On the continuity of the pressure for monotonic mod one transformations

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Abstract. If $f : [0,1] \to \mathbb{R}$ is strictly increasing and continuous define $T_f x = f(x) \pmod 1$. A transformation $\tilde{T} : [0,1] \to [0,1]$ is called ε-close to T_f, if $\tilde{T}x = \tilde{f}(x) \pmod 1$ for a strictly increasing and continuous function $\tilde{f} : [0,1] \to \mathbb{R}$ with $\|\tilde{f} - f\|_\infty < \varepsilon$. It is proved that the topological pressure $p(T_f, g)$ is lower semi-continuous, and an upper bound for the jumps up is given. Furthermore the continuity of the maximal measure is shown, if a certain condition is satisfied. Then it is proved that the topological pressure is upper semi-continuous for every continuous function $g : [0,1] \to \mathbb{R}$, if and only if 0 is not periodic or 1 is not periodic. Finally it is shown that the topological entropy is continuous, if $h_{\mathrm{top}}(T_f) > 0$.

Keywords: mod one transformation, topological pressure, topological entropy, maximal measure, perturbation

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Introduction

Consider a strictly increasing and continuous function $f : [0,1] \to \mathbb{R}$, and define $T_f x := f(x) \pmod 1$. Then the map $T_f : [0,1] \to [0,1]$ is piecewise monotonic. A monotonic mod one transformation $\tilde{T} : [0,1] \to [0,1]$ is called ε-close to T_f, if there exists a strictly increasing and continuous function $\tilde{f} : [0,1] \to \mathbb{R}$, such that $\tilde{T}x = \tilde{f}(x) \pmod 1$ and $\|\tilde{f} - f\|_\infty < \varepsilon$. We investigate the influence of small perturbations of T_f on the topological pressure $p(T_f, g)$ and the topological entropy $h_{\mathrm{top}}(T_f)$.

Perturbations of piecewise monotonic maps have been considered in many papers, e.g. in [1], [4], [5], [7], [9], [10], [11] and [13]. The topology considered in these papers is the R^0-topology. A piecewise monotonic map $\tilde{T}$ is said to be close to T in the R^0-topology, if $\tilde{T}$ and T have the same number of intervals of monotonicity and the graph of $\tilde{T}$ is contained in a small neighbourhood of the graph of T considered as subsets of $\mathbb{R}^2$ (a more detailed description is given in Section 1). In particular, if $\tilde{T}$ is close to T in the R^0-topology, then $\tilde{T}$ cannot have more intervals of monotonicity than T. On the other hand, if $\|\tilde{f} - f\|_\infty$ is small, then $T_{\tilde{f}}x := \tilde{f}(x) \pmod 1$ can have up to two intervals of monotonicity more than T_f (see e.g. the example given in (3.1) of this paper). Hence the results of the papers mentioned above are not applicable in our situation. A certain kind of perturbations allowing the number of intervals of monotonicity to increase is considered e.g. in [8], but also these results are not applicable in our situation.

Next we describe results known for the R^0-topology. For a general piecewise monotonic map $T : [0,1] \to [0,1]$ the lower semi-continuity of the topological pressure is treated in Theorem 1 of [7]. Upper bounds for the jumps up of the topological pressure are given in Theorem 2 of [7]. For the special case of the topological entropy this result has been earlier obtained in Theorem 2 of [5]. The continuity of the maximal measure is investigated in Theorem 3 of [10] and Theorem 1 of [12]. In [13] conditions are given, which are equivalent to the upper semi-continuity of the pressure for all continuous functions $g : [0,1] \to \mathbb{R}$. For the special case of a monotonic mod one transformation T_f it is proved in Theorem 1 of [9] that the topological entropy is continuous at T_f, if $h_{\mathrm{top}}(T_f) > 0$.

We will see that similar results hold also in our situation. The example given in (3.1) shows that we have to modify these results. In order to obtain continuity results for the pressure we prove Lemma 4, which is as important in our proofs as Lemma 6 of [7] is in the proofs of [7] and [10]. Using Lemma 4 we obtain in Theorem 1 a result on the lower semi-continuity of the topological pressure and on upper bounds of the jumps up for the topological pressure. The continuity of the maximal measure is treated in Theorem 2. In Theorem 3 we prove that the upper semi-continuity of the topological pressure for every continuous function $g : [0,1] \to \mathbb{R}$ is equivalent to $\lim_{x\to 0^+} {T_f}^n x \neq 0$ for all $n \in \mathbb{N}$ or $\lim_{x\to 1^-} {T_f}^n x \neq 1$ for all $n \in \mathbb{N}$. Finally in Theorem 4 conditions equivalent to the continuity of the topological entropy are given. This result implies Corollary 4.1, which states that the topological entropy is continuous at T_f, if $h_{\mathrm{top}}(T_f) > 0$.

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1. Monotonic mod one transformations

Assume that X is a finite union of closed intervals. We call $\mathcal{Z}$ a *finite partition* of X, if $\mathcal{Z}$ consists of finitely many pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = X$. A map $T : X \to \mathbb{R}$ is called *piecewise monotone*, if there exists a finite partition $\mathcal{Z}$ of X, such that $T|_Z$ is strictly monotone and continuous for all $Z \in \mathcal{Z}$. If $f : [0,1] \to \mathbb{R}$ is a strictly increasing and continuous function, then define $T_f : [0,1] \to [0,1]$ in the following way. For $x \in [0,1)$ set $T_f x := f(x) - [f(x)]$, where $[y]$ denotes the largest integer smaller than or equal to y. Furthermore set $T_f 1 := \lim_{x\to 1^-} T_f x$. Let $\mathcal{Z}_f$ be the set of all maximal open subintervals of $[0,1] \setminus f^{-1}(\mathbb{Z})$. Obviously $T_f : [0,1] \to [0,1]$ is a piecewise monotonic map with respect to the finite partition $\mathcal{Z}_f$ of $[0,1]$. We call T a *monotonic mod one transformation*, if there exists a strictly increasing and continuous function $f : [0,1] \to \mathbb{R}$ with $T = T_f$. If $x \in [0,1)$ and $n \in \mathbb{N}_0$, then let ${T_f}^n x$ be the n-th iterate of T_f evaluated at x. Note that ${T_f}^n 0 := \lim_{x\to 0^+} {T_f}^n x$. For $n \in \mathbb{N}_0$ we define ${T_f}^n 1 := \lim_{x\to 1^-} {T_f}^n x$. Observe that it may happen that $T_f({T_f}^{n-1} 1) \neq$

${T_f}^n 1$.

In [9] monotonic mod one transformations are investigated under small perturbations with respect to the R^0-topology. The R^0-topology considered in [7], [9], [10] and [13] is equivalent to the following topology on piecewise monotonic maps. Assume that X and $\tilde{X}$ are finite unions of closed intervals. Let $T : X \to \mathbb{R}$ be piecewise monotonic with respect to $\mathcal{Z}$ and $\tilde{T} : \tilde{X} \to \mathbb{R}$ be piecewise monotonic with respect to $\tilde{\mathcal{Z}}$. Suppose that $\mathcal{Z} = \{Z_1, Z_2, \dots, Z_K\}$ with $Z_1 < Z_2 < \cdots < Z_K$, and $\tilde{\mathcal{Z}} = \{\tilde{Z}_1, \tilde{Z}_2, \dots, \tilde{Z}_{\tilde{K}}\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \cdots < \tilde{Z}_{\tilde{K}}$. Then we say that $(\tilde{T}, \tilde{\mathcal{Z}})$ is *ε-close* to $(T, \mathcal{Z})$ in the *R^0-topology*, if $\operatorname{card} \tilde{\mathcal{Z}} = \operatorname{card} \mathcal{Z}$ and for every $j \in \{1, 2, \dots, K\}$ the graph of $\tilde{T}|_{\tilde{Z}_j}$ is contained in an ε-neighbourhood of the graph of $T|_{Z_j}$ considered as a subset of $\mathbb{R}^2$. In particular this definition implies that T and $\tilde{T}$ have the same number of intervals of monotonicity.

Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function. We are interested in the dynamics of $T_{\tilde{f}}$, if $\tilde{f} : [0,1] \to \mathbb{R}$ is a strictly increasing and continuous function, and $\|\tilde{f} - f\|_\infty$ is sufficiently small. Unfortunately this does not imply that $T_{\tilde{f}}$ is close to T_f in the R^0-topology. For example, suppose that $f(0) = 0$ and $\tilde{f}(x) = f(x) - \varepsilon$. Then $\|\tilde{f} - f\|_\infty = \varepsilon$. The interval $\bigl(0, f^{-1}(\varepsilon)\bigr)$ is an interval of monotonicity for the map $T_{\tilde{f}}$. Hence $T_{\tilde{f}}$ has more intervals of monotonicity than T_f, and therefore $T_{\tilde{f}}$ is not close to T_f in the R^0-topology.

