfrak{J}$ is the class of countably based filters, atomic $\mathfrak{J}$-based topologies are metrizable.

Now I give an explicit description of the reflector $A_{\mathfrak{J}}$. By definition,

$$\mathrm{ad}_\xi A = \bigcup_{a\in A} \lim_\xi(a), \quad \mathrm{ad}_{T\xi} A = \bigcup_{a\in A} \mathrm{cl}_\xi\, a.$$

Let $\mathcal{H}_{\mathrm{ad}_{T\xi}}$ denote the filter generated by $\{\mathrm{ad}_{T\xi} H : H \in \mathcal{H}\}$ and let $(\mathfrak{J})_{\mathrm{ad}_{T\xi}}$ denote the class of $\mathfrak{J}$-filters $\mathcal{H}$ for which $\mathcal{H} = \mathcal{H}_{\mathrm{ad}_{T\xi}}$.

Lemma 2.2. *Let $\mathfrak{J}$ be a composable class of filters. A filter on $|\xi|$ is the reduced filter of a $\mathfrak{J}$-filter on $|\$[\xi]|$ if and only if it is a $(\mathfrak{J})_{\mathrm{ad}_{T\xi}}$-filter.*

Proof: If $\mathcal{H}$ is the reduced filter of a $\mathfrak{J}$-filter $\mathcal{G}$, it is a $\mathfrak{J}$-filter because $\mathcal{H}$ is the inverse image of $\mathcal{G}$ under the relation $\{(x, A) \in |\xi| \times |\$[\xi]| : x \in A\}$. Moreover, $\mathcal{H} = \mathcal{H}_{\mathrm{ad}_{T\xi}}$, because a union of closed sets is closed for $\mathrm{ad}_{T\xi}$.

Conversely, if $\mathcal{H} \in (\mathfrak{J})_{\mathrm{ad}_{T\xi}}$ then the filter $\widetilde{\mathcal{H}}$ generated by $\{\mathrm{cl}_\xi\, h : h \in H\}_{H\in\mathcal{H}}$ is the image of $\mathcal{H}$ under the relation $\{(x, \mathrm{cl}_\xi\, x) : x \in |\xi|\}$. Hence $\widetilde{\mathcal{H}}$ is a $\mathfrak{J}$-filter such that $\mathcal{H} = |\widetilde{\mathcal{H}}|$. □

Theorem 2.3. *If $\mathfrak{J}$ is a composable class of filters, then the reflector $A_{\mathfrak{J}}$ is given by*

$$\lim_{A_{\mathfrak{J}}\xi} \mathcal{F} = \bigcap_{(\mathfrak{J})_{\mathrm{ad}_{T\xi}}\ni\mathcal{H}\#\mathcal{F}} \mathrm{cl}_\xi(\mathrm{adh}_\xi\, \mathcal{H}). \tag{2.7}$$

Proof: By definition, $x_0 \in \lim_{A_{\mathfrak{J}}\xi} \mathcal{F}$ if and only if $1 \in \lim_\$ w(\mathcal{F} \times \mathcal{G})$, whenever $\mathcal{G}$ is a $\mathfrak{J}$-filter that $\$[\xi]$-converges to a ξ-closed set A such that $x_0 \notin A$. In view of (2.3), there exists a ξ-closed set A not containing x_0 such that $A \in \lim_{\$[\xi]} \mathcal{G}$ if and only if $x_0 \notin \mathrm{cl}_\xi(\mathrm{adh}_\xi\, |\mathcal{G}|)$. Equivalently, if $\mathcal{G}$ is a $\mathfrak{J}$-filter on $|\$[\xi]|$ such that $|\mathcal{G}|\#\mathcal{F}$, then $x_0 \in \mathrm{cl}_\xi(\mathrm{adh}_\xi\, |\mathcal{G}|)$. Consequently, (2.7) holds, by Lemma 2.2. □

Concerning the behavior of $\mathfrak{J}$-Antoine convergences under product, more can be said than Theorem 2.1 ([6]).

