Some results and problems about weakly pseudocompact spaces

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Abstract. A space X is *truly weakly pseudocompact* if X is either weakly pseudocompact or Lindelöf locally compact. We prove: (1) every locally weakly pseudocompact space is truly weakly pseudocompact if it is either a generalized linearly ordered space, or a proto-metrizable zero-dimensional space with $\chi(x, X) > \omega$ for every $x \in X$; (2) every locally bounded space is truly weakly pseudocompact; (3) for $\omega < \kappa < \alpha$, the κ-Lindelöfication of a discrete space of cardinality α is weakly pseudocompact if $\kappa = \kappa^\omega$.

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All spaces considered in this paper are assumed to be Tychonoff (= completely regular Hausdorff). A *zero set* in X is a set of the form $Z(f) = f^{-1}(0)$ for some continuous function $f \colon X \to \mathbb{R}$. A *z-ultrafilter* is an ultrafilter of z-sets. For each zero set Z in X, we denote by $\tilde{Z}$ the set of all z-ultrafilters containing Z. A space Y is a *compactification* of a space X if Y is compact and X is a dense subspace of Y. If X is a space, βX denotes the Stone-Čech compactification of X, that is, βX is the space of all the z-ultrafilters on X where a base for its closed subsets is $\{\, \tilde{Z} : Z \text{ is a zero set of } X \,\}$. Notation and terminology not explained in the text are as in [Eng], with the exception that the closure of a set A in a space X is denoted as $\mathrm{cl}_X A$ (or simply $\mathrm{cl}\, A$).

A space X is called *weakly pseudocompact* if there is a compactification bX of X such that X is G_δ*-dense* in bX, which means that every nonempty G_δ-set in bX meets X. The notion of weak pseudocompactness, introduced in [GG], is a natural generalization of a well-known characterization of pseudocompact spaces as spaces that are G_δ-dense in their Stone-Čech compactifications (which implies G_δ-density in *every* compactification). Unlike pseudocompactness, weak pseudocompactness in combination with realcompactness does not imply compactness (it is easy to see that every locally compact non-Lindelöf space is weakly pseudocompact), but every Lindelöf weakly pseudocompact space is compact. Nice properties of weak pseudocompactness are that it is multiplicative (in any numbers), and implies the Baire property.

Eckertson proved [Eck] that every open set in a weakly pseudocompact space is either weakly pseudocompact or (Lindelöf) locally compact. This indicates that it is natural to consider the class of spaces that are either weakly pseudocompact or locally compact; we call such spaces *truly weakly pseudocompact*. Thus, every open subset of a truly weakly pseudocompact space is truly weakly pseudocompact. A natural and, as it turns out, nontrivial question (still open in the general case) is whether every locally weakly pseudocompact space is truly weakly pseudocompact (a space is *locally weakly pseudocompact* if each of its points has a weakly pseudocompact neighborhood; by the above, this is equivalent to the requirement that every point has a truly weakly pseudocompact *open* neighborhood). In [OT] the true weak pseudocompactness of a locally weakly pseudocompact space was proved for the class of paracompact generalized linearly ordered spaces. The main result of Section 2 of this article is to prove that the requirement of paracompactness is superfluous.

The above problem, of course, is a particular case of a general question: When is a union of weakly pseudocompact spaces weakly pseudocompact? In particular, it is not even clear whether a space which is the union of two closed weakly pseudocompact subspaces must be weakly pseudocompact. In Section 1 we give some steps in this direction. It is worth mentioning that the free topological sum of weakly pseudocompact spaces is truly weakly pseudocompact ([Eck]).

In Section 3 we look at some generalizations of the fact that every locally compact space is truly weakly pseudocompact and obtain answers to questions about the behavior of weak pseudocompactness under "nice" mappings in some special classes of spaces. In Section 4 we study relations between local and global weak pseudocompactness in some "very disconnected" spaces. Finally, in Section 5 we prove weak pseudocompactness (or its absence) for some individual spaces; questions of this type turn out to be surprisingly difficult for some very simple spaces. For example, it appears to be unknown to date whether $\mathbb{R}^{\omega_1}$ is weakly pseudocompact.

We often abbreviate "weakly pseudocompact" to "w.p.", "truly weakly pseudocompact" to "t.w.p.", and "locally weakly pseudocompact" to "l.w.p.".

1. Unions of truly weakly pseudocompact subspaces

In this section we give some sufficient conditions on subsets A and B in order to guarantee that $A \cup B$ is truly weakly pseudocompact.

We will need the following lemma for construction of compactifications.

1.1 Lemma. *Let X be a normal* (*Hausdorff*) *space, and let $q\colon X \to Y$ be a continuous* (*continuous and with compact fibers*) *function. If q is closed, then Y is a Hausdorff space.*

We divide our first theorem into several lemmas. These lemmas will refer to the following objects:

Let $X = A \cup B$ be a topological space. Let bA and bB be Hausdorff extensions of A and B. Let Y be the space $(bA \times \{0\}) \oplus (bB \times \{1\})$, and let R be the equivalence relation on Y defined by the rule: $(x, i)R(y, j)$ iff either $(x, i) = (y, j)$ or $x = y$ and $x, y \in X$. Let X_0 be the space $(A \times \{0\}) \cup (B \times \{1\})$, and consider $h\colon X_0 \to X$

defined by $h(x, i) = x$. Obviously, h is continuous. We denote by ϕ the function from X to Y/R defined as $\phi(x) = p((x, 0))$ if $x \in A$, and $\phi(x) = p((x, 1))$ if $x \in B \setminus A$ where $p\colon Y \to Y/R$ is the natural projection. Clearly, $\phi \circ h = p$.

1.2 Lemma. *If $A \cap B$ has a locally finite in Y compact cover (in particular, if $A \cap B$ is compact), then p is closed.*

Proof: Let F be a closed subset of Y. Let $F_1 = F \cap (bA \times \{0\})$ and $F_2 = F \cap (bB \times \{1\})$. We have to prove that $p^{-1}(p(F))$ is closed in Y. Let $\mathcal{D}$ be a locally finite (in Y) compact cover of $A \cap B$. We have

$$p^{-1}(p(F)) = \big(p^{-1}(p(F)) \cap (bA \times \{0\})\big) \cup \big(p^{-1}(p(F)) \cap (bB \times \{1\})\big).$$

Clearly, $p^{-1}(p(F)) \cap (bA \times \{0\}) = F_1 \cup (F_2 \cap A \cap B)$. But $(F_2 \cap A \cap B)$ is the union of a locally finite family of compact sets, so it is closed in Y. Thus, $p^{-1}(p(F)) \cap (bA \times \{0\})$ is closed in Y. Similarly, we conclude that $p^{-1}(p(F)) \cap (bB \times \{1\})$ is closed in Y. Since p is quotient, $p(F)$ is closed in Y/R. $\square$

1.3 Lemma. *If A and B are G_δ-dense in bA and bB, then $\phi(X) = p(A \times \{0\} \cup B \times \{1\})$ is G_δ-dense in Y/R.*

Proof: X_0 is G_δ-dense in Y, and $\phi(X) = p(X_0)$. $\square$

The following result is a generalization of Lemma 2.7 in [OT].

1.4 Theorem. *Let A and B be weakly pseudocompact spaces. Let $X = A \cup B$. Assume that there are two compactifications bA and bB of A and B, respectively, such that A and B are G_δ-densely embedded, and such that $A \cap B$ can be covered by a locally finite (in $Y = (bA \times \{0\}) \cup (bB \times \{1\})$) family of compact subsets. If h is a quotient (in particular, if A and B are both open or closed), then X is weakly pseudocompact.*

Proof: By Lemma 1.2, p is closed. So ϕ is closed, because $p = \phi \circ h$. By assumption, h is quotient, so ϕ is continuous. On the other hand, ϕ is one-to-one. Hence, ϕ is an embedding. Therefore, X is G_δ-dense in the compact space Y/R by Lemma 1.3. $\square$

1.5 Theorem. *Let $X = Y \cup \{x_0\}$ where Y is weakly pseudocompact and x_0 has a compact neighborhood in X. Then X is weakly pseudocompact.*

Proof: Let U be an open neighborhood of x_0 in X such that the closure of U in X is compact. Let $P = \mathrm{cl}_Y(U \setminus \{x_0\}) = \mathrm{cl}_X(U) \cap Y$. Obviously, P is a locally compact closed set in Y. Let bY be a compactification of Y in which Y is G_δ-dense, and let $F = \mathrm{cl}_{bY} P \setminus P$. Since P is locally compact, it is open in its compactification $\mathrm{cl}_{bY} P$; hence F is compact; $F \cap Y = \emptyset$ because P is closed in Y.