Observe that the following fact is true, if $T_f 0 \neq 0$ and $T_f 1 \neq 1$ (as above f and $\tilde{f}$ are strictly increasing and continuous functions $[0,1] \to \mathbb{R}$). For every $\varepsilon > 0$ there exists a $\delta > 0$, such that $\|\tilde{f} - f\|_\infty < \delta$ implies $(T_{\tilde{f}}, \mathcal{Z}_{\tilde{f}})$ is ε-close to $(T_f, \mathcal{Z}_f)$ in the R^0-topology.

We can use a standard doubling points construction as in [7] (cf. also [9]) in order to get a dynamical system (a dynamical system is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a continuous map). In fact we would have to do it for the exact definitions of the topological entropy, the pressure (see [9] for details) and the maximal measure (see [10] for details). For our purpose it is enough to replace each $c \in f^{-1}(\mathbb{Z}) \setminus \{0, 1\}$ by c^- and c^+, and define ${T_f}^n c^- := {T_f}^{n-1} 1$ and ${T_f}^n c^+ := {T_f}^{n-1} 0$ for $n \in \mathbb{N}$. Nevertheless it is important to notice that this doubling points construction can be done with respect to a finite partition $\mathcal{Y}$ of [0,1] refining $\mathcal{Z}_f$. By Lemma 2 in [6] the definition of the pressure does not depend on the partition $\mathcal{Y}$. Hence the topological pressure can be defined for a function $g : [0,1] \to \mathbb{R}$, for which there exists a finite partition $\mathcal{Y}$ of $[0,1]$, such that for every $Y \in \mathcal{Y}$ the function $g|_Y$ can be extended to a continuous function on the closure of Y. In particular the topological pressure can be defined for piecewise constant functions (the definition of the notion piecewise constant function will be given later).

Consider a continuous map $T : X \to X$, where (X, d) is a compact metric space. For $\varepsilon > 0$ and $n \in \mathbb{N}$ a set $E \subseteq X$ is called (n, ε)-separated, if for every

$x \neq y \in E$ there exists a $j \in \{0, 1, \ldots, n-1\}$ with $d(T^j x, T^j y) > \varepsilon$. If $g : X \to \mathbb{R}$ is a continuous function, then the *topological pressure* $p(T, g)$ is defined by

$$p(T,g) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp\Bigl(\sum_{j=0}^{n-1} g(T^j x)\Bigr), \tag{1.1}$$

where the supremum is taken over all (n, ε)-separated subsets E of X. We define the *topological entropy* $h_{\mathrm{top}}(T)$ by

$$h_{\mathrm{top}}(T) := p(T, 0). \tag{1.2}$$

For some alternative definitions see e.g. [14]. A T-invariant Borel probability measure μ is called *maximal measure*, if

$$h_\mu(T) = h_{\mathrm{top}}(T), \tag{1.3}$$

where $h_\mu(T)$ denotes the measure-theoretic entropy of (X, T, μ) (see e.g. [14] for the definition).

Next we describe the Markov diagram $(\mathcal{D}, \to)$ of a piecewise monotonic map $T : X \to \mathbb{R}$. This is an at most countable oriented graph describing the orbit structure of T, which was introduced by Franz Hofbauer in [2]. Let $T : X \to \mathbb{R}$ be a piecewise monotonic map with respect to $\mathcal{Z}$, and suppose that $\mathcal{Y}$ is a finite partition of X refining $\mathcal{Z}$. Suppose that $D \subseteq Y_0$ for a $Y_0 \in \mathcal{Y}$. Then a nonempty C is called *successor* of D, if there exists a $Y \in \mathcal{Y}$ with $C = TD \cap Y$. In this case we write $D \to C$. Now let $\mathcal{D}$ be the smallest set with $\mathcal{Y} \subseteq \mathcal{D}$ and so that $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. We call $(\mathcal{D}, \to)$ the *Markov diagram* of T with respect to $\mathcal{Y}$. Note that the Markov diagram does not only depend on T, but also on $\mathcal{Y}$. If we set $\mathcal{D}_0 := \mathcal{Y}$ and $\mathcal{D}_n := \mathcal{D}_{n-1} \cup \{\, D \in \mathcal{D} : \exists C \in \mathcal{D}_{n-1} \text{ with } C \to D\}$, then $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots$ and $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$.

In the proofs we will need also the notion *variant* $(\mathcal{A}, \to)$ *of the Markov diagram* of T with respect to $\mathcal{Y}$ (in particular Lemma 4 would not be true, if we used Markov diagrams instead of variants of the Markov diagram). The definition of it is given on pp. 107–108 of [7]. We describe its most important properties shortly. If $(\mathcal{A}, \to)$ is a variant of the Markov diagram of T with respect to $\mathcal{Y}$, then $(\mathcal{A}, \to)$ is an oriented graph and there exists a function $A : \mathcal{A} \to \mathcal{D}$, such that the following properties are satisfied.

(1) The property $c \to d$ in $(\mathcal{A}, \to)$ implies that $A(c) \to A(d)$ in $(\mathcal{D}, \to)$.
(2) For every $c \in \mathcal{A}$ the map A is bijective from $\{d \in \mathcal{A} : c \to d\}$ to $\{D \in \mathcal{D} : A(c) \to D\}$.
(3) We can write $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ with $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ and $A(\mathcal{A}_n) = \mathcal{D}_n$ for every $n \in \mathbb{N}_0$.

Observe that $A : \mathcal{A} \to \mathcal{D}$ is surjective, but not necessarily injective. Furthermore, note that $(\mathcal{D}, \to)$ can be considered as a variant of the Markov diagram of T with respect to $\mathcal{Y}$. A subset $\mathcal{C} \subseteq \mathcal{A}$ is called *closed*, if $c \in \mathcal{C}$, $d \in \mathcal{A}$ and $c \to d$ in $(\mathcal{A}, \to)$

imply $d \in \mathcal{C}$. For every $c \in \mathcal{A}$ there are at most two different successors of c, which are not contained in $\{d \in \mathcal{A}_0 : \inf A(d) > \inf X \text{ and } \sup A(d) < \sup X\}$. If $n \in \mathbb{N}$, $k \in \{1, 2, \ldots, n\}$ and $c \in \mathcal{A} \setminus \mathcal{A}_n$, then there are at most two different paths $c_0 \to c_1 \to \cdots \to c_k$ of length k in $\mathcal{A} \setminus \mathcal{A}_n$ with $c_0 = c$.

Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function. The Markov diagram of T_f with respect to $\mathcal{Z}_f$ is described in Lemma 1 of [9] (see also [3]).

Suppose that $\mathcal{Y}$ is a finite partition of $[0,1]$ refining $\mathcal{Z}_f$. We call a function $g : [0,1] \to \mathbb{R}$ *piecewise constant* with respect to $\mathcal{Y}$, if $g|_Y$ is constant for all $Y \in \mathcal{Y}$. Let $g : [0,1] \to \mathbb{R}$ be a piecewise constant function with respect to $\mathcal{Y}$, and let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Y}$. If $c \in \mathcal{A}$, then let g_c be the unique real number with $g(x) = g_c$ for all $x \in A(c)$. For $c, d \in \mathcal{A}$ define

$$F_{c,d}(g) := \begin{cases} e^{g_c} & \text{if } c \to d \text{ in } (\mathcal{A}, \to), \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Set $F_{\mathcal{C}}(g) := \big(F_{c,d}(g)\big)_{c,d \in \mathcal{C}}$, if $\mathcal{C} \subseteq \mathcal{A}$, and set $F(g) := F_{\mathcal{A}}(g)$. As in [6] and [7] $u \mapsto uF_{\mathcal{C}}(g)$ is a continuous linear operator on $\ell^1(\mathcal{C})$ and $v \mapsto F_{\mathcal{C}}(g)v$ is a continuous linear operator on $\ell^\infty(\mathcal{C})$, where both operators have the same norm $\|F_{\mathcal{C}}(g)\|$ and the same spectral radius $r\big(F_{\mathcal{C}}(g)\big)$. Furthermore we have (cf. [7])

$$\|F_{\mathcal{C}}(g)^n\| = \sup_{c \in \mathcal{C}} \sum_{c_0 = c \to c_1 \to \cdots \to c_n} \exp\Big(\sum_{j=0}^{n-1} g_{c_j}\Big) \tag{1.5}$$

for every $n \in \mathbb{N}$, where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{C}$ with $c_0 = c$, and

$$r\big(F_{\mathcal{C}}(g)\big) = \lim_{n\to\infty} \|F_{\mathcal{C}}(g)^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|F_{\mathcal{C}}(g)^n\|^{\frac{1}{n}}. \tag{1.6}$$

By Lemma 6 in [6] (cf. also (2.12) of [7]) we have

$$p(T_f, g) = \log r\big(F(g)\big). \tag{1.7}$$

2. The graph $(\mathcal{G}', \to)$ associated to T_f

As in [7] we introduce an oriented graph in order to describe the jumps up of the pressure. Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function. Then set

$$\begin{aligned} \mathcal{G} := \{&{T_f}^n c^-, {T_f}^n c^+ : n \in \mathbb{N}_0, c \in f^{-1}(\mathbb{Z}) \setminus \{0,1\}\} \cup \\ &\{{T_f}^n 0, {T_f}^n 1 : n \in \mathbb{N}_0\}. \end{aligned} \tag{2.1}$$

For $a, b \in \mathcal{G}$ we introduce an arrow $a \to b$ in $(\mathcal{G}, \to)$, if and only if $T_f a = b$ or there exists a $c \in f^{-1}(\mathbb{Z}) \setminus \{0,1\}$ with $T_f a = c$ and $b \in \{c^-, c^+\}$.