[6]Theorems 2.1 and 2.4 are corollaries of more general results of [8]. I give here proofs for the sake of completeness.

Theorem 2.4. *If $\mathfrak{J}$ is a composable class of filters, then*

$$A_{\mathfrak{J}}\sigma \times A\mathrm{Base}_{\mathfrak{J}}\theta \geq A_{\mathfrak{J}}(\sigma \times \theta). \tag{2.8}$$

Proof: It suffices to prove

$$A_{\mathfrak{J}}\sigma \times \mathrm{Base}_{\mathfrak{J}}\theta \geq A_{\mathfrak{J}}(\sigma \times \theta), \tag{2.9}$$

because it proves that A commutes with finite products ([7]) so that applying A to (2.9) we get (2.8). Let $y \in \lim_{\mathrm{Base}_{\mathfrak{J}}\theta} \mathcal{G}$, $x \in \lim_{A_{\mathfrak{J}}\sigma} \mathcal{F}$ and $h \in \lim_{\mathrm{Base}_{\mathfrak{J}}\$[\sigma\times\theta]} \mathcal{M}$. Denote by $\omega : |(\sigma \times \theta) \times \$[\sigma \times \theta]| \to |\$|$ the evaluation map. I need to show that $h(x, y) \in \lim_{\$} \omega\big((\mathcal{F} \times \mathcal{G}) \times \mathcal{M}\big)$. Without loss of generality I can assume $\mathcal{G}$ and $\mathcal{M}$ to be $\mathfrak{J}$-filters. Thus ${}^t\mathcal{M}$ is a $\mathfrak{J}$-filter and ${}^th \in \lim_{\$[\sigma][\theta]} {}^t\mathcal{M}$, by the exponential law (2.1). Let $\omega_1 : |\theta \times \$[\sigma][\theta]| \to |\$[\sigma]|$ be the evaluation map. This is a continuous map so that ${}^th(y) \in \lim_{\$[\sigma]} \mathcal{H}$, where $\mathcal{H} = \omega_1(\mathcal{G} \times {}^t\mathcal{M})$ is a $\mathfrak{J}$-filter. Since $x \in \lim_{A_{\mathfrak{J}}\sigma} \mathcal{F}$, one has ${}^th(y)(x) \in \lim_{\$} \omega_2(\mathcal{F} \times \mathcal{H}) = \lim_{\$} \omega\big((\mathcal{F} \times \mathcal{G}) \times \mathcal{M}\big)$, where $\omega_2 : \sigma \times \$[\sigma] \to \$$ is the evaluation map. Consequently, $h(x, y) \in \lim_{\$} \omega\big((\mathcal{F} \times \mathcal{G}) \times \mathcal{M}\big)$. □

3. Strongly sequential convergences

Using Theorem 2.1 with $\alpha = \xi$, $\theta = \mathrm{First}\,\xi$ and $\mathrm{Base}_{\mathfrak{J}} = \mathrm{First}$, in view of (1.7), I conclude that a convergence ξ is a solution for Problem 0.2 if and only if

$$\xi \geq A_{\omega}\,\mathrm{First}\,\xi. \tag{3.1}$$

I call such a convergence *strongly sequential.*

Now we are in a position to answer Problem 0.2.

Theorem 3.1. *The following are equivalent:*

1. *ξ is strongly sequential;*
2. *$\mathrm{adh}_{\xi}\,\mathcal{H} \subset \mathrm{cl}_{\mathrm{First}\,\xi}(\mathrm{adh}_{\mathrm{First}\,\xi}\,\mathcal{H})$ for each countably based $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_{\mathrm{ad}_{T\xi}}$;*
3. *$\xi \times \tau$ is sequential for each first-countable convergence τ;*
4. *$\xi \times \tau$ is sequential for each metrizable atomic topology τ;*
5. *$\xi \times \tau$ is strongly sequential for each quasi-bisequential convergence τ.*