Let $B = bY/F$, $p\colon bY \to B$ the projection, $\{z_0\} = p(F)$, and $Z = p(Y) \cup \{z_0\}$. Obviously, B is a compactification of Z in which Z is G_δ-dense; to end the proof it suffices to verify that Z is homeomorphic to X. Note that Z is homeomorphic to $(Y \cup F)/F$, because p is closed and $Y \cup F = p^{-1}(Z)$. Hence, it suffices to check that the mapping $p_1 \colon Y \cup F \to X$ defined by the rule

$$p_1(x) = \begin{cases} x & \text{if } x \in Y, \\ x_0 & \text{if } x \in F \end{cases}$$

is quotient. To that end it is enough to verify that the restrictions of p_1 to the elements of the closed cover $\{P \cup F, Y \setminus V\}$ of $Y \cup F$ are closed. But the restriction of p_1 to $Y \setminus V$ is a homeomorphism, and the restriction of p_1 to $P \cup F$ is the standard mapping of the compactification $P \cup F = \mathrm{cl}_{bY} P$ of the locally compact space P to its Alexandroff compactification $P \cup \{x_0\} = \mathrm{cl}_X U$. $\square$

2. GLOTS and weak pseudocompactness

Our main goal of this section is to prove that every locally weakly pseudocompact GLOTS is truly weakly pseudocompact. Recall that a topological space X is called *generalized linearly ordered* (GLOTS) if it is homeomorphic to a subspace of some linearly ordered space Z. In what follows we assume the linearly ordered space $(Z, <)$ fixed; we have thus a linear order fixed on X. The symbols (a, b), $[a, b)$, etc. refer to intervals with respect to this order (we do not require $a, b \in X$); we denote

$$\begin{aligned} (\leftarrow, a] &= \{\, x \in X : x \leq a \,\}, \\ (\leftarrow, a) &= \{\, x \in X : x < a \,\}, \\ [a, \rightarrow) &= \{\, x \in X : a \leq x \,\}, \\ (a, \rightarrow) &= \{\, x \in X : a < x \,\} \end{aligned}$$

A point $x_0 \in X$ is *locally compact at the left* if x is the minimum of X or there is an $a \in (\leftarrow, x)$ such that the interval $[a, x]$ is compact. The notion of a point locally compact at the right is defined symmetrically.

The following results were proved in [OT].

2.1 Proposition. *Let X be a truly weakly pseudocompact GLOTS, and suppose x_0 is a point of X that is locally compact at the right. Then $(\leftarrow, x_0]$ is truly weakly pseudocompact.*

2.2 Proposition. *Let X be a GLOTS. If X has a point x_0 such that both $[x_0, \rightarrow)$ and $(\leftarrow, x_0]$ are truly weakly pseudocompact, then X is truly weakly pseudocompact.*

2.3 Proposition. *Let X be a GLOTS which is not locally compact at any point. Then X is w.p. iff for every $x \in X$, $[x_0, \rightarrow)$ and $(\leftarrow, x_0]$ are weakly pseudocompact.*

2.4 Proposition. *Let X be a GLOTS. Then the following statements are equivalent:*

(1) *X is truly weakly pseudocompact;*
(2) *for every $x, y \in X$ with $x < y$, (x, y) is truly weakly pseudocompact;*
(3) *for every $a, b \in Z$ with $a < b$, $(a, b) \cap X$ is truly weakly pseudocompact;*
(4) *for every $x \in X$, $(x, \rightarrow)$ and $(\leftarrow, x)$ are truly weakly pseudocompact;*
(5) *every proper open subset of X is truly weakly pseudocompact;*
(6) *there exists $x_0 \in X$ such that $(\leftarrow, x_0]$ and $[x_0, \rightarrow)$ are truly weakly pseudocompact.*

Using the above, we are now able to prove

2.5 Proposition. *Let X be a GLOTS and A and B two open intervals in X. If A and B are truly weakly pseudocompact, then $A \cup B$ is truly weakly pseudocompact.*

Proof: If $A \cap B = \emptyset$, then $A \cup B$ is the free topological sum of two t.w.p. spaces, so it is t.w.p. Suppose $A \cap B \neq \emptyset$. Let Z be a compact LOTS that contains X as a dense subset. Let $a, b, c, d \in Z$ be such that $A = (a, b) \cap X$ and $B = (c, d) \cap X$. Without loss of generality we assume that $a \leq c$. If $d \leq b$, then there is nothing to prove, because in this case $B \cup A = A$. We can even assume that $a < c < b < d$, because if $a = c$ or $b = d$, then $A \cup B = A$ or $A \cup B = B$.

Case 1. Suppose there is a point $x_0 \in A \cap B$ with a compact neighborhood. Then there is a compact neighborhood of x_0 contained in $A \cap B = (c, b) \cap X$. Then x_0 is a point of the t.w.p. GLOTS $(a, b) \cap X = A$ which is locally compact, so, by Proposition 2.1, $(a, x_0] \cap X$ is t.w.p. Similarly, $[x_0, d) \cap X$ is t.w.p. We conclude that $(a, d) \cap X = A \cup B$ is t.w.p. by Proposition 2.2.

Case 2. Suppose now that $E = (c, b) \cap X$ has no points of local compactness. Observe that a point x in E cannot have an immediate predecessor and an immediate successor in X, because in this case x would have a compact neighborhood ($\{x\}$). Take $z \in E$. If z does not have an immediate predecessor in X, then there is a point $e \in Z \setminus X$ with $c < e < z$. By Proposition 2.3, $(c, z] \cap X$ is t.w.p. Since open subsets of t.w.p. spaces are t.w.p., $(e, z] \cap X$ and $(a, e) \cap X$ are t.w.p. Therefore, $(a, z] \cap X = \big((a, e) \oplus (e, z]\big) \cap X$ is t.w.p. If z has an immediate successor w, then $[z, d) \cap X$ is the free topological sum of two t.w.p. spaces: $\{z\}$ and $(z, d) \cap X$. Using similar arguments we can prove that, in any case, $(a, z] \cap X$ and $[z, d) \cap X$ are t.w.p. By Proposition 2.2, $(a, d) \cap X = A \cup B$ is t.w.p. $\square$

Let X be a GLOTS. For each $x \in X$ consider the collection $\mathcal{W}(x)$ of all open truly weakly pseudocompact intervals in X that contain x. Let $W_x = \bigcup \mathcal{W}(x)$.

2.6 Proposition. *Let X be a l.w.p. GLOTS and let $x \in X$. Then W_x is a clopen t.w.p. interval containing x.*

Proof: We have $x \in W_x$, because X is l.w.p. Besides, it is open as the union of open intervals. If $|W_x| = 1$, we have already finished. Now, assume that

$|W_x| > 1$, and take $z, y \in W_x$ with $z < y$. Then there are open intervals A and B containing x and containing z and y respectively, which are t.w.p. By Proposition 2.5, $A \cup B$ is a t.w.p. open interval containing x, y and z. Thus, W_x is an interval, and $(z, y) \cap X$ is t.w.p., because it is an open subset of the t.w.p. space $A \cup B$. By Proposition 2.4, W_x is t.w.p. Now let us check that W_x is closed. Let $y \in \mathrm{cl}_X W_x$. Since X is l.w.p., there is an open t.w.p. interval V containing y. Since $y \in \mathrm{cl}_X W_x$, there is a t.w.p. open interval W containing x and intersecting V. By Proposition 2.5, $V \cup W$ is t.w.p., so $V \cup W \in \mathcal{W}(x)$, and $y \in W_x$. □

Thus we obtain

2.7 Theorem. *If X is a l.w.p. GLOTS, then X is t.w.p.*

Proof: Let $x, y \in X$. Consider the t.w.p. spaces W_x and W_y. Because of Propositions 2.5 and 2.6, if $W_x \cap W_y \neq \emptyset$, then $W_x \cup W_y$ is a t.w.p. open interval containing x and y. Thus, $W_x = W_y$. That is, X is the free topological sum of t.w.p. spaces. In fact, the relation $x \sim y$ iff $W_x = W_y$ is an equivalence relation in X. If Z is a subset containing one and only one element of each class of equivalence, then $X = \bigoplus_{z \in Z} W_z$. Therefore, X is t.w.p. □

3. Locally bounded spaces

We devote this section to exhibit some other classes of spaces where the property of being locally weakly pseudocompact implies the property of being truly weakly pseudocompact. The section is divided in two parts; in the first one, we are going to prove that every locally pseudocompact space is t.w.p., and in the second part we prove that "l.w.p. ⇒ t.w.p." holds for spaces X such that $\mathcal{K}(X)$ is a b-lattice.