Now set $\mathcal{G}' := \mathcal{G}$, and let $a, b \in \mathcal{G}'$. We introduce an arrow $a \to b$ in $(\mathcal{G}', \to)$, if and only if $a \to b$ in $(\mathcal{G}, \to)$, or $a = 0$, $b \in \{0,1\}$ and $T_f 0 = 0$, or $a = 1$, $b \in \{0,1\}$ and $T_f 1 = 1$. Note that the sets $\mathcal{G}$ and $\mathcal{G}'$ are the same, but $(\mathcal{G}', \to)$ may have more arrows than $(\mathcal{G}, \to)$. Furthermore observe that for every $a \in \mathcal{G}'$ there exist at most two different $b \in \mathcal{G}'$ with $a \to b$ in $(\mathcal{G}', \to)$. Define $P(\mathcal{G}') := \big(\mathcal{G}' \setminus \{c^-, c^+ : c \in f^{-1}(\mathbb{Z}) \setminus \{0,1\}\}\big) \cup \big(f^{-1}(\mathbb{Z}) \setminus \{0,1\}\big)$. Roughly spoken the only difference between $\mathcal{G}'$ and $P(\mathcal{G}')$ is, that for $c \in f^{-1}(\mathbb{Z})$ the elements c^- and c^+ are considered to be different in $\mathcal{G}'$, but they are identified in $P(\mathcal{G}')$.

Let $g : [0,1] \to \mathbb{R}$ be a continuous function. For $a, b \in \mathcal{G}'$ we define

$$G'_{a,b}(g) := \begin{cases} e^{g(a)} & \text{if } a \to b \text{ in } (\mathcal{G}', \to), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Set $G'(g) := \big(G'_{a,b}(g)\big)_{a,b \in \mathcal{G}'}$. As in [7] and [9] the map $u \mapsto uG'(g)$ is a continuous linear operator on $\ell^1(\mathcal{G}')$, and the map $v \mapsto G'(g)v$ is a continuous linear operator on $\ell^\infty(\mathcal{G}')$. Both operators have the same norm $\|G'(g)\|$ and the same spectral radius $r\big(G'(g)\big)$. Furthermore we have (cf. [7] and [9])

$$\|G'(g)\| = \sup_{a \in \mathcal{G}'} \sum_{b \in \mathcal{G}'} G'_{a,b}(g), \tag{2.3}$$

$$\|G'(g)^n\| = \sup_{a \in \mathcal{G}'} \sum_{b_0 = a \to b_1 \to \cdots \to b_n} \exp\Big(\sum_{j=0}^{n-1} g(b_j)\Big) \quad \text{for every } n \in \mathbb{N}, \tag{2.4}$$

where the sum is taken over all paths $b_0 \to b_1 \to \cdots \to b_n$ of length n in $\mathcal{G}'$ with $b_0 = a$, and

$$r\big(G'(g)\big) = \lim_{n \to \infty} \|G'(g)^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|G'(g)^n\|^{\frac{1}{n}}. \tag{2.5}$$

The proof of the next result is completely analogous to the proof of Lemma 2 in [9].

Lemma 1. *Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function. Suppose that ${T_f}^n 0 \neq 0$ for all $n \in \mathbb{N}$ or ${T_f}^n 1 \neq 1$ for all $n \in \mathbb{N}$. Then*

$$\log r\big(G'(g)\big) \le \lim_{n \to \infty} \frac{1}{n} \sup_{x \in [0,1]} \sum_{j=0}^{n-1} g\big({T_f}^j x\big) \le p(T_f, g)$$

for every continuous function $g : [0,1] \to \mathbb{R}$.

Next assume that ${T_f}^p 0 = 0$ and ${T_f}^q 1 = 1$ for some $p, q \in \mathbb{N}$. We construct a family of strictly increasing and continuous functions $f_s : [0,1] \to \mathbb{R}$. Let $s \in (0,1)$ be arbitrary. Choose a $u > 0$, such that $u < s$, $|f(y) - f(x)| < s$ for all $x, y \in [0,1]$

with $|x-y|<u$, and every interval of length $2u$ contains at most one element of $P(\mathcal{G}')$. Now choose a $t>0$ with $t<u$ and $t<\min_{x\in[0,1-u]}|f(x+u)-f(x)|$.

If $x\in[0,1]$ satisfies $|x-y|\ge u$ for all $y\in P(\mathcal{G}')$, then set $f_s(x):=f(x)$. Let $y\in P(\mathcal{G}')$ and suppose that $x\in[0,1]$ and $|x-y|\le t$. Assume at first that $y\notin\{0,1\}$. Define $f_s(x):=f(y)+x-y$. If $f(y)\notin\mathbb{Z}$, then set $D_s(y):=(y-t,y+t)$, and if $f(y)\in\mathbb{Z}$, then set $D_s(y^-):=(y-t,y)$ and $D_s(y^+):=(y,y+t)$. Now consider the case $y=0$. For $x\in[0,t]$ define $f_s(x):=f(0)-t+2x$, and set $D_s(0):=(0,t)$. In the case $y=1$ define $f_s(x):=f(1)+t-2+2x$ for $x\in[1-t,1]$, and set $D_s(1):=(1-t,1)$. It remains to define f_s on the finitely many intervals $[a,b]$ of the form $[y-u,y-t]$ or $[y+t,y+u]$ for a suitable $y\in P(\mathcal{G}')$. We define in this case $f_s(x):=f_s(a)+\frac{f_s(b)-f_s(a)}{b-a}(x-a)$.

Using the above construction simple calculations show the following result.

Lemma 2. *Let $f:[0,1]\to\mathbb{R}$ be a strictly increasing and continuous function. Suppose that there exist $p,q\in\mathbb{N}$ with ${T_f}^p0=0$ and ${T_f}^q1=1$. Then for every $s\in(0,1)$ there exists a strictly increasing and continuous function $f_s:[0,1]\to\mathbb{R}$, and for each $a\in\mathcal{G}'$ there exists an open interval $D_s(a)\subseteq[0,1]$, such that the following properties hold.*

(1) *We have $\|f_s-f\|_\infty\le s$.*
(2) *If $a\in\mathcal{G}'$ and $x\in\overline{D_s(a)}$, then $|x-a|<s$.*
(3) *If $a,b\in\mathcal{G}'$ and $a\ne b$, then $\overline{D_s(a)\cap D_s(b)}=\emptyset$.*
(4) *For $a\in\mathcal{G}'$ we have $\overline{T_{f_s}D_s(a)}=\bigcup_b\overline{D_s(b)}$, where the union is taken over all $b\in\mathcal{G}'$ with $a\to b$ in $(\mathcal{G}',\to)$.*

We will also need the following result.

Lemma 3. *Let $f:[0,1]\to\mathbb{R}$ be a strictly increasing and continuous function. Suppose that there exist $p,q\in\mathbb{N}$ with ${T_f}^p0=0$ and ${T_f}^q1=1$. Assume that $(f_s)_{s\in(0,1)}$ are as in Lemma 2. Then*

$$\limsup_{\|\tilde f-f\|_\infty\to0}p(T_{\tilde f},g)\ge\liminf_{s\to0}p(T_{f_s},g)\ge\log r\big(G'(g)\big)$$

for every continuous function $g:[0,1]\to\mathbb{R}$.

Proof: Assume that $\varepsilon>0$ is arbitrary. Then there exists an $\eta>0$, such that $x,y\in[0,1]$ and $|x-y|<\eta$ imply $|g(x)-g(y)|<\frac{\varepsilon}{2}$. Now fix an $s\in(0,\eta)$. Choose a finite partition $\mathcal{Y}$ of $[0,1]$ refining $\mathcal{Z}_{f_s}$, such that $\sup_{Y\in\mathcal{Y}}\sup_{x,y\in Y}|x-y|<\eta$ and $\{D_s(a):a\in\mathcal{G}'\}\subseteq\mathcal{Y}$. Let $Y\in\mathcal{Y}$. If $Y=D_s(a)$ for an $a\in\mathcal{G}'$, then set $x_Y:=a$. Otherwise choose an $x_Y\in Y$. Define $\hat g(x):=g(x_Y)$, if $Y\in\mathcal{Y}$ and $x\in Y$. Then $\hat g:[0,1]\to\mathbb{R}$ is piecewise constant with respect to $\mathcal{Y}$. As $g(x)\ge\hat g(x)-\frac{\varepsilon}{2}$ for all $x\in[0,1]$, we obtain that $p(T_{f_s},g)\ge p(T_{f_s},\hat g)-\frac{\varepsilon}{2}$.