A convergence ξ is *quasi-bisequential* whenever $\xi \geq A\,\mathrm{First}\,\xi$. Recall that a topology ξ is *bisequential* if there exists a countably based filter $\mathcal{H}\#\mathcal{F}$ such that $x \in \lim_{\xi} \mathcal{H}$ whenever $x \in \lim_{\xi} \mathcal{F}$ (see [6]). This definition can be extended to convergences via $\xi \geq S\,\mathrm{First}\,\xi$ ([3]), where S denotes the reflector on the category of *pseudotopologies* defined by G. Choquet in [2] ([8]). Since $A\xi = S\xi$

[7] Apply (2.9) two times with $\mathfrak{J}$ the class of all filters.

for each Hausdorff convergence ξ (see for example [1]), quasi-bisequentiality and bisequentiality coincide for Hausdorff convergences.

PROOF: (1) $\iff$ (2) follows immediately from Theorem 2.3.

(1) $\iff$ (3) $\iff$ (4) follows from Theorem 2.1 applied with $\alpha = \xi$, $\theta = \mathrm{First}\,\xi$ and $\mathrm{Base}_{\mathfrak{J}} = \mathrm{First}$.

(5) $\Longrightarrow$ (3) because each strongly sequential convergence is sequential while each first-countable convergence is quasi-bisequential.

(1) $\Longrightarrow$ (5) follows from Theorem 2.4 applied with $\mathfrak{J}$ the class of countably based filters, $\sigma = \mathrm{First}\,\xi$ and $\theta = \mathrm{First}\,\tau$. □

In view of [3, Theorem 5.2], each strongly sequential convergence is a A_ω-quotient ([9]) image of a first countable convergence. Such quotient maps are studied and characterized in [4] and [8]. Each countably biquotient map is A_ω-quotient while each A_ω-quotient map is quotient. More precisely, the following characterizations of A_ω-quotient arise from a combination of [4, Theorems 11.3 and 11.7].

Corollary 3.2. *Let $f : \xi \to \tau$ be a continuous surjection. Then the following are equivalent:*

1. *f is A_ω-quotient;*
2. *if $y \in \lim_\tau \mathcal{F}$, then $\mathcal{F}$ is countably $A_\omega(f\xi)$-compactoid ([10]) in $\mathcal{N}_{f\xi}(y)$;*
3. *if $y \in \lim_\tau \mathcal{F}$, V is a $f\xi$-open set containing y, and $\mathcal{S}$ is a countable ξ-cover of $f^- V$, there exists a finite subfamily $\mathcal{P} \subset \mathcal{S}$ such that the intersection of all τ-open sets containing $\bigcup_{P \in \mathcal{P}} f(P)$ is an element of $\mathcal{F}$;*
4. *$f \times \mathrm{Id}_\theta$ is quotient for each first-countable convergence (equivalently each metrizable atomic topology) θ.*
5. *$f \times g$ is quotient for every A_ω-quotient map g with quasi-bisequential domain ([11]).*

Since each first countable convergence being an almost open image of a Hausdorff metrizable topology,

Theorem 3.3. *A convergence is strongly sequential if and only if it is a A_ω-quotient image of a Hausdorff metrizable topology.*

Notice that $\mathfrak{J}_{\mathrm{ad}_{T\xi}} = \mathfrak{J}$ for each T_1-convergence ξ ([12]), so that a T_1 convergence is strongly sequential if $\mathrm{adh}_\xi \mathcal{H} \subset \mathrm{cl}_{\mathrm{First}\,\xi}(\mathrm{adh}_{\mathrm{First}\,\xi} \mathcal{H})$ for each countably based $\mathcal{H}$. Observe also that $\mathrm{adh}_{\mathrm{First}\,\xi} \mathcal{H} = \mathrm{adh}_{\mathrm{Seq}\,\xi} \mathcal{H}$ for every countably based filter so that

$$\mathrm{cl}_{\mathrm{First}\,\xi}(\mathrm{adh}_{\mathrm{First}\,\xi} \mathcal{H}) = \mathrm{cl}_{\mathrm{Seq}\,\xi}(\mathrm{adh}_{\mathrm{Seq}\,\xi} \mathcal{H}).$$