3.1 Definition. A subset B of a space X is called *bounded in X* if for every continuous function $f\colon X \to \mathbb{R}$, $f(B)$ is bounded in $\mathbb{R}$.

A space X is *locally bounded* if every point of X has a bounded neighborhood.

A space X is *locally pseudocompact* if every point of X has a pseudocompact neighborhood.

Obviously, the closure of a bounded (pseudocompact) set is bounded (pseudocompact), in particular, in a locally bounded (locally pseudocompact) space, every point has a closed bounded (pseudocompact) neighborhood.

3.2 Lemma. *Let F be a set in a space X. If F is G_δ-dense in $\mathrm{cl}_{\beta X} F$, then F is bounded in X.*

Proof: Suppose there is a continuous function $f\colon X \to \mathbb{R}$ such that f is not bounded on F. Let $C = \mathbb{R} \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}$. Then ∞ is a limit point for $f(F)$. Let $\tilde{f}\colon \beta X \to C$ be the continuous extension of f. By the compactness of $\mathrm{cl}_{\beta X} F$, $\infty \in \tilde{f}(\mathrm{cl}_{\beta X} F)$, hence, $P = \tilde{f}^{-1}(\infty)$ is a G_δ-set that meets $\mathrm{cl}_{\beta X} F$. By the G_δ-density of F in $\mathrm{cl}_{\beta X} F$, $P \cap F \neq \emptyset$, and there is a point in F such that the value of f in it is ∞; a contradiction. □

3.3 Lemma. *If F is a bounded zero-set in X, then F is G_δ-dense in $\mathrm{cl}_{\beta X} F$.*

PROOF: Let $x_0 \in \mathrm{cl}_{\beta X} F$, and suppose there is a G_δ-set in βX that contains x_0 and is disjoint with F. Then there is a continuous function $g\colon X \to [0,1]$ such that $\tilde{g}(x_0) = 0$ and $\tilde{g}(F) \subset (0,1]$ where $\tilde{g}\colon \beta X \to [0,1]$ is the continuous extension of g. The sets F and $F_1 = g^{-1}(0)$ are disjoint zero-sets in X, hence there is a continuous function $h\colon X \to [0,1]$ such that $h(F) \subset \{0\}$ and $h(F_1) \subset \{1\}$. Let $f_0 = g + h$ and $\tilde{f}_0\colon \beta X \to [0,2]$ its continuous extension. Then $f_0|F = g|F$, whence $f_0(F) \subset (0,1]$ and $\tilde{f}_0(x_0) = 0$. Furthermore, $f_0(x) > 0$ for all $x \in X$. The function $f = 1/f_0$ is well-defined and continuous on X and is unbounded on F. □

3.4 Lemma. *Let $F \subset X$ and let bX be a compactification of X. If F is G_δ-dense in $\mathrm{cl}_{\beta X} F$, then F is G_δ-dense in $\mathrm{cl}_{bX} F$.*

PROOF: Let $f\colon \beta X \to bX$ be the continuous extension of the embedding $X \hookrightarrow bX$. Since f is continuous and $\mathrm{cl}_{\beta X} F$ is compact, $f(\mathrm{cl}_{\beta X} F) = \mathrm{cl}_{bX} F$. Thus, $f|\,\mathrm{cl}_{\beta X} F\colon \mathrm{cl}_{\beta X} F \to \mathrm{cl}_{bX} F$ is continuous and onto. Now if we had a nonempty G_δ-set G in $\mathrm{cl}_{bX} F$ disjoint with F, then $f^{-1}(G)$ would be a nonempty G_δ-set in $\mathrm{cl}_{\beta X} F$ disjoint with F. □

3.5 Theorem. *Every locally bounded space is truly weakly pseudocompact.*

PROOF: If X is locally compact, then, obviously, X is t.w.p. Now assume that X is not locally compact.

For each point $x \in X$ choose an open neighborhood V_x of x so that $\mathrm{cl}_X V_x$ is a bounded zero-set in X. For each $x \in X$ choose an open set $\tilde{V}_x$ in βX so that $\tilde{V}_x \cap X = V_x$. Consider the subset $Y = \bigcup_{x\in X} \tilde{V}_x$ of βX. Y is not compact and is locally compact, because it is open in βX. Let $\alpha Y = Y \cup \{z_0\}$ be the one-point compactification of Y where $z_0 \notin Y$. Since X is not locally compact, there is a point $x_0 \in X$ such that $\tilde{V}_{x_0} \setminus X \neq \emptyset$. Let z_1 be an element of $\tilde{V}_{x_0} \setminus X$, and let Z be the space obtained from αY by identifying z_0 with z_1. Let $q\colon \alpha(Y) \to Z$ be the projection. Then Z is a Hausdorff (Lemma 1.1) compact space containing X as a dense subset. Let us verify that X is G_δ-dense in Z. Indeed, let G be a nonempty G_δ-set in Z. We have $Z = \bigcup_{x\in X} q(\tilde{V}_x)$, and for each $x \in X$, $q(\tilde{V}_x) \subset \mathrm{cl}_Z \mathrm{cl}_X V_x$. Thus, $Z = \bigcup_{x\in X} \mathrm{cl}_Z \mathrm{cl}_X V_x$. Hence, there is $a \in X$ such that $G \cap \mathrm{cl}_Z \mathrm{cl}_X V_a$ is a nonempty G_δ-set in $\mathrm{cl}_Z \mathrm{cl}_X V_a$. On the other hand, $\mathrm{cl}_X V_a$ is a bounded zero-set in X, so it is G_δ-dense in $\mathrm{cl}_{\beta X} \mathrm{cl}_X V_a$ by Lemma 3.3. Then $\mathrm{cl}_X V_a$ is G_δ-dense in $\mathrm{cl}_Z \mathrm{cl}_X V_a$ by Lemma 3.4. Therefore, $G \cap \mathrm{cl}_X V_a \neq \emptyset$. We conclude that G meets X. □

3.6 Corollary. *If X is locally pseudocompact, then X is t.w.p.*

3.7 Corollary. *Let X be a space. The following statements are equivalent:*

(1) *X is t.w.p.;*
(2) *X is G_δ-densely embedded in a locally bounded space;*
(3) *X is G_δ-densely embedded in a locally pseudocompact space.*

Proof: Of course, if X is w.p., then it is G_δ-dense in a compactification bX of X, which is locally pseudocompact. If X is locally compact, then X is G_δ-dense in itself. Suppose now that X is G_δ-dense in a locally bounded space Y. By the previous theorem, there is a compactification bY of Y in which Y is G_δ-dense. Therefore, bY is a compactification of X in which X is G_δ-dense. □

As usual, we denote by υX the Hewitt realcompactification of X. The space υX can be identified with the subset of βX of all z-ultrafilters on X with the countable intersection property (this kind of z-ultrafilters are the so called *real* z-ultrafilters).

Let bX and cX be two compactifications of the space X. We write $bX \leq cX$ if there is a continuous function $f : cX \to bX$ such that $f(x) = x$ for every $x \in X$. The compactifications bX and cX are equivalent if $bX \leq cX$ and $cX \leq bX$. We denote by $\mathcal{K}(X)$ the set of equivalence classes of compactifications of X, and call a compactification $bX \in \mathcal{K}(X)$ *simple* if the standard mapping $\beta X \to bX$ has at most one fiber that is not a singleton; thus, a simple compactification of X is the quotient space $\beta X/K$ where $K \subset \beta X \setminus X$ is a compact set. The family $\mathcal{K}(X)$ is said to be a *b-lattice* (see [U]) if the simple compactifications of X are dense in the ordered set $\mathcal{K}(X)$, that is, if for each $bX \in \mathcal{K}(X)$ there is a simple compactification $b'X$ such that $b'X \leq bX$.