Denote by $(\mathcal{D},\to)$ the Markov diagram of T_{f_s} with respect to $\mathcal{Y}$, and set $\mathcal{C}:=\{D_s(a):a\in\mathcal{G}'\}$. Then (1.5), (1.6) and (1.7) imply

$$p(T_{f_s},g)+\frac{\varepsilon}{2}\ge p(T_{f_s},\hat g)=\log r\big(F(\hat g)\big)\ge\log r\big(F_{\mathcal{C}}(\hat g)\big).\tag{2.6}$$

Using (1.4) and (2.2) we obtain by Lemma 2 that $F_{\mathcal{C}}(\hat{g}) = G'(g)$. Therefore (2.6) implies $p(T_{f_s}, g) > \log r\big(G'(g)\big) - \varepsilon$.

By (1) of Lemma 2 we get $\limsup_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) \geq \liminf_{s\to 0} p(T_{f_s}, g)$. $\square$

3. Continuity of the topological pressure

Considering the R^0-topology the lower semi-continuity of the pressure is treated in Theorem 1 of [7]. Further upper bounds for the jumps up of the pressure are given in Theorem 2 of [7]. These upper bounds are related to the oriented graph $(\mathcal{G}, \to)$. A result on the continuity of the maximal measure is proved in Theorem 3 of [10] and in Theorem 1 of [12]. In this section we will show that similar results hold for monotonic mod one transformations, where we consider $T_{\tilde{f}}$ to be close to T_f, if $\|\tilde{f} - f\|_\infty$ is small. The conditions on $(\mathcal{G}, \to)$ have to be replaced by conditions on $(\mathcal{G}', \to)$.

We assume throughout this paper that the functions $\tilde{f} : [0,1] \to \mathbb{R}$ are strictly increasing and continuous.

At first we prove a result analogous to Lemma 6 in [7]. Consider a strictly increasing and continuous function $f : [0,1] \to \mathbb{R}$. Denote by N_1 the largest integer smaller than or equal to $f(0)$, and denote by N_2 the smallest integer larger than or equal to $f(1)$. For $\tilde{f}$ define $X_{\tilde{f}} := \tilde{f}^{-1}[N_1, N_2]$. If $X_{\tilde{f}} \neq \emptyset$, then $X_{\tilde{f}} \subseteq [0,1]$ is a closed interval. Let $\mathcal{W}(\tilde{f})$ be the collection of the nonempty sets among $\big(0, \inf X_{\tilde{f}}\big)$ and $\big(\sup X_{\tilde{f}}, 1\big)$. Suppose that $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ is a finite partition of $[0,1]$ refining $\mathcal{Z}_f$ with $Y_1 < Y_2 < \cdots < Y_N$, and $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_N\} \cup \mathcal{W}(\tilde{f})$ is a finite partition of $[0,1]$ refining $\mathcal{Z}_{\tilde{f}}$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$. Set $\tilde{\mathcal{Y}}|_{X_{\tilde{f}}} := \{\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_N\}$ and $\mathcal{Z}_{\tilde{f}}|_{X_{\tilde{f}}} := \mathcal{Z}_{\tilde{f}} \setminus \mathcal{W}(\tilde{f})$. Now let $(\tilde{\mathcal{A}}, \to)$ be a variant of the Markov diagram of $T_{\tilde{f}}$ with respect to $\tilde{\mathcal{Y}}$. Define $\tilde{\mathcal{A}}_0|_{X_{\tilde{f}}} := \big\{\tilde{c} \in \tilde{\mathcal{A}}_0 : \tilde{A}(\tilde{c}) \in \tilde{\mathcal{Y}}|_{X_{\tilde{f}}}\big\}$, $\tilde{\mathcal{A}}_n|_{X_{\tilde{f}}} := \tilde{\mathcal{A}}_{n-1}|_{X_{\tilde{f}}} \cup \big\{\tilde{d} \in \tilde{\mathcal{A}} : \exists\, \tilde{c} \in \tilde{\mathcal{A}}_{n-1}|_{X_{\tilde{f}}} \text{ with } \tilde{c} \to \tilde{d} \text{ and } \tilde{A}(\tilde{d}) \subseteq X_{\tilde{f}}\big\}$ for $n \in \mathbb{N}$, and $\tilde{\mathcal{A}}|_{X_{\tilde{f}}} := \bigcup_{n=0}^{\infty} \tilde{\mathcal{A}}_n|_{X_{\tilde{f}}}$. Then $(\tilde{\mathcal{A}}|_{X_{\tilde{f}}}, \to)$ is a variant of the Markov diagram of $T_{\tilde{f}}|_{X_{\tilde{f}}}$ with respect to $\tilde{\mathcal{Y}}|_{X_{\tilde{f}}}$. If $\tilde{c} \in \tilde{\mathcal{A}}_n \setminus \tilde{\mathcal{A}}_n|_{X_{\tilde{f}}}$, then there exists a $k \leq n$ and there exists a path $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_k$ in $(\tilde{\mathcal{A}}, \to)$ with $\tilde{c}_0 \in \tilde{\mathcal{A}}_0$, $\tilde{A}(\tilde{c}_0) \in \mathcal{W}(\tilde{f})$ and $\tilde{A}(\tilde{c}) \subseteq \tilde{A}(\tilde{c}_k)$. Observe that for every variant $(\tilde{\mathcal{C}}, \to)$ of the Markov diagram of $T_{\tilde{f}}|_{X_{\tilde{f}}}$ with respect to $\tilde{\mathcal{Y}}|_{X_{\tilde{f}}}$ there exists a variant $(\tilde{\mathcal{A}}, \to)$ of the Markov diagram of $T_{\tilde{f}}$ with respect to $\tilde{\mathcal{Y}}$, such that $\tilde{\mathcal{A}}|_{X_{\tilde{f}}} = \tilde{\mathcal{C}}$.

Given $\varepsilon > 0$ two intervals B and C are called to be ε-close in the *Hausdorff metric*, if $|\inf B - \inf C| < \varepsilon$ and $|\sup B - \sup C| < \varepsilon$. Let $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ be as above. We call $\tilde{\mathcal{Y}}$ to be ε-close to $\mathcal{Y}$, if $\tilde{Y}_j$ is ε-close to Y_j in the Hausdorff metric for all $j \in \{1, 2, \ldots, N\}$.

Lemma 4. *Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function, and suppose that $\mathcal{Y}$ is a finite partition of $[0,1]$ refining $\mathcal{Z}_f$. Then for every $r \in \mathbb{N}$*

and for every $\eta > 0$ *there exists a* $\delta > 0$*, such that the following property holds. Suppose that* $\tilde{f} : [0,1] \to \mathbb{R}$ *is a strictly increasing and continuous function with* $\|\tilde{f} - f\|_\infty < \delta$*, and assume that* $\tilde{\mathcal{Y}}$ *is a finite partition of* $[0,1]$ *refining* $\mathcal{Z}_{\tilde{f}}$*, which is* δ*-close to* $\mathcal{Y}$*. Then there exists a variant* $(\mathcal{A}, \to)$ *of the Markov diagram of* T_f *with respect to* $\mathcal{Y}$ *and a variant* $(\tilde{\mathcal{A}}, \to)$ *of the Markov diagram of* $T_{\tilde{f}}$ *with respect to* $\tilde{\mathcal{Y}}$ *with the following properties.*

(1) *We can write* $\tilde{\mathcal{A}}_r$ *as a disjoint union* $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$*, such that* $\mathcal{B}_1 \cup \mathcal{B}_2$ *and* $\mathcal{B}_2$ *are closed. If* $\tilde{c} \in \tilde{\mathcal{A}}_0 \setminus \mathcal{B}_0$*, then* $\tilde{c} \in \mathcal{B}_2$ *and* $\tilde{A}(\tilde{c}) \in \mathcal{W}(\tilde{f})$*.*
(2) *Every* $\tilde{c} \in \tilde{\mathcal{A}}_r$ *has at most two successors in* $\mathcal{B}_1 \cup \mathcal{B}_2$*.*
(3) *There exists a bijective function* $\varphi : \mathcal{A}_r \to \mathcal{B}_0$*, and there exists a function* $\psi : \mathcal{B}_2 \to \mathcal{G}'$*.*
(4) *For* $c, d \in \mathcal{A}_r$ *the property* $c \to d$ *in* $(\mathcal{A}, \to)$ *is equivalent to* $\varphi(c) \to \varphi(d)$ *in* $(\tilde{\mathcal{A}}, \to)$*. If* $\tilde{c}, \tilde{d} \in \mathcal{B}_2$ *and* $\tilde{c} \to \tilde{d}$ *in* $(\tilde{\mathcal{A}}, \to)$*, then* $\psi(\tilde{c}) \to \psi(\tilde{d})$ *in* $(\mathcal{G}', \to)$*.*
(5) *Let* $c \in \mathcal{A}_0$*. Then there is a* $j \in \{1, 2, \ldots, N\}$ *with* $A(c) = Y_j$*. Furthermore we have* $\varphi(c) \in \tilde{\mathcal{A}}_0$ *and* $\tilde{A}\big(\varphi(c)\big) = \tilde{Y}_j$*.*
(6) *If* $c \in \mathcal{A}_r$*, then* $\tilde{A}\big(\varphi(c)\big)$ *and* $A(c)$ *are* η*-close in the Hausdorff metric. The properties* $c \in \mathcal{A}_r$*,* $d \in \mathcal{A}_0$ *and* $A(c) \subseteq A(d)$ *imply* $\tilde{A}\big(\varphi(c)\big) \subseteq \tilde{A}\big(\varphi(d)\big)$*. We have* $\big|x - \psi(\tilde{c})\big| < \eta$ *for all* $x \in \tilde{A}(\tilde{c})$*, if* $\tilde{c} \in \mathcal{B}_2$*.*
(7) *Fix an* $s \in \mathbb{N}$ *with* $s \le r$*. Let* $\mathcal{P}_s$ *be the set of all paths* $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s$ *in* $(\tilde{\mathcal{A}}, \to)$ *with* $\tilde{c}_0 \in \tilde{\mathcal{A}}_0$*, and set* $\mathcal{N}_s := \big\{(d_0, d_1, \ldots, d_s) : d_j \in \mathcal{A}_r \cup \mathcal{G}'$ *for* $j \in \{0, 1, \ldots, s\}\big\}$*. Then there exists a function* $\chi_s : \mathcal{P}_s \to \mathcal{N}_s$ *satisfying the properties* (8), (9) *and* (10) *below.*
(8) *Suppose* $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s \in \mathcal{P}_s$*,*