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[8] $\lim_{S\xi} \mathcal{F} = \bigcap_{\mathcal{U} \in \beta(\mathcal{F})} \lim_\xi \mathcal{U}$, where $\beta(\mathcal{F})$ denotes the set of ultrafilters of $\mathcal{F}$.

[9] That is, a quotient in the category of A_ω-convergences.

[10] In other words, $\mathrm{adh}_{A_\omega(f\xi)} \mathcal{H} \# \mathcal{N}_{f\xi}(y)$ whenever $\mathcal{H}$ is a countably based filter such that $\mathcal{H} \# \mathcal{F}$.

[11] Notice that a A_ω-quotient map is quotient. For quotient maps, $\mathcal{N}_{f\xi}(.) = \mathcal{N}_\tau(.)$ and $f\xi$-open and τ-open sets coincide.

[12] A convergence is T_1 if each point is closed.

Hence, ξ is strongly sequential if and only if $\xi \geq A_\omega \operatorname{Seq}\xi$. In other words, ξ is strongly sequential if whenever a decreasing sequence of subsets (A_n) accumulates at x, the point x belongs to the (idempotent) sequential closure of the set of limit points of convergent sequences $(x_n)_n$ such that $x_n \in A_n$.

4. Strong sequentiality and strong Fréchetness

Recall that a topology (or convergence) ξ is *strongly Fréchet* if for each countably based filter $\mathcal{H}$ with $x_0 \in \operatorname{adh}_\xi \mathcal{H}$ there exists a sequence meshing $\mathcal{H}$ that converges to x_0. In [10] Y. Tanaka introduced the condition (C) that can be rephrased as follows.

Condition 4.1 (C)**.** For each countably based filter $\mathcal{H}$, if $\operatorname{adh}_\xi \mathcal{H} \neq \emptyset$ then $\operatorname{adh}_{\operatorname{Seq}\xi} \mathcal{H} \neq \emptyset$.

Notice that strong sequentiality is weaker than strong Fréchetness and stronger than Condition (C).

Strong sequentiality appears as an extension of the concept of strong Fréchetness from the class of Fréchet spaces to the whole class of sequential spaces.

Proposition 4.2. *A T_1 strongly sequential and Fréchet convergence is strongly Fréchet.*

Proof: Let ξ be a T_1 strongly sequential and Fréchet convergence. By Theorem 3.1,

$$\operatorname{adh}_\xi \mathcal{H} \subset \operatorname{cl}_{\operatorname{Seq}\xi}(\operatorname{adh}_{\operatorname{Seq}\xi} \mathcal{H}) \tag{4.1}$$

for every countably based filter $\mathcal{H}$. I need to show that $\operatorname{adh}_\xi \mathcal{H} \subset \operatorname{adh}_{\operatorname{Seq}\xi} \mathcal{H}$. Let (H_n) be a decreasing base of $\mathcal{H}$ and let $x \in \operatorname{adh}_\xi \mathcal{H}$. By (4.1) and since ξ is T_1, there exists on $\operatorname{adh}_{\operatorname{Seq}\xi} \mathcal{H}$ a sequence of distinct terms (x_n) that converges to x. By Fréchetness and the fact that $\mathcal{H}$ is countably based, for each n, there exists a sequence $(x_{(n,k)})_k$ that converges to x_n and such that $x_{(n,k)} \in H_{n+k}$. As ξ is Fréchet and x belongs to the sequential closure of $\{x_{n,k} : n \in \omega, k \in \omega\}$, there exists a sequence $(x_{n_j,k_j})_j$ such that $x \in \lim_\xi (x_{n_j,k_j})_j$; because ξ is T_1, the sequence $(n_j + k_j)_j$ tends to infinity. Therefore $(x_{n_j,k_j})_j$ is finer than $\mathcal{H}$, hence $x \in \operatorname{adh}_{\operatorname{Seq}\xi} \mathcal{H}$. □