In [GFS] the authors determine when υX is w.p. in terms of Wallman bases. In what follows we characterize weak pseudocompactness for spaces X such that $\mathcal{K}(X)$ is a b-lattice.

The following lemma is immediate from one of the equivalent definitions of υX.

3.8 Lemma. *A space X is G_δ-dense in a subset Y of βX containing X if and only if $Y \subset \upsilon X$.*

It is not difficult to prove the next two statements.

3.9 Lemma. *Let X be a space. If X is contained in $\mathrm{Int}_{\beta X}\, \upsilon X$, then X is locally bounded.*

3.10 Lemma. *Let X be a space. Then, X is contained in $\mathrm{Int}_{\beta X}\, \upsilon X$ if and only if there is a locally compact subspace Y of βX such that $X \subset Y \subset \upsilon X$.*

3.11 Theorem. *Let X be a space such that $\mathcal{K}(X)$ is a b-lattice. Then the following statements are equivalent.*

(1) *X is t.w.p.*
(2) *X is locally bounded.*
(3) *There is a locally compact space Y such that $X \subset Y \subset \upsilon X$.*
(4) *$X \subset \mathrm{Int}_{\beta X}\, \upsilon X$.*

Moreover, if X is normal, we can add to the previous list the following:

(5) *X is locally pseudocompact.*

Proof: The equivalence (3) ⇔ (4) is Lemma 3.10, (4) $\implies$ (2) is Lemma 3.9, and the implication (2) $\implies$ (1) is Theorem 3.5. Let us prove (1) $\implies$ (3): If X is locally compact and Lindelöf, then X is realcompact and $\upsilon X = X$ is open in βX. Assume now that υX does not contain an open subset of βX that contains X. We will prove that X is not w.p. Since $\mathcal{K}(X)$ is a b-lattice, it is enough to prove that for every compact $K \subset \beta X \setminus X$, X is not G_δ-dense in $\beta X/K$.

By our hypothesis, we can pick a $p \in \beta X \setminus (\upsilon X \cup K)$. There is a G_δ-subset G of βX such that $p \in G$ and $G \cap (\upsilon X \cup K) = \emptyset$. Let $q\colon \beta X \to \beta X/K$ be the natural projection. The set $q(G)$ is a nonempty G_δ subset of $\beta(X)/K$ that does not meet X. Therefore, X is not w.p.

Obviously, every locally pseudocompact space is locally bounded. If X is normal, then every closed bounded set in X is C-embedded, and hence pseudocompact, so a normal locally bounded space is locally pseudocompact. This shows the equivalence of (5) to the other conditions in the case where X is normal. $\square$

As a consequence of the previous result we obtain Theorem 3.2 in [Eck]:

3.12 Corollary. *Let X be a realcompact space such that $\mathcal{K}(X)$ is a b-lattice. Then X is t.w.p. iff X is locally compact.*

3.13 Lemma. *Let X be a space. Then υX is locally compact if and only if for each real z-ultrafilter p on X there are zero-sets $Z \in p$ and $W \notin p$ such that $Z \supset X \setminus W$ and Z is bounded in X.*

Proof: Assume that υX is locally compact, and let $p \in \upsilon X$ (that is, let p be a real z-ultrafilter). Let V be a compact neighborhood of p contained in υX. Thus, there exist $Z \in p$ and $W \notin p$ such that $Z \supset X \setminus W$ and $Z \subset V \cap X$ (see [GJ, 7.12]). Since V is compact and $V \subset \upsilon X$, Z is bounded in X.

Now suppose that $p \in \upsilon X$, and let Z and W two zero sets such that

(1) $Z \in p$, $W \notin p$, $Z \supset X \setminus W$, and
(2) Z is bounded in X.

By (1), $\mathrm{cl}_{\beta X} Z$ is a neighborhood of p in βX ([GJ, 7.12]), and because of (2), $\mathrm{cl}_{\beta X} Z \subset \upsilon X$. $\square$

Combining Corollary 3.12 and Lemma 3.13, we obtain

3.14 Corollary. *Let X be a space such that $\mathcal{K}(X)$ is a b-lattice. Then the following statements are equivalent:*

(1) *υX is t.w.p.,*
(2) *υX is locally compact,*
(3) *for each real z-ultrafilter p on X there are zero-sets $Z \in p$ and $W \notin p$ such that $Z \supset X \setminus W$ and Z is bounded in X.*

In [GFS] and [Eck] questions were posed about the invariance of weak pseudocompactness under perfect or open maps, and whether X is w.p. whenever $X \times X$ is w.p. Using Theorems 3.5 and 3.11, we obtain affirmative answers to these questions for the spaces X such that $\mathcal{K}(X)$ is a b-lattice. Indeed,

3.15 Lemma.

(1) *If $f\colon X \to Y$ is a perfect or an open map, and X is locally bounded, then Y is locally bounded.*

(2) *If the product space $\prod_{\lambda<\alpha} X_\lambda$ is locally bounded, then X_λ is locally bounded for every $\lambda < \alpha$.*

Proof: Since the projections from a product space to each of its factors are open, (1) implies (2).

So let us prove (1).

Suppose f is perfect, and let $y \in Y$. Since X is locally bounded and $f^{-1}(y)$ is compact, there is a bounded neighborhood V of $f^{-1}(y)$. Then $Y \setminus f(X \setminus V)$ is a bounded neighborhood of y in Y.

If f is open and $y \in Y$, pick $x \in f^{-1}(y)$, and let V be a bounded neighborhood of x in X. Then $f(V)$ is a bounded neighborhood of y in Y. □

3.16 Theorem. *Let $f\colon X \to Y$ be a perfect map onto Y, and assume that $\mathcal{K}(X)$ is a b-lattice. If X is t.w.p., then Y is t.w.p.*

Proof: By Theorem 3.11, X is locally bounded. Applying the previous lemma, we obtain that Y is locally bounded. Y is t.w.p. by Theorem 3.11. □

A similar argument proves

3.17 Theorem. *If $\mathcal{K}\left(\prod_{\lambda<\alpha} X_\lambda\right)$ is a b-lattice and $\prod_{\lambda<\alpha} X_\lambda$ is t.w.p., then X_λ is t.w.p. for every $\lambda < \alpha$.*

4. Zero-dimensional spaces

We begin with the following remark, which will be very useful for our purposes.

4.1 Lemma. *The union of a countable family of clopen t.w.p. subsets of a space X is t.w.p.*

4.2 Definition. A collection $\mathcal{N}$ of subsets of a set X is called *Noetherian* if $\mathcal{N}$ contains no infinite increasing sequence (with respect to the inclusion).

A collection $\mathcal{N}$ of subsets of a set X is $\aleph_0$-*Noetherian* if $\mathcal{N}$ contains no uncountable increasing sequences.

A collection $\mathcal{N}$ of subsets of a set X is said to be *of subinfinite rank* (*of countable rank, of rank n*) if for each $x \in X$, every antichain of the elements of $\mathcal{N}$ that contain x is finite (resp., countable, has cardinality $\leq n$).

The *non-archimedean spaces* are the spaces having a base of rank 1. These spaces are zero-dimensional, and are GLOTS ([NR]). Therefore, every non-archimedean l.w.p. space is t.w.p. (Theorem 2.7). An important subclass of non-archimedean spaces is the class of ω_μ-metrizable spaces with $\mu > 0$, which are the spaces with a compatible uniformity that has an uncountable totally ordered base (see [NR]). There is a subclass of zero-dimensional spaces wider than that of non-archimedean spaces in which l.w.p. implies t.w.p.: a space X is called *ultraparacompact* if every open cover of X has a clopen disjoint refinement. Observe that a space X is ultracompact if and only if every open cover of X has a locally finite clopen refinement. We can generalize this concept as follows:

4.3 Definition. Let α be a cardinal. A space X is *α-paracompact* if every open cover of X can be refined by a clopen cover $\mathcal{C}$ such that for each $C \in \mathcal{C}$,

$$\big|\{\, D \in \mathcal{C} : C \cap D \neq \emptyset \ \text{ and } \ C \cap (X \setminus D) \neq \emptyset \,\}\big| \leq \alpha.$$

Obviously, ultraparacompactness is equivalent to 1-paracompactness, every α-paracompact space is zero-dimensional, and if α, κ are two cardinals and $\alpha < \kappa$, then α-paracompactness implies κ-paracompactness.