$$\chi_s(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s) = (d_0, d_1, \ldots, d_s),$$

and let $j \in \{0, 1, \ldots, s\}$*. We have* $\tilde{c}_j \in \mathcal{B}_2$*, if and only if* $d_j \in \mathcal{G}'$*. Moreover,* $\tilde{c}_j \in \mathcal{B}_2$ *implies* $\psi(\tilde{c}_j) = d_j$*. If* $\tilde{c}_j \in \mathcal{B}_0$*, then* $\tilde{c}_j = \varphi(d_j)$*. In the case* $\tilde{c}_j \in \mathcal{B}_0 \cup \mathcal{B}_1$ *we have that for every* $x \in \tilde{A}(\tilde{c}_j)$ *there exists a* $y \in A(d_j)$ *with* $|x - y| < \eta$*. Furthermore* $\tilde{c}_j \in \mathcal{B}_0 \cup \mathcal{B}_1$ *and* $j \ge 1$ *imply* $d_{j-1} \to d_j$ *in* $(\mathcal{A}, \to)$*.*
(9) *For a fixed* $\tilde{c} \in \tilde{\mathcal{A}}_0$ *and for a fixed* $(d_0, d_1, \ldots, d_s) \in \mathcal{N}_s$ *there are at most* $2s + 1$ *different* $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s \in \mathcal{P}_s$ *with* $\tilde{c}_0 = \tilde{c}$ *and* $\chi_s(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s) = (d_0, d_1, \ldots, d_s)$*. If* $s < r$ *and* $(d_0, d_1, \ldots, d_s) \in \mathcal{N}_s$*, then there exist at most 4 different* $d_{s+1} \in \mathcal{G}'$ *with* $(d_0, d_1, \ldots, d_s, d_{s+1}) \in \chi_{s+1}(\mathcal{P}_{s+1})$*.*
(10) *Fix a* $\tilde{d}_0 \to \tilde{d}_1 \to \cdots \to \tilde{d}_s \in \mathcal{P}_s$*, let* $k \le s$*, and assume that* $\tilde{d}_k \in \mathcal{B}_1$*. In the case* $\tilde{d}_s \in \mathcal{B}_2$ *we have* $\tilde{c}_s \in \mathcal{B}_2$ *for every* $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s \in \mathcal{P}_s$ *with* $\tilde{c}_j = \tilde{d}_j$ *for* $j \in \{0, 1, \ldots, k\}$*. If* $\tilde{d}_s \in \mathcal{B}_1$*, then* $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s \in \mathcal{P}_s$ *with* $\tilde{c}_j = \tilde{d}_j$ *for* $j \in \{0, 1, \ldots, k\}$ *implies* $\chi_s(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s) = \chi_s(\tilde{d}_0 \to \tilde{d}_1 \to \cdots \to \tilde{d}_s)$*.*

Proof: Set $\mathcal{G}_r := \{{T_f}^n c^-, {T_f}^n c^+ : n \in \{0,1,\dots,r+1\}, c \in f^{-1}(\mathbb{Z}) \setminus \{0,1\}\} \cup \{{T_f}^n 0, {T_f}^n 1 : n \in \{0,1,\dots,r+1\}\}$, and define $P(\mathcal{G}_r) := (\mathcal{G}_r \setminus \{c^-, c^+ : c \in f^{-1}(\mathbb{Z}) \setminus \{0,1\}\}) \cup (f^{-1}(\mathbb{Z}) \setminus \{0,1\})$. Denote by $(\mathcal{D}, \to)$ the Markov diagram of T_f with respect to $\mathcal{Y}$. Now define $E := \{\inf C, \sup C : C \in \mathcal{D}_{r+1}\}$. Then $P(\mathcal{G}_r) \subseteq E$. We may assume that η is so small that $|a-b| > 2\eta$, whenever $a \neq b \in E$. Set $\alpha_0 := \eta$, and for $j \in \{1,2,\dots,r+1\}$ let $\alpha_j > 0$ be so that $\alpha_j < \frac{\alpha_{j-1}}{4}$ and $|x-y| < \alpha_j$ implies $|f(x) - f(y)| < \frac{\alpha_{j-1}}{4}$. By the proof of Lemma 6 in [7] there exists a $\delta_0 > 0$ with $\delta_0 \leq \alpha_{r+1}$, such that the conclusions of Lemma 6 in [7] and the properties (a)–(o) in the proof of Lemma 6 in [7] hold, whenever $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ_0-close to $(T_f, \mathcal{Z}_f)$ in the R^0-topology. Then $\beta := \inf_{x \in [0, 1-\delta_0]} |f(x+\delta_0) - f(x)| > 0$. Now define $\delta := \min\{\delta_0, \beta\}$.

Assume that $\|\tilde{f} - f\|_\infty < \delta$, and suppose that $\tilde{\mathcal{Y}}$ is a finite partition of $[0,1]$ refining $\mathcal{Z}_{\tilde{f}}$, which is δ-close to $\mathcal{Y}$. Then $X_{\tilde{f}} \subseteq [0,1]$ is a closed interval with $\inf X_{\tilde{f}} < \delta_0$ and $\sup X_{\tilde{f}} > 1 - \delta_0$. Furthermore $\inf X_{\tilde{f}} \neq 0$ implies $T_f 0 = 0$, and $\sup X_{\tilde{f}} \neq 1$ implies $T_f 1 = 1$. Moreover we have that $(T_{\tilde{f}}|_{X_{\tilde{f}}}, \mathcal{Z}_{\tilde{f}}|_{X_{\tilde{f}}})$ is δ_0-close to $(T_f, \mathcal{Z}_f)$ in the R^0-topology. Therefore Lemma 6 in [7] gives the existence of a variant $(\mathcal{A}, \to)$ of the Markov diagram of T_f with respect to $\mathcal{Y}$ and a variant $(\tilde{\mathcal{A}}, \to)$ of the Markov diagram of $T_{\tilde{f}}$ with respect to $\tilde{\mathcal{Y}}$, such that $(\mathcal{A}, \to)$ and $(\tilde{\mathcal{A}}|_{X_{\tilde{f}}}, \to)$ satisfy the conclusions of Lemma 6 in [7] and the properties (a)–(o) in the proof of Lemma 6 in [7]. Hence (5) holds. Denote the sets occurring in (1) of Lemma 6 in [7] by $\mathcal{B}_0$, $\mathcal{B}_1$ and ${\mathcal{B}_2}'$. Now set $\mathcal{B}_2 := {\mathcal{B}_2}' \cup (\tilde{\mathcal{A}}_r \setminus \tilde{\mathcal{A}}_r|_{X_{\tilde{f}}})$.

Consider a $\tilde{c} \in \tilde{\mathcal{A}}_r \setminus \tilde{\mathcal{A}}_r|_{X_{\tilde{f}}}$. Then there exists a $k \leq r$ and there exists a path $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_k$ in $(\tilde{\mathcal{A}}, \to)$ with $\tilde{c}_0 \in \tilde{\mathcal{A}}_0$, $\tilde{A}(\tilde{c}_0) \in \mathcal{W}(\tilde{f})$ and $\tilde{A}(\tilde{c}) \subseteq \tilde{A}(\tilde{c}_k)$. Hence there is a unique $y \in P(\mathcal{G}_r)$ with $|x-y| < \alpha_{r+1-k} < \eta$ for all $x \in \tilde{A}(\tilde{c})$. Then $|T_{\tilde{f}}x - T_f y| < \alpha_{r-k} \leq \eta$ for all $x \in T_{\tilde{f}}\tilde{A}(\tilde{c})$, which completes the proof of (1). If $y \notin f^{-1}(\mathbb{Z}) \setminus \{0,1\}$, then define $\psi(\tilde{c}) := y$. Otherwise we have either $\tilde{f}(x) < f(y)$ for all $x \in \tilde{A}(\tilde{c})$ or $\tilde{f}(x) > f(y)$ for all $x \in \tilde{A}(\tilde{c})$. In the first case we define $\psi(\tilde{c}) := y^-$, and in the second case set $\psi(\tilde{c}) := y^+$. This implies (3) and (6). Now simple calculations yield that $\tilde{c} \to \tilde{d}$ in $(\tilde{\mathcal{A}}, \to)$ implies $\psi(\tilde{c}) \to \psi(\tilde{d})$ in $(\mathcal{G}', \to)$, whenever $\tilde{c}, \tilde{d} \in \mathcal{B}_2$. Therefore we have proved (4).