Consequently, examples of sequential non strongly sequential topologies are well known. For example the countable fan S_ω is Fréchet Hausdorff but not strongly Fréchet, hence not strongly sequential. By [9, Example 6.6], the product of two strongly Fréchet topologies, hence of two strongly sequential topologies, need not be sequential, under $2^{\aleph_0} < 2^{\aleph_1}$. On the other hand, a strongly sequential topology can be of arbitrary large sequential order (between one and ω_1). For example, a convergent free bisequence with its usual topology is a strongly sequential topology of sequential order 2. Moreover, there are also structural analogies between strongly sequential and strongly Fréchet spaces. The following problem is analogous to Problem 0.2.

Problem 4.3. Characterize topologies (or convergences) ξ such that $\xi \times \tau$ is Fréchet for every first-countable topology (convergence) τ.

The structure of its solution (Theorem 4.4 below) is very similar to that of Problem 0.2 (Theorem 3.1).

Theorem 4.4. *The following are equivalent:*

1. ξ *is strongly Fréchet;*
2. $\operatorname{adh}_\xi \mathcal{H} \subset \operatorname{adh}_{\mathrm{First}\,\xi} \mathcal{H}$ *for each countably based* $\mathcal{H}$*;*
3. $\xi \times \tau$ *is Fréchet for each first-countable convergence* τ*;*
4. $\xi \times \tau$ *is Fréchet for each metrizable atomic topology* τ*;*
5. $\xi \times \tau$ *is strongly Fréchet for each bisequential convergence* τ*.*

This last theorem is just an extension to convergences of a combination of well-known results of E. Michael [6, Proposition 4.D.4] and [6, Proposition 4.D.5] ([13]). However, Theorem 4.4 can be proved by the same methods as Theorem 3.1. Moreover, both Theorems 3.1 and 4.4 are corollaries of a single abstract theorem stated in its general form in [8]. Indeed, S. Dolecki observed in [3] that both concepts of Fréchetness and strong Fréchetness can be extended from topologies to convergences via

$$\xi \geq P\,\mathrm{First}\,\xi \text{ and } \xi \geq P_\omega\,\mathrm{First}\,\xi,$$

respectively. The reflector P_ω is introduced in the same paper [3] and I show in [8] that $P_\omega \xi = i^-(\Lambda[\mathrm{First}(\Lambda[\xi])])$.

In view of (1.4), it suffices to apply the mechanism of continuous duality at work in the proof of Theorem 3.1, in order to prove Theorem 4.4.

5. A characterization of the pairs with sequential product under a first-countability assumption

Theorem 5.1 answers Problem 0.3 only in the case of first-countable T_1 regular topologies τ. It refines Theorem 0.1 of Tanaka ([14]).

Theorem 5.1. *Let τ be a first-countable regular T_1 topology. Then $\xi \times \tau$ is sequential if and only if ξ is strongly sequential or τ is locally countably compact.*

To prove Theorem 5.1, I follow the method of Y. Tanaka in [10]. In particular the following lemma is a refinement of [10, Lemma 2.2] obtained by an adaptation of Tanaka's proof.

Lemma 5.2. *If a sequential convergence ξ is not strongly sequential, then there exists a countable metrizable atomic topology τ_0 such that $\xi \times \tau_0$ is not sequential.*

Proof: If $\xi \not\geq A_\omega\,\mathrm{First}\,\xi$ then there exists a countably based filter $\mathcal{H} = \mathcal{H}_{\mathrm{ad}_{T\xi}}$ such that $x_0 \in \operatorname{adh}_\xi \mathcal{H} \setminus \operatorname{cl}_{\mathrm{First}\,\xi}(\operatorname{adh}_{\mathrm{First}\,\xi} \mathcal{H})$. Let $(H_n)_{n\in\omega}$ be a decreasing base

[13] In [6], E. Michael uses the term *countably bisequential* for strongly Fréchet.