4.4 Theorem. *Every ω-paracompact l.w.p. space is t.w.p.*

Proof: For each $x \in X$ let V_x be a t.w.p. open set containing x. Let $\mathcal{C}$ be a clopen refinement of $\{\, V_x : x \in X \,\}$ that satisfies the condition in the definition of ω-paracompactness of X. Fix an $x \in X$. Put $A_1 = \bigcup\{\, C \in \mathcal{C} : x \in C \,\}$. There is a countable subcollection $\mathcal{C}_1$ of $\mathcal{C}$ such that $A_1 = \bigcup \mathcal{C}_1$. Let

$$\mathcal{C}_2 = \{\, C \in \mathcal{C} : C \cap D \neq \emptyset \ \text{ and } \ C \cap (X \setminus D) \neq \emptyset \ \text{ for some } \ D \in \mathcal{C}_1 \,\}.$$

By induction, let

$$\mathcal{C}_{n+1} = \{\, C \in \mathcal{C} : C \cap D \neq \emptyset \ \text{ and } \ C \cap (X \setminus D) \neq \emptyset \ \text{ for some } \ D \in \mathcal{C}_n \,\},$$

and

$$\mathcal{D} = \bigcup_{n<\omega} \mathcal{C}_n.$$

Obviously, $|\mathcal{D}| \leq \omega$.

Let $W_x = \bigcup_{n<\omega} \bigcup \mathcal{C}_n$. We claim that W_x is a clopen t.w.p. subspace of X. Indeed, let $\mathcal{D} = \{\, D_n : n < \omega \,\}$. Because each D_n is t.w.p. and clopen, $D_0, D_0 \cup D_1, D_0 \cup D_1 \cup D_2, \ldots$ are t.w.p. Besides, W_x is the free topological sum of the collection $\{\, E_n : n < \omega \,\}$ where $E_n = D_{n+1} \setminus (D_0 \cup \cdots \cup D_n)$. Since each E_n is t.w.p., the set W_x is t.w.p. Now suppose $y \in \mathrm{cl}_X W_x$. Then $y \in C$ for some $C \in \mathcal{C}$, and there is $n < \omega$ such that $C \cap D_n \neq \emptyset$. There is $k < \omega$ such that $D_n \in \mathcal{C}_k$. If $C \cap (X \setminus D_n) = \emptyset$, then $y \in C \subset D_n \subset W_x$. If $C \cap (X \setminus D_n) \neq \emptyset$, then $C \in \mathcal{C}_{k+1}$, and this implies that $y \in C \subset W_x$. Therefore, W_x is clopen in X.

Using the same argument than before, we can prove that $W_x \cap W_y = \emptyset$ if W_x and W_y are not equal. So, X is a free topological sum of t.w.p. spaces. Therefore, X is t.w.p. $\square$

4.5 Corollary. *Every ultraparacompact l.w.p. is t.w.p. In particular, if X is paracompact, extremely disconnected and l.w.p., then X is t.w.p.*

4.6 Definition. A collection $\mathcal{C}$ of subsets of a space X is *of rank* α ($< \alpha$) *with respect to a collection* $\mathcal{F}$ if for each $F \in \mathcal{F}$, the cardinality of any antichain $\mathcal{C}' \subset \mathcal{C}$ such that each element of $\mathcal{C}'$ meets F, has cardinality $\leq \alpha$ ($< \alpha$).

A space X is called $\aleph_0$-*N-refinable* if every open cover of X has an open refinement which is $\aleph_0$-Noetherian and is of rank ω with respect to itself.

4.7 Theorem. *Let X be a zero-dimensional $\aleph_0$-N-refinable l.w.p. space. Then X is t.w.p.*

Proof: Let $\mathcal{V}$ be a clopen cover of X consisting of t.w.p. subspaces. Let $\mathcal{C}$ be an open refinement of $\mathcal{V}$ which is $\aleph_0$-Noetherian and of rank ω with respect to itself. Let $x \in X$. Let $E_0 = C_0 \in \mathcal{C}$ where $x \in C_0$. Put $E_1 = St(E_0, \mathcal{C}), \dots, E_{n+1} = St(E_n, \mathcal{C}), \dots$, and $E_x = \bigcup_{n<\omega} E_n$.

Claims.

(1) There exists a sequence $\{ C_n : n < \omega \}$ in $\mathcal{C}$ such that $E_x = \bigcup_{n<\omega} C_n$.
(2) E_x is t.w.p.
(3) Let $z \in E_x$, and let $C_1, \dots, C_k \in \mathcal{C}$ such that $z \in C_1$ and $C_i \cap C_{i+1} \neq \emptyset$ for every $i = 1, \dots, k-1$. Then, $C_k \subset E_x$.
(4) If $y \in E_x$, then $E_y \subset E_x$.
(5) If $x, y \in X$ are distinct points, then either $E_x \cap E_y = \emptyset$, or $E_x = E_y$.

Condition (1) holds because $\mathcal{C}$ is $\aleph_0$-Noetherian of rank ω with respect to itself. Now, we prove condition (2): For each $n < \omega$ let $V_n \in \mathcal{V}$ be such that $C_n \subset V_n$. Then $E_x = \bigcup_{n<\omega} C_n \subset \bigcup_{n<\omega} V_n = V$. E_x is open, and V is t.w.p. as the countable union of t.w.p. clopen subsets of X. So E_x is t.w.p.

Claim (3) implies (4), and this one implies condition (5). Let us prove (3). If $z \in E_x$, then there is $n < \omega$ such that $z \in E_n$. Therefore, $A_k \subset \bigcup \{ E_i : n \leq i \leq n+k \} \subset E_x$.

Let $\sim$ be the equivalence relation defined by the rule: $x \sim y$ if and only if $E_x = E_y$. Let Y be a subset of X containing exactly one element of each class of equivalence. Then $X = \bigoplus \{ E_y : y \in Y \}$. Therefore, X is t.w.p. $\square$

4.8 Definition. A base $\mathcal{B}$ of a space X is called an *ortho-base* if for every $\mathcal{B}' \subset \mathcal{B}$, either $\bigcap \mathcal{B}'$ is open, or for some $x \in X$, $\bigcap \mathcal{B}' = \{x\}$ and $\mathcal{B}'$ is a base of neighborhoods of x. A space X is called *proto-metrizable* if it is paracompact and has an ortho-base.

The class of proto-metrizable spaces contains all non-archimedean spaces and all metrizable spaces.

4.9 Theorem. *Let X be a zero-dimensional proto-metrizable space with $\chi(x, X) > \omega$ for every $x \in X$. If X is l.w.p., then X is t.w.p.*

Proof: Let $\mathcal{B}$ be an ortho-base for X, and let $\mathcal{C}$ be a clopen cover of X where each $C \in \mathcal{C}$ is t.w.p. We take a countable collection $\{ \mathcal{D}_n : n < \omega \}$ where $\mathcal{D}_{n+1}$ is a subcollection of $\mathcal{B}$ which is a star refinement of $\mathcal{D}_n$, and $\mathcal{D}_0 \subset \mathcal{B}$ is a star refinement of $\mathcal{C}$. For each $x \in X$, put $A_x = \bigcap_{n<\omega} St(x, \mathcal{D}_n)$. Since $\mathcal{B}$ is an ortho-base and $\chi(x, X) > \omega$ for every $x \in X$, each A_x is an open set. Let $\mathcal{A} = \{ A_x : x \in X \}$. Fix a point $x \in X$. Consider the sequence $E_0 = St(x, \mathcal{A})$, $E_1 = St(E_0, \mathcal{A}), \dots, E_{n+1} = St(E_n, \mathcal{A}), \dots$. For every $n < \omega$ there exist $D_n \in \mathcal{D}_n$ and $C_n \in \mathcal{C}$ such that $E_n \subset D_n \subset C_n$. Of course, each E_n is open. Besides, $E_x = \bigcup_{n<\omega} E_n$ is an open subset of $\bigcup_{n<\omega} C_n = C$, and C is t.w.p., because it is a countable union of clopen sets. Hence, E_x is t.w.p.

Reasoning as in the proof of Theorem 4.9, we can show the following

Claims.