For every $\tilde{c} \in \tilde{\mathcal{A}}$ at most two successors of $\tilde{c}$ are not contained in $\mathcal{C} := \{\tilde{d} \in \tilde{\mathcal{A}}_0 : \tilde{A}(\tilde{d}) = \tilde{Y}_j \text{ for a } j \in \{1,2,\dots,N\}\}$. By (1) we have $\mathcal{C} \subseteq \mathcal{B}_0$, completing the proof of (2).

Next assume that $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s \in \mathcal{P}_s$. If $\tilde{c}_j \in \tilde{\mathcal{A}}|_{X_{\tilde{f}}}$ for all $j \in \{0,1,\dots,s\}$, then $\chi_s(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s)$ is defined in the proof of Lemma 6 in [7]. Otherwise let j be the minimal number in $\{0,1,\dots,s\}$ with $\tilde{c}_j \notin \tilde{\mathcal{A}}|_{X_{\tilde{f}}}$. By (1) we have $\tilde{c}_j, \tilde{c}_{j+1}, \dots, \tilde{c}_s \in \mathcal{B}_2$. In the case $j > 1$ define $\chi_s(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s) := (d_0, d_1, \dots, d_{j-1}, \psi(\tilde{c}_j), \psi(\tilde{c}_{j+1}), \dots, \psi(\tilde{c}_s))$, where $(d_0, d_1, \dots, d_{j-1}) =$

$\chi_{j-1}(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_{j-1})$, for $j = 1$ define $\chi_s(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s) := \big(d_0, \psi(\tilde{c}_1), \psi(\tilde{c}_2), \ldots, \psi(\tilde{c}_s)\big)$, where $d_0 \in \mathcal{A}_0$ satisfies $\varphi(d_0) = \tilde{c}_0$, and for $j = 0$ define $\chi_s(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s) := \big(\psi(\tilde{c}_0), \psi(\tilde{c}_1), \ldots, \psi(\tilde{c}_s)\big)$. Now easy calculations give (8).

Suppose that $\tilde{d}_0 \to \tilde{d}_1 \to \cdots \to \tilde{d}_s \in \mathcal{P}_s$, $k \leq s$ and $\tilde{d}_k \in \mathcal{B}_1 \cup \mathcal{B}_2$. Then $\sup_{x,y \in \tilde{A}(\tilde{d}_k)} |x - y| < \alpha_{r+1-k}$. Denote by $\mathcal{P}$ the set of all $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s \in \mathcal{P}_s$ with $\tilde{c}_j = \tilde{d}_j$ for all $j \in \{0, 1, \ldots, k\}$ and $\chi_s(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s) = \chi_s(\tilde{d}_0 \to \tilde{d}_1 \to \cdots \to \tilde{d}_s)$. Using induction we get that $\operatorname{card} \mathcal{P} \leq s - k + 1$, and $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s \in \mathcal{P}$, $x \in \tilde{A}(\tilde{c}_s)$ and $y \in \tilde{A}(\tilde{d}_s)$ imply $|x - y| < \alpha_{r+1-s}$. If $\tilde{d}_s \in \mathcal{B}_2$, then let $l \geq k$ be minimal with $\tilde{d}_l \in \mathcal{B}_2$. Assume that $\tilde{d}_l \notin \tilde{\mathcal{A}}|_{X_{\tilde{f}}}$. Then there exists a $y \in P(\mathcal{G}_r)$ with $|x - y| < \alpha_{r+1-l}$ for all $x \in \tilde{A}(\tilde{c}_l)$, where $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_s \in \mathcal{P}_s$ satisfies $\tilde{c}_j = \tilde{d}_j$ for all $j \in \{0, 1, \ldots, k\}$. Hence using also (1) and the proof of Lemma 6 in [7] we get (10). Observing that every $a \in \mathcal{G}'$ has at most two successors in $\mathcal{G}'$ we obtain (9). This completes also the proof of (7). □

Now we prove the result on the behaviour of the pressure under small perturbations.

Theorem 1. *Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function.*

(1) *If $g : [0,1] \to \mathbb{R}$ is a continuous function, then*

$$\limsup_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) \leq \max\{p(T_f, g), \log r\big(G'(g)\big)\}.$$

(2) *Suppose that $g : [0,1] \to \mathbb{R}$ is a continuous function with $p(T_f, g) > \lim_{n\to\infty} \frac{1}{n} \sup_{x \in [0,1]} \sum_{j=0}^{n-1} g(T_f{}^j x)$. Then*

$$\liminf_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) \geq p(T_f, g) \quad \textit{and}$$

$$\limsup_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) = \max\{p(T_f, g), \log r\big(G'(g)\big)\}.$$

Proof: Using Lemma 4 a proof completely analogous to the proof of Theorem 2 in [7] shows (1).

Assume that $g : [0,1] \to \mathbb{R}$ is a continuous function with

$$p(T_f, g) > \lim_{n\to\infty} \frac{1}{n} \sup_{x \in [0,1]} \sum_{j=0}^{n-1} g(T_f{}^j x).$$

Using Lemma 4 again, a proof completely analogous to the proof of Theorem 1 in [7] gives $\liminf_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) \geq p(T_f, g)$. If $\log r\big(G'(g)\big) \leq p(T_f, g)$, then

(1) implies $\lim_{\|\tilde{f}-f\|_\infty\to 0} p(T_{\tilde{f}}, g) = p(T_f, g)$. Otherwise $\log r\big(G'(g)\big) > p(T_f, g)$, and (1) and Lemma 3 imply $\limsup_{\|\tilde{f}-f\|_\infty\to 0} p(T_{\tilde{f}}, g) = \log r\big(G'(g)\big)$. □

The following example shows that the pressure (and the topological entropy) is not upper semi-continuous in general (examples for this fact are also given in Section 4 of [9]). Moreover it illustrates the difference between Theorem 2 in [7] and Theorem 1. For $s \in [0, \frac{1}{2}]$ define

$$f_s(x) := \begin{cases} 2x - s & \text{if } x \in [0, s], \\ x & \text{if } x \in [s, 1-s], \\ 2x - 1 + s & \text{if } x \in [1-s, 1]. \end{cases} \tag{3.1}$$

If $s < \delta$, then $\|f_s - f_0\|_\infty < \delta$. Obviously $h_{\text{top}}(T_{f_0}) = 0$, $\mathcal{G} = \{0, 1\}$ and the only arrows in $(\mathcal{G}, \to)$ are $0 \to 0$ and $1 \to 1$. Hence given $\varepsilon > 0$ Theorem 2 in [7] (or Theorem 2 in [5]) implies that $|h_{\text{top}}(\tilde{T}) - h_{\text{top}}(T_{f_0})| < \varepsilon$, whenever $(\tilde{T}, \tilde{\mathcal{Z}})$ is sufficiently close to $(T_{f_0}, \mathcal{Z}_{f_0})$ in the R^0-topology. For $s > 0$ we get $T_{f_s}x = x$ for all $x \in [s, 1-s]$, $T_{f_s}[0, s] = [0, s] \cup [1-s, 1]$ and $T_{f_s}[1-s, 1] = [0, s] \cup [1-s, 1]$. Therefore (1.7) gives $h_{\text{top}}(T_{f_s}) = \log 2$.

Remark. At this point we like to mention that we can obtain results analogous to Theorem 2, Theorem 4 and Corollary 4.1 in [11]. The proofs are completely analogous to the proofs in [11]. As the statements of these results are technical, and as these results are not needed in this paper, we omit them.

Next we prove a result on the continuity of the maximal measure. This result is analogous to Theorem 1 in [12].