[14] Recall that in [10] all spaces are supposed to be regular T_1 topologies.

of $\mathcal{H}$. By the sequentiality of ξ, for each n, there exists a countable subset C_n of H_n such that $x_0 \in \mathrm{cl}_\xi C_n$. Consider the set $\{x_0\} \bigcup_{n\in\omega}(C_n \times \{n\})$ endowed with the atomic topology τ_0 with $\mathcal{N}_{\tau_0}(x_0)$ generated by $\{\bigcup_{i\geq n}(C_i \times \{i\})\}_{n\in\omega} \wedge (x_0)$. The convergence τ_0 is a countable metrizable atomic topology, and $\xi \times \tau_0$ is not sequential. Let $A = \bigcup_{x\in C_n, n\in\omega} \left(\mathrm{cl}_\xi\, x \times (x,n) \right)$. Obviously, $(x_0,x_0) \in \mathrm{cl}_{\xi\times\tau_0} A$, but $(x_0,x_0) \notin \mathrm{cl}_{\mathrm{First}(\xi\times\tau_0)} A$. Indeed, consider two countably based filters $\mathcal{F}$ and $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{F}$, $y \in \lim_{\tau_0} \mathcal{G}$ and $(\mathcal{F}\times\mathcal{G})\#A$. If $y \neq x_0$, there exists $n \in \omega$ and $z \in C_n$ such that $y = (z,n)$. This point being isolated in τ_0, $\mathcal{G}$ is its principal ultrafilter. From $(\mathcal{F}\times\mathcal{G})\#A$, we get $x \in \mathrm{adh}_\xi\, z$, so that $(x,(z,n)) \in A$. If $y = x_0$, then $\mathcal{F}\#A^{-}\mathcal{N}_{\tau_0}(x_0)$. In other words, $\mathcal{F}\#\mathcal{H}_{\mathrm{ad}_\xi}$, so that $x \in \mathrm{adh}_{\mathrm{First}\,\xi}\,\mathcal{H}$. Hence $\mathrm{adh}_{\mathrm{First}(\xi\times\tau_0)} A \subset A \cup (\mathrm{adh}_{\mathrm{First}\,\xi}\,\mathcal{H} \times \{x_0\})$. Since $x_0 \notin \mathrm{cl}_{\mathrm{First}\,\xi}(\mathrm{adh}_{\mathrm{First}\,\xi}\,\mathcal{H})$, I conclude that $(x_0,x_0) \notin \mathrm{cl}_{\mathrm{First}(\xi\times\tau_0)} A$. □

Proof of Theorem 5.1: The necessity follows from Theorem 3.1 and [7, Theorem 4.2] mentioned in the introduction.

Assume that $\xi\times\tau$ is sequential. If τ is not locally countably compact, then it is easy to check (see [10, Lemma 2.3]) that the metrizable topology τ_0 of Lemma 5.2 is homeomorphic to a closed subset of ξ. Thus, in view of Lemma 5.2, ξ is strongly sequential, because $\xi \times \tau_0$ is sequential as a closed subspace of $\xi \times \tau$. □

To conclude, notice that other results of [10] can be improved on replacing Condition 4.1 by strong sequentiality. For example, compare the following with [10, Proposition 4.1] (the proofs are completely similar). Denote by ξ^ω the countable power of a convergence ξ.

Theorem 5.3. *If ξ^ω is sequential, then ξ is strongly sequential.*

The converse is false under (MA). Indeed, by [11, p. 301], there exists, under (MA), a strongly Fréchet topology (hence strongly sequential) whose countable product is not even a k-space.

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