(1) Let $z \in E_x$, and let $A_1, \dots, A_k \in \mathcal{A}$ such that $z \in A_1$ and $A_i \cap A_{i+1} \neq \emptyset$ for every $i = 1, \dots, k-1$. Then $A_k \subset E_x$.
(2) If $y \in E_x$, then $E_y \subset E_x$.
(3) Let $x, y \in X$. If $E_x \cap E_y \neq \emptyset$, then $E_x = E_y$.

Thus, X is the free topological sum of some of its E_x subsets. We conclude that X is t.w.p. □

Problem. Is restriction on the character necessary in Theorem 4.9?

4.10 Problems. 1. Is every l.w.p. zero-dimensional metrizable space t.w.p.?

2. Is every l.w.p. zero-dimensional paracompact space t.w.p.?

3. Is every l.w.p. zero-dimensional metacompact developable space t.w.p.?

4. Is every l.w.p. zero-dimensional spaces with a Noetherian base of subinfinite rank t.w.p.?

5. Is every l.w.p. zero-dimensional space with a base of rank 2 t.w.p.?

4.11 Problems. 1. Let X be a non-archimedean space. Let Y be a zero-dimensional metrizable space, and $f\colon X \to Y$ a perfect mapping onto Y. Is Y t.w.p. if X is t.w.p.?

2. Let X be a non-archimedean space. Let Y be a zero-dimensional metrizable space, and $f\colon X \to Y$ a perfect mapping onto Y. Is X t.w.p. if Y is t.w.p.?

5. Some examples

We have already proved the following:

5.1 Theorem. *A non-Lindelöf space X is w.p. if and only if it is homeomorphic to a G_δ-dense subspace of a locally pseudocompact space.*

So, if $\mathcal{P}$ is a topological property such that compactness implies $\mathcal{P}$ and $\mathcal{P}$ implies pseudocompactness, then the following are equivalent for a non-Lindelöf space X.

(1) X is w.p.
(2) X can be embedded as a G_δ-dense subspace of a space satisfying $\mathcal{P}$ locally.

An example of a property $\mathcal{P}$ between compactness and pseudocompactness is initial α-compactness (every open cover of cardinality $\leq \alpha$ has a finite subcover). In particular, countable compactness is such a property. Recall that a space X is *countably compact* if every countable subset of X has a limit point. These remarks motivate the following concepts.

5.2 Definition. Let α be a cardinal.

A space X is *α-w.p.* if it can be G_δ-densely embedded into a Tychonoff space Y in which each subset of cardinality α has a limit point.

A space X is *completely-α-w.p.* if it can be G_δ-densely embedded into a Tychonoff space Y in which each subset of cardinality α has a complete accumulation point.

A space X is *relatively-α-w.p.* if it can be G_δ-densely embedded into a Tychonoff space Y so that every set of cardinality α contained in X has a limit point in Y.

A space X is *completely-relatively-α-w.p.* if it can be G_δ-densely embedded into a Tychonoff space Y so that every set of cardinality α contained in X has a complete accumulation point in Y.

5.3 Definition. The *degree of w.p.* (of c.w.p., r.w.p., c.r.w.p) of a space X is defined as the minimum of cardinals α such that X is α-w.p. (completely-α-w.p., relatively-α-w.p., completely-relatively-α-w.p.); we denote these by $wp(X)$, $cwp(X)$, $rwp(X)$, and $rcwp(X)$.

Observe that a space X is ω-w.p. iff X is w.p. Obviously, if X is α-w.p. (relatively-α-w.p.) and $\gamma > \alpha$, then X is γ-w.p. (relatively-γ-w.p.). Also, completely-α-w.p. implies α-w.p., and α-w.p. implies relatively-α-w.p. Finally, $rwp(X) \leq wp(X) \leq e(X)^+ \leq l(X)^+$, $wp(X) \leq cwp(X) \leq |X|^+$, and $rwp(X) \leq rcwp(X) \leq cwp(X)$, where $e(X)$ and $l(X)$ are the extent and the Lindelöf number of X. Thus, every noncompact Lindelöf space X satisfies $wp(X) = \omega_1$. The next result guarantees in particular that for every Lindelöf space X, $wp(X) = rwp(X)$.

5.4 Theorem. *A space X is w.p. if an only if it is ω-r.w.p.*

Proof: Let Y be a space such that X is G_δ-dense in Y and every countable subset of X has a limit point in Y. We are going to prove that Y is a pseudocompact space.

Let $f\colon Y \to \mathbb{R}$ be a continuous function. If f is unbounded, then f is unbounded on X, and for every $n \in \mathbb{N}$ there is a point $x_n \in X$ such that $|f(x)| \geq n$. The set $\{ x_n : n \in \mathbb{N} \}$ is infinite, so by the hypothesis it has a limit point $y \in Y$. This is a contradiction, since by the continuity of f we then must have that $|f(x_n)| \leq |f(y)| + 1$ for infinitely many $n \in \omega$.

Thus, Y is G_δ-dense in βY; since X is G_δ-dense in Y, it is also G_δ-dense in βY. □

Remark. The spaces Y that have a dense subspace X with the property that every infinite subset of X has a limit point in Y are called *relatively countably com-*

pact; the above argument proves a well-known fact that every relatively countably compact space is pseudocompact.

5.5 Example. The Moore-Niemytsky Plane N satisfies $cwp(X) = wp(X) = rwp(X) = rcwp(X) = \omega_1$.

Proof: Consider the following subsets of N: $N_0 = \{ (x,y) \in \mathbb{R}^2 : y > 0\}$ and $N_1 = \{ (x,0) : x \in \mathbb{R} \}$. The space N is not w.p., because the subspace $N_0 \cup \{ (r,0) : r \in \mathbb{Q} \}$ is an open Lindelöf non-locally compact subspace of N_0 (recall that every open set of a w.p. space is t.w.p.). Therefore, $\omega < rwp(N) \leq \min\{wp(N), rcwp(X)\} \leq \max\{wp(N), rcwp(X)\} \leq cwp(N)$. We will finish the proof by showing that $cwp(X) \leq \omega_1$. Let $Z = N \cup \{a\}$ where $a \notin N$. We consider the topology τ in Z generated by all open subsets of N and all the sets of the form $\{a\} \cup V$ where V is an open subset of N such that $|N_1 \setminus V| < \omega$. The space (Z, τ) is Tychonoff, every of its subset of cardinality ω_1 has a complete accumulation point, and N is G_δ-dense in Z. □

In order to prove the assertion in the following example we need some results and definitions. The first definition and theorem can be found in [CN, pp. 286–288].

5.6 Definition. Let $\omega \leq \kappa \leq \alpha$. A family $\mathcal{C}$ of subsets of α is a *κ-almost disjoint family* on α if $|C| \geq \kappa$ for all $C \in \mathcal{C}$, and $|C_0 \cap C_1| < \kappa$ for all distinct $C_0, C_1 \in \mathcal{C}$.

An *ω-almost disjoint family* is called simply an *almost disjoint family.*

We denote by $S(\alpha, \kappa)$ the smallest cardinal β such that there is no κ-almost disjoint family on α of cardinality β.

For two cardinals α and κ, we denote by

$$\alpha^{<\kappa} = \sup\{ \alpha^\lambda : \lambda < \kappa \}.$$

5.7 Theorem. *Let κ be an infinite cardinal and $\alpha \geq 2$. Then $S(\alpha^{<\kappa}, \kappa) = (\alpha^\kappa)^+$.*

The following result is a variation of the last theorem.

5.8 Lemma. *Let γ, κ and α be cardinals such that $\omega \leq \gamma \leq \kappa \leq \alpha^{<\gamma}$. Then there exists a family $\mathcal{C}$ of subsets of $\alpha^{<\gamma}$ such that*

(1) *$|C| = \kappa$ for every $C \in \mathcal{C}$,*
(2) *$|C_1 \cap C_2| < \gamma$ for all distinct $C_1, C_2 \in \mathcal{C}$,*
(3) *if $D \subset \alpha^{<\gamma}$ and $|D| = \kappa$, then there is $C \in \mathcal{C}$ such that $|C \cap D| \geq \gamma$,*
(4) *$\alpha^{<\gamma} \leq |\mathcal{C}| \leq \alpha^\gamma$.*

Proof: Let $\mathcal{D}$ be a partition of $\alpha^{<\gamma}$ into subsets of cardinality κ with $|\mathcal{D}| = \alpha^{<\gamma}$. Let Υ be the collection of all families of subsets of $\alpha^{<\gamma}$ satisfying (1) and (2) and containing $\mathcal{D}$. Consider Υ with the order defined by inclusion. By Zorn Lemma, there exists a family $\mathcal{C}$ containing $\mathcal{D}$ and satisfying (1), (2) and (3). Besides, $\mathcal{C}$ must satisfy (4) because of Theorem 5.7 and the fact that $\mathcal{D} \subset \mathcal{C}$ and $|\mathcal{D}| = \alpha^{<\gamma}$.