Theorem 2. *Let $f : [0, 1] \to \mathbb{R}$ be a strictly increasing and continuous function. Suppose that $\log r\big(G'(0)\big) < h_{\text{top}}(T_f)$ and that μ is the unique maximal measure of T_f. Then for every neighbourhood U of μ in the weak star-topology there exists a $\delta > 0$, such that $\tilde{f} : [0, 1] \to \mathbb{R}$ is a strictly increasing and continuous function with $\|\tilde{f} - f\|_\infty < \delta$ implies that $T_{\tilde{f}}$ has a unique maximal measure $\tilde{\mu}$, and $\tilde{\mu} \in U$.*

Proof: Choose a $\beta \in \mathbb{R}$ with

$$\log r\big(G'(0)\big) < \log \beta < h_{\text{top}}(T_f). \tag{3.2}$$

Now let $\varrho \in \mathbb{R}$ with

$$r\big(G'(0)\big) < \varrho < \beta. \tag{3.3}$$

By (2.5) there exists a $B \geq 1$ such that

$$\|G'(0)^n\| \leq B\varrho^n \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{3.4}$$

Next we choose an $s \in \mathbb{N}$ with

$$\varrho \sqrt[s]{65Bs(s+1)} < \beta. \tag{3.5}$$

Assume that $\mathcal{Y}$ is a finite partition of $[0,1]$ refining $\mathcal{Z}_f$, that $\tilde{f} : [0,1] \to \mathbb{R}$ is strictly increasing and continuous, and that $\tilde{\mathcal{Y}}$ is a finite partition of $[0,1]$ refining $\mathcal{Z}_{\tilde{f}}$. Suppose that $(\mathcal{A}, \to)$ is a variant of the Markov diagram of T_f with respect to $\mathcal{Y}$ and $(\tilde{\mathcal{A}}, \to)$ is a variant of the Markov diagram of $T_{\tilde{f}}$ with respect to $\tilde{\mathcal{Y}}$, such that the conclusions of Lemma 4 hold with r replaced by $2s$ and η replaced by 1 (the existence of these variants is an assumption). For $\mathcal{C} \subseteq \tilde{\mathcal{A}}$, $\tilde{c} \in \mathcal{C}$ and $n \in \mathbb{N}$ let $\mathcal{P}_{\mathcal{C}}(\tilde{c}, n)$ be the set of all paths $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_n$ of length n in $\mathcal{C}$ with $\tilde{c}_0 = \tilde{c}$.

Fix a $\tilde{c} \in \tilde{\mathcal{A}}_s \cap (\mathcal{B}_1 \cup \mathcal{B}_2)$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are the sets described in the conclusions of Lemma 4. Let $q \in \mathbb{N}$ with $1 \le q \le s$. As $\tilde{c} \in \tilde{\mathcal{A}}_s$, there is an $l \in \{0, 1, \ldots, s\}$ and there is a path $\tilde{d}_0 \to \tilde{d}_1 \to \cdots \to \tilde{d}_l$ in $(\tilde{\mathcal{A}}, \to)$ with $\tilde{d}_0 \in \tilde{\mathcal{A}}_0$ and $\tilde{d}_l = \tilde{c}$. Set $\mathcal{N}(\tilde{c}) := \{\chi_{l+q}(\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_{l+q}) : \tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_{l+q} \in \mathcal{P}_{l+q}, \tilde{c}_j = \tilde{d}_j \text{ for } j \in \{0, 1, \ldots, l\}\}$. By (9) of Lemma 4 we get

$$\operatorname{card} \mathcal{P}_{\tilde{\mathcal{A}}}(\tilde{c}, q) \le (2(l+q)+1) \operatorname{card} \mathcal{N}(\tilde{c}) \le 4(s+1) \operatorname{card} \mathcal{N}(\tilde{c}).$$

Using (1) and (10) of Lemma 4 we obtain that either there is a minimal $p \in \{0, 1, \ldots, q\}$ with $\tilde{c}_{l+p} \in \mathcal{B}_2$ for every $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_{l+q} \in \mathcal{P}_{l+q}$ with $\tilde{c}_j = \tilde{d}_j$ for $j \in \{0, 1, \ldots, l\}$, or $\tilde{c}_{l+q} \in \mathcal{B}_1$ for every $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_{l+q} \in \mathcal{P}_{l+q}$ with $\tilde{c}_j = \tilde{d}_j$ for $j \in \{0, 1, \ldots, l\}$. In the first case, (2.4) and (4), (9) and (10) of Lemma 4 yield

$$\operatorname{card} \mathcal{N}(\tilde{c}) \le 4 \| G'(0)^{q-p} \|.$$

This implies $\operatorname{card} \mathcal{N}(\tilde{c}) \le 4B\varrho^{q-p} \le 4B\varrho^q$ by (3.4). Otherwise (10) of Lemma 4 gives $\operatorname{card} \mathcal{N}(\tilde{c}) = 1 \le 4B\varrho^q$. Hence we obtain

$$\operatorname{card} \mathcal{P}_{\tilde{\mathcal{A}}}(\tilde{c}, q) \le 16B(s+1)\varrho^q. \tag{3.6}$$

Next let $\mathcal{C} \subseteq (\tilde{\mathcal{A}} \setminus \tilde{\mathcal{A}}_s) \cup \mathcal{B}_1 \cup \mathcal{B}_2$. Fix a $\tilde{c} \in \mathcal{C}$, and let $p \in \mathbb{N}$ with $1 \le p \le s$. At first assume that $\tilde{c} \notin \tilde{\mathcal{A}}_s$. We have that there are at most two different paths $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_p$ in $\tilde{\mathcal{A}} \setminus \tilde{\mathcal{A}}_s$ with $\tilde{c}_0 = \tilde{c}$. Fix a path $\tilde{d}_0 \to \tilde{d}_1 \to \cdots \to \tilde{d}_p$ in $\tilde{\mathcal{A}} \setminus \tilde{\mathcal{A}}_s$ with $\tilde{d}_0 = \tilde{c}$, and fix an $l \in \{0, 1, \ldots, p-1\}$. Then (2) of Lemma 4 implies that there are at most two different $\tilde{d} \in \tilde{\mathcal{A}}_s \cap (\mathcal{B}_1 \cup \mathcal{B}_2)$ with $\tilde{d}_l \to \tilde{d}$. Therefore (3.6) gives that there are at most $32B(s+1)\varrho^{p-l-1} \le 32B(s+1)\varrho^p$ different paths $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_p$ in $\mathcal{C}$ with $\tilde{c}_j = \tilde{d}_j$ for $j \in \{0, 1, \ldots, l\}$ and $\tilde{c}_{l+1} \in \tilde{\mathcal{A}}_s$. Hence there are at most $64Bp(s+1)\varrho^p \le 64Bs(s+1)\varrho^p$ different paths $\tilde{c}_0 \to \tilde{c}_1 \to \cdots \to \tilde{c}_p$ in $\mathcal{C}$ with $\tilde{c}_0 = \tilde{c}$ and $\tilde{c}_j \in \tilde{\mathcal{A}}_s$ for a $j \in \{1, 2, \ldots, p\}$. This implies

$$\operatorname{card} \mathcal{P}_{\mathcal{C}}(\tilde{c}, p) \le 65Bs(s+1)\varrho^p. \tag{3.7}$$

Now (3.6) gives that (3.7) holds also in the case $\tilde{c} \in \tilde{\mathcal{A}}_s$.

Using (1.5) and (3.7) we obtain $\|F_{\mathcal{C}}(0)^p\| \leq 65Bs(s+1)\varrho^p$. For $k \in \mathbb{N}_0$ and $p \in \{1, 2, \ldots, s\}$ we get by induction that $\|F_{\mathcal{C}}(0)^{ks+p}\| \leq \big(65Bs(s+1)\big)^{k+1} \varrho^{ks+p}$. Hence $\|F_{\mathcal{C}}(0)^n\| \leq \big(65Bs(s+1)\big)^{\frac{n}{s}+1} \varrho^n$ for every $n \in \mathbb{N}_0$. Therefore, by (3.5)

$$\|F_{\mathcal{C}}(0)^n\| \leq \beta^s \beta^n \tag{3.8}$$

for every $n \in \mathbb{N}_0$.

Now the proof of Theorem 2 is completely analogous to the proof of Theorem 3 in [10]. We have only to replace $s(s+1)$ with β^s, $\mathcal{B}_1$ with $\mathcal{B}_1 \cup \mathcal{B}_2$, (4.8) of [10] with (3.8), and Lemma 6 in [7] with Lemma 4. □

4. Stability conditions for the topological pressure

In this section we prove that the upper semi-continuity of the pressure for every continuous function g is equivalent to 0 is not periodic or 1 is not periodic. This result is analogous to Theorem 8 in [13]. We will need the following result, which is proved in [13] (Corollary 1.1 in [13]).

Lemma 5. *Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function. Assume that $r > 0$ is a real number, and suppose that $P \subseteq [0,1]$ is finite. Then for every $\varepsilon > 0$ there exists a continuous function $g : [0,1] \to \mathbb{R}$ with $0 \leq g(x) \leq r$ for all $x \in [0,1]$, $g(x) = r$ is equivalent to $x \in P$, and*

$$h_{\mathrm{top}}(T_f) \leq p(T_f, g) < \max\big\{r, h_{\mathrm{top}}(T_f)\big\} + \varepsilon.$$

Now we are able to prove the main result of this section.