Now assume that $\alpha = \alpha^\gamma$. In this case we can take the partition $\mathcal{D}$ of cardinality α; the collection $\mathcal{C}$ will satisfy $\alpha \leq |\mathcal{C}| \leq \alpha^\gamma = \alpha$.

□

5.9 Lemma. *Let γ, κ and α be cardinals such that $\omega \le \gamma \le \kappa \le \alpha = \alpha^\gamma$. Then there exists a family $\mathcal{C}$ of subsets of α such that*

(1) *$|C| = \kappa$ for every $C \in \mathcal{C}$,*
(2) *$|C_1 \cap C_2| < \gamma$ for all distinct $C_1, C_2 \in \mathcal{C}$,*
(3) *if $D \subset \alpha$ and $|D| = \gamma$, then there is $C \in \mathcal{C}$ such that $|C \cap D| = \gamma$.*

Proof: Let $\mathcal{D}$ be a family of subsets of α satisfying (1)–(3) in Lemma 5.8 and of cardinality α^γ. Let $\mathcal{E}$ be the set of all subsets of α of cardinality equal to γ. We have then $|\mathcal{E}| = \alpha^\gamma$. Enumerate faithfully $\mathcal{D}$ and $\mathcal{E}$:

$$\mathcal{D} = \{ D_\lambda : \lambda < \alpha^\gamma \},$$
$$\mathcal{E} = \{ E_\lambda : \lambda < \alpha^\gamma \}.$$

We will construct the family $\mathcal{C}$ by an inductive process. If for every $E \in \mathcal{E}$ there is $D \in \mathcal{D}$ such that $|E \cap D| = \gamma$, then $\mathcal{D}$ satisfies all the requirements. If not, let ξ_0 be the first ordinal such that for every $D \in \mathcal{D}$, $|D \cap E_{\xi_0}| < \gamma$. Put $\tilde{D}_0 = D_0 \cup E_{\xi_0}$. Assume that we have already constructed the sets E_{ξ_λ} and $\tilde{D}_\lambda$ for every $\lambda < \eta < \alpha^\gamma$. If for every $\delta \ge \sup\{ \xi_\lambda : \lambda < \eta \}$ there is a $D \in \{ \tilde{D}_\lambda : \lambda < \eta \} \cup \{ D_\lambda : \eta \le \lambda < \alpha^\gamma \}$ such that $|E_\delta \cap D| \ge \gamma$, then we put $\mathcal{C} = \{ \tilde{D}_\lambda : \lambda < \eta \} \cup \{ D_\lambda : \eta \le \lambda < \alpha^\gamma \}$. Otherwise, let ξ_η be the first ordinal greater or equal to $\sup\{ \xi_\lambda : \lambda < \eta \}$ such that for every $D \in \{ \tilde{D}_\lambda : \lambda < \eta \} \cup \{ D_\lambda : \eta \le \lambda < \alpha^\gamma \}$, $|E_{\xi_\eta} \cap D| < \gamma$. Let $\tilde{D}_\eta = D_\eta \cup E_{\xi_\eta}$. This process will stop after at most α^γ steps. □

In the following example we answer affirmatively a question posed in [Eck].

5.10 Example. Let α and κ be two cardinals such that $\omega \le \kappa \le \alpha$. Let $D(\alpha)$ be the discrete space of cardinality α, and let $A_\kappa(\alpha) = D(\alpha) \cup \{o\}$, where $o \notin D(\alpha)$, be the space with the topology generated by all the subsets of $D(\alpha)$ and all the sets of the form $\{o\} \cup V$ where $V \subset D(\alpha)$ and $|\alpha \setminus V| \le \kappa$. Then we have:

if $\kappa = \omega$, then $A_\kappa(\alpha)$ is a Lindelöf non-locally compact space;

if $\kappa = \alpha$, then $A_\kappa(\alpha)$ is a discrete space of cardinality α; and

if $\omega < \kappa < \alpha$, and $\kappa^\omega = \kappa$ then $A_\kappa(\alpha)$ is weakly pseudocompact.

Proof: The first two assertions are trivial. We will prove the third one by cases.

Case 1. Assume that $\alpha = \alpha^\omega$. By Lemma 5.9, there is an almost disjoint family $\mathcal{C}$ of subsets of α such that $|C| = \omega_1$ for every $C \in \mathcal{C}$, $|C_1 \cap C_2| < \omega$ for different $C_1, C_2 \in \mathcal{C}$, and for every infinite $D \subset \alpha$, $D \cap C$ is infinite for some $C \in \mathcal{C}$.

For each $C \in \mathcal{C}$, let e_C be an element not belonging to $A_\kappa(\alpha)$ and such that $e_C \ne e_D$ if $C, D \in \mathcal{C}$ and $C \ne D$. Consider the space $Z = Z_\kappa(\alpha) = A_\kappa(\alpha) \cup Y$ where $Y = \{ e_C : C \in \mathcal{C} \}$, with the topology generated by all the subsets of $D(\alpha)$, the sets of the form $e_C \cup V$ where $V \subset C$ and $|C \setminus V| < \omega$, and the sets of the form $\{o\} \cup V \cup W$ where $V \subset D(\alpha)$ and $|D(\alpha) \setminus V| \le \kappa$, and $W \subset Y$ such that

$e_C \in W$ iff $|C \setminus V| < \omega$. It is not difficult to prove that these sets form a base of a Hausdorff topology τ on Z. Besides, $A_\kappa(\alpha)$ is a G_δ-dense subspace of Z, and every subset of $A_\kappa(\alpha)$ of cardinality ω has a limit point in Z because of the properties of $\mathcal{C}$. By Theorem 5.4, to finish the proof it is enough to show that Z is a Tychonoff space. We are going to prove that in fact, Z is a zero-dimensional space.

In order to do this it is enough to verify that every standard neighborhood of o contains a clopen subset that contains o. Let $O = \{o\} \cup V \cup W$ be a standard neighborhood of o as described above. Let $V_0 = D(\alpha) \setminus V$. V_0 has cardinality $\leq \kappa$. Put

$$\mathcal{C}_0 = \{\, C \in \mathcal{C} : |C \cap V_0| \geq \omega\},$$
$$\mathcal{D}_0 = \{\, C \cap V_0 : C \in \mathcal{C}_0 \,\}$$

and

$$W_0 = \{\, e_C : C \in \mathcal{C}_0 \,\}.$$

Then $|\mathcal{D}_0| = |\mathcal{C}_0|$, because if $C, D \in \mathcal{C}_0$ and $C \neq D$, then $|C \cap D| < \omega$ and $\min\{\, |C \cap V_0|, |D \cap V_0| \,\} \geq \omega$. Hence, $C \cap V_0$ and $D \cap V_0$ must be different. Besides, $|W_0| = |\mathcal{C}_0|$, $|V_0| \leq \kappa$, and $\mathcal{D}_0$ is an almost disjoint family in V_0. Thus, $|W_0| \leq \kappa^\omega = \kappa$. Consider $V_1 = V_0 \cup \bigcup \mathcal{C}_0$. The cardinality of V_1 is less or equal to $\kappa^\omega = \kappa$. Assume that we have constructed sequences $\{\, V_\lambda : \lambda < \eta \,\}$, $\{\, W_\lambda : \lambda < \eta \,\}$, $\{\, \mathcal{C}_\lambda : \lambda < \eta \,\}$ and $\{\, \mathcal{D}_\lambda : \lambda < \eta \,\}$ such that

$$\begin{aligned} V_\lambda &= \bigcup \{\, V_\gamma : \gamma < \lambda \,\} \qquad \text{if } \lambda \text{ is a limit ordinal, and} \\ V_\lambda &= V_\gamma \cup \bigcup \mathcal{C}_\gamma \qquad \text{if } \lambda = \gamma + 1, \\ \mathcal{C}_\lambda &= \{\, C \in \mathcal{C} : |C \cap V_\lambda| \geq \omega \,\}, \\ \mathcal{D}_\lambda &= \{\, C \cap V_\lambda : C \in \mathcal{C}_\lambda \,\}, \qquad \text{and} \\ W_\lambda &= \{\, e_C : C \in \mathcal{C}_\lambda \,\}. \end{aligned}$$

These sequences have the following properties.

1) $V_\lambda \subset \alpha$, $W_\lambda \subset Y$, and $\mathcal{C}_\lambda \subset \mathcal{C}$ for all $\lambda < \eta$;
2) $\mathcal{D}_\lambda$ is an almost disjoint family on V_λ for every $\lambda < \eta$;
3) $|V_\lambda| = |W_\lambda| = |\mathcal{C}_\lambda| = |\mathcal{D}_\lambda| \leq \kappa^\omega = \kappa$ for every $0 < \lambda < \eta$, and $|V_0| \leq |W_0| = |\mathcal{C}_0| \leq \kappa^\omega = \kappa$;
4) if $\xi < \lambda < \eta$, then $V_\xi \subset V_\lambda$, $W_\xi \subset W_\lambda$, and $\mathcal{C}_\xi \subset \mathcal{C}_\lambda$;

If $\eta < \kappa^+$, we can continue this process by putting $V_\eta = \bigcup_{\lambda<\eta} V_\lambda$ if η is a limit ordinal, and $V_\eta = V_\zeta \cup \bigcup \mathcal{C}_\zeta$ if $\eta = \zeta + 1$, and

$$\mathcal{C}_\eta = \{\, C \in \mathcal{C} : |C \cap V_\eta| \geq \omega \,\},$$
$$\mathcal{D}_\eta = \{\, C \cap V_\eta : C \in \mathcal{C}_\eta \,\}$$

and $W_\eta = \{ e_C : C \in \mathcal{C}_\eta \}$. It is not difficult to prove, as was done when $\eta = 0$ and using the fact that $\eta < \kappa^+$, that the new families $\{ V_\lambda : \lambda \leq \eta \}$, $\{ W_\lambda : \lambda \leq \eta \}$, $\{ \mathcal{C}_\lambda : \lambda \leq \eta \}$ and $\{ \mathcal{D}_\lambda : \lambda \leq \eta \}$ satisfy the properties 1), 2), 3) and 4) above.

Continue this inductive construction to ω_1, and put $\tilde{V} = \bigcup \{ V_\lambda : \lambda < \omega_1 \}$, $\tilde{W} = \bigcup \{ V_\lambda : \lambda < \omega_1 \}$ and $F = F(V_0) = \tilde{V} \cup \tilde{W}$. We claim that $Z \setminus F$ is a clopen neighborhood of o contained in O.

Indeed, suppose $e_C \in F$. Then there is $\lambda < \omega_1$ such that $e_C \in W_\lambda$. The union $\{e_C\} \cup C$ is an open set containing e_C and contained in $V_{\lambda+1} \cup W_{\lambda+1} \subset F$. Besides, $o \notin F$. Thus, F is an open set. Now suppose $e_B \in \operatorname{cl}_Z F$. This means that $|B \cap \tilde{V}| \geq \omega$. Let E be an infinite countable subset of $B \cap \tilde{V}$. There is a $\lambda < \omega_1$ such that $E \subset V_\lambda$. The collection $\{ C \cap V_\lambda : C \in \mathcal{C}_{\lambda+1} \}$ is a maximal almost disjoint family in V_λ, so there exist $D \in \mathcal{C}_{\lambda+1}$ such that $|B \cap D| \geq |E \cap D| \geq \omega$. This means that $e_D \in W_{\lambda+1} \subset F$ is a limit point for D. But this is possible only if $e_B = e_D$. Thus, $e_B \in F$. Besides, $o \notin \operatorname{cl}_Z F$, because $|\tilde{V}| \leq \kappa^\omega \cdot \omega_1 = \kappa$, and, as we have already seen, if $e_C \notin \tilde{W}$, then $|C \cap \tilde{V}| < \aleph_0$. Thus, $Z \setminus F$ is a clopen subset of Z containing o and contained in O. Therefore, Z is zero-dimensional.

Case 2. Assume only that $\alpha > \omega$.

Let $\beta = 2^\alpha$. Then $\beta^\omega = \beta$. Let $Z = Z_\kappa(\beta)$ as constructed in the proof in Case 1. Then Z is zero-dimensional, hence Tychonoff.

For every subset K of $D(\beta)$ of cardinality κ we can take the clopen subset $F(K)$ of $A_\kappa(\beta)$ constructed from K as in the proof in Case 1. Let K_0 be a subset of $D(\beta)$ of cardinality κ. Then $|F(K_0) \cap D(\beta)| < \beta$, so we can take $K_1 \subset D(\beta) \setminus F(K_0)$ of cardinality κ. In this manner we can construct two α-sequences $\{ K_\lambda : \lambda < \alpha \}$ and $\{ F(K_\lambda) : \lambda < \alpha \}$ such that if $\lambda < \xi$, then $K_\xi \subset D(\beta) \setminus F(K_\lambda)$. Let $F = \bigcup \{ F(K_\lambda) : \lambda < \alpha \}$. The set F is open in $A_\kappa(\beta)$.

Claim 1. $(F \cap D(\beta)) \cup \{o\}$ is G_δ-dense in $\operatorname{cl}_Z F$.

Indeed, suppose $p \in \operatorname{cl}_Z F \cap \bigcap_{n<\omega} A_n$ where A_n is open in Z. Assume that $p \notin (F \cap D(\beta)) \cup \{o\}$. We can assume that $A_n \subset A_m$ if $n > m$ and $o \notin A_n$ for all $n < \omega$. Then for each $n < \omega$, $B_n = A_n \cap \operatorname{cl}_Z F$ is open in Z, and $p \in B_n$ for every $n < \omega$. Hence, $\bigcap_{n<\omega} B_n$ is a G_δ-set in Z. Then $\bigcap_{n<\omega} B_n \cap D(\beta) \neq \emptyset$. But $B_n \subset \operatorname{cl}_Z F$ for every $n < \omega$, so if $z \in \bigcap_{n<\omega} B_n \cap D(\beta)$, then $z \in F \cap D(\beta)$.

Claim 2. Every countable subset of $(F \cap D(\beta)) \cup \{o\}$ has an accumulation point in $\operatorname{cl}_Z F$.

In fact, this is trivial, because if N is a countable infinite subset of $(F \cap D(\beta)) \cup \{o\}$, then $N \setminus \{o\} \subset F$ has a limit point in Z, and it is clear that this point must be in $\operatorname{cl}_Z F$.

Thus, $\operatorname{cl}_Z F$ is relatively countably compact, and therefore pseudocompact. Since $(F \cap D(\beta)) \cup \{o\}$ is G_δ-dense in $\operatorname{cl}_Z F$, $(F \cap D(\beta)) \cup \{o\}$ is weakly pseudocompact.

Claim 3. $A_\kappa(\alpha)$ is homeomorphic to $(F \cap D(\beta)) \cup \{o\}$.

Indeed, $F \cap D(\beta)$ is a discrete space of cardinality α, and V is a neighborhood of o in $(F \cap D(\beta)) \cup \{o\}$ if and only if $o \in V$ and $|(F \cap D(\beta)) \setminus V| \leq \kappa$. $\square$

5.11 Problems.

1. Is $A_\kappa(\alpha)$ w.p. without any restriction on uncountable κ and α?
2. What are $rcw(X)$, $wp(X)$, $rcwp(X)$ and $cwp(X)$ when X is
   (1) the Michael line?
   (2) the square of the Sorgenfrey line?
   (3) $X = V_\gamma(A_\kappa(\alpha))$, the quotient spaces obtained by identifying the points (λ, o), $\lambda < \alpha$, in the space $D(\gamma) \times A_\kappa(\alpha)$ to a single point?
3. Find examples of spaces X, Y and Z such that $rwp(X) < wp(X)$, $wp(Y) < cwp(Y)$ and $rcwp(Z) < cwp(Z)$.

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