Theorem 3. *Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function. Then the following properties are equivalent.*

(1) *For every continuous function $g : [0,1] \to \mathbb{R}$ we have*

$$\limsup_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) \leq p(T_f, g).$$

(2) *For every continuous function $g : [0,1] \to \mathbb{R}$ satisfying*

$$\liminf_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) \geq p(T_f, g)$$

we have

$$\lim_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) = p(T_f, g).$$

(3) *We have $\log r\big(G'(0)\big) = 0$.*

(4) *We have that ${T_f}^n 0 \neq 0$ for all $n \in \mathbb{N}$ or ${T_f}^n 1 \neq 1$ for all $n \in \mathbb{N}$.*

PROOF: It is obvious that (1) implies (2). In order to show that (3) implies (4) assume that (3) holds, and that there are $p, q \in \mathbb{N}$ with ${T_f}^p 0 = 0$ and ${T_f}^q 1 = 1$. Then $\mathcal{G}'$ is finite by the definition of $\mathcal{G}'$. Furthermore there is a $j \in \mathbb{N}_0$, such that ${T_f}^j 0$ has exactly two successors in $(\mathcal{G}', \to)$. Hence the Perron-Frobenius Theorem gives $r\big(G'(0)\big) > 1$, which contradicts (3).

Next suppose that (4) holds. Then Lemma 1 yields $\log r\big(G'(0)\big) \leq 0$, which shows (3). Let $g : [0,1] \to \mathbb{R}$ be a continuous function. By Lemma 1 we obtain $\log r\big(G'(g)\big) \leq p(T_f, g)$. Therefore (1) of Theorem 1 gives (1).

Hence it remains to show that (2) implies (4). Assume that (2) holds. Suppose that there are $p, q \in \mathbb{N}$ with ${T_f}^p 0 = 0$ and ${T_f}^q 1 = 1$. As (3) implies (4) we get $\log r\big(G'(0)\big) > 0$. Furthermore $\mathcal{G}'$ is finite. If $h_{\mathrm{top}}(T_f) = 0$, then Theorem 1 and Lemma 3 give $\limsup_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, 0) = \log r\big(G'(0)\big)$, and this contradicts (2), as $p(T_{\tilde{f}}, 0) = h_{\mathrm{top}}(T_{\tilde{f}}) \geq 0$ is trivial. Therefore suppose $h_{\mathrm{top}}(T_f) > 0$. Choose an r with $0 < r < h_{\mathrm{top}}(T_f)$ and

$$r + \log r\big(G'(0)\big) > h_{\mathrm{top}}(T_f).$$

By Lemma 5 there exists a continuous function $g : [0,1] \to \mathbb{R}$ with $g(x) \leq r$ for all $x \in [0,1]$, $g(x) = r$ for all $x \in P(\mathcal{G}')$ and

$$h_{\mathrm{top}}(T_f) \leq p(T_f, g) < r + \log r\big(G'(0)\big). \tag{4.1}$$

Using (2.2) and (2.5) we get $\log r\big(G'(g)\big) = r + \log r\big(G'(0)\big)$. Since

$$\lim_{n\to\infty} \frac{1}{n} \sup_{x\in[0,1]} \sum_{j=0}^{n-1} g({T_f}^j x) \leq r < p(T_f, g)$$

(2) of Theorem 1 gives $\liminf_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) \geq p(T_f, g)$ and

$$\limsup_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) = r + \log r\big(G'(0)\big).$$

By (4.1) this contradicts (2). $\square$

For $x \in [0,1]$ define

$$p(x) := \min\{n \in \mathbb{N} : {T_f}^n x = x\}, \tag{4.2}$$

where we set $p(x) := \infty$, if ${T_f}^n x \neq x$ for all $n \in \mathbb{N}$. In the situation considered in this paper there is a continuous function $g : [0,1] \to \mathbb{R}$ with $\limsup_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) > p(T_f, g)$, if and only if $p(0) < \infty$ and $p(1) < \infty$. By Theorem 8 in [13] there exists a continuous $g : [0,1] \to \mathbb{R}$, such that the pressure is not upper semi-continuous with respect to the R^0-topology, if and only if $2 \leq p(0) < \infty$ and $2 \leq p(1) < \infty$. Setting $f := f_0$ for the function f_0 defined in (3.1) we obtain an explicit example, where the pressure is upper semi-continuous with respect to the R^0-topology for all continuous $g : [0,1] \to \mathbb{R}$, but there is a continuous $g : [0,1] \to \mathbb{R}$ with $\limsup_{\|\tilde{f}-f\|_\infty \to 0} p(T_{\tilde{f}}, g) > p(T_f, g)$.

5. Continuity of the topological entropy

Finally we investigate the continuity of the entropy. We prove that the entropy is continuous, if $h_{\text{top}}(T_f) > 0$. This result is analogous to Theorem 1 and Theorem 2 in [9]. Recall the definition of $p(x)$ given in (4.2).

Theorem 4. *Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function. Then the following properties are equivalent.*

(1) *We have $\limsup_{\|\tilde{f}-f\|_\infty \to 0} h_{\text{top}}(T_{\tilde{f}}) > h_{\text{top}}(T_f)$.*

(2) *There exist $p, q \in \mathbb{N}$ with ${T_f}^p 0 = 0$ and ${T_f}^q 1 = 1$, and we have $h_{\text{top}}(T_f) = 0$.*

(3) *We have $p(0) = p(1) < \infty$, $h_{\text{top}}(T_f) = 0$, $\operatorname{card}\big(f^{-1}(\mathbb{Z})\big) \le 2$, and there exists a subset $\mathcal{C}$ of the Markov diagram $(\mathcal{D}, \to)$ of T_f with respect to $\mathcal{Z}_f$, such that for every $C \in \mathcal{C}$ there exists exactly one $D \in \mathcal{D}$ with $C \to D$, and $C \to D$ implies $D \in \mathcal{C}$.*

Proof: Obviously (3) implies (2). Now assume that (2) holds. Then $\mathcal{G}'$ is finite and there is a $j \in \mathbb{N}_0$, such that ${T_f}^j 0$ has exactly two successors in $(\mathcal{G}', \to)$. By the Perron-Frobenius Theorem we obtain $r\big(G'(0)\big) > 1$. Now Lemma 3 implies (1).

It remains to show that (1) implies (3).

Assume that (1) holds. Let $(\mathcal{D}, \to)$ be the Markov diagram of T_f with respect to $\mathcal{Z}_f$. Suppose that $\operatorname{card}\big(f^{-1}(\mathbb{Z})\big) > 2$. Then there are $c_1, c_2, c_3 \in [0,1]$ with $c_1 < c_2 < c_3$, $f(c_1) \in \mathbb{Z}$, $f(c_2) = f(c_1) + 1$ and $f(c_3) = f(c_1) + 2$. Set $E_1 := (c_1, c_2)$, $E_2 := (c_2, c_3)$ and $\mathcal{C} := \{E_1, E_2\}$. Then $\mathcal{C} \subseteq \mathcal{D}$ and $E_j \to E_k$ in $(\mathcal{D}, \to)$ for all $j, k \in \{1, 2\}$. Hence (1.4) gives $r\big(F_{\mathcal{C}}(0)\big) = 2$. By (1.5) and (1.6) we get $r\big(F(0)\big) \ge r\big(F_{\mathcal{C}}(0)\big) = 2$. Now (1.7) implies

$$h_{\text{top}}(T_f) \ge \log 2. \tag{5.1}$$

As every $a \in \mathcal{G}'$ has at most two successors in $(\mathcal{G}', \to)$, (2.2), (2.3) and (2.5) give

$$r\big(G'(0)\big) \le 2. \tag{5.2}$$

Using (5.1) and (5.2) we get by (1) of Theorem 1 that $\limsup_{\|\tilde{f}-f\|_\infty \to 0} h_{\text{top}}(T_{\tilde{f}}) \le h_{\text{top}}(T_f)$, which contradicts (1). Therefore we have shown that $\operatorname{card}\big(f^{-1}(\mathbb{Z})\big) \le 2$.

By Theorem 3 we get $p(0) < \infty$ and $p(1) < \infty$. If $p(0) \ge 2$ and $p(1) \ge 2$, then $\|\tilde{f} - f\|_\infty \to 0$ implies $(T_{\tilde{f}}, \mathcal{Z}_{\tilde{f}}) \to (T_f, \mathcal{Z}_f)$ in the R^0-topology. Hence Theorem 2 in [9] shows that $h_{\text{top}}(T_f) = 0$, and that there exists a $\mathcal{C} \subseteq \mathcal{D}$, such that for every $C \in \mathcal{C}$ there exists exactly one $D \in \mathcal{D}$ with $C \to D$, and $C \to D$ implies $D \in \mathcal{C}$. Now Lemma 3 in [9] gives $p(0) = p(1)$, which completes the proof of (3) in this case.

Suppose that $p(0) = 1$ (an analogous proof works in the case $p(1) = 1$). At first we consider the case $p(0) = p(1) = 1$. As $\operatorname{card}\big(f^{-1}(\mathbb{Z})\big) \le 2$ we get $\mathcal{D} = \big\{(0,1)\big\}$

with $(0,1) \to (0,1)$. Therefore (1.4) and (1.7) give $h_{\mathrm{top}}(T_f) = 0$, and (3) is satisfied in this case.

Now suppose that $q := p(1) > 1$. Then $\mathrm{card}\big(f^{-1}(\mathbb{Z})\big) = 2$. Let $c \in (0,1)$ be the unique element with $f(c) \in \mathbb{Z}$. For $j \in \{0,1,\ldots,q-1\}$ let $D_j \in \mathcal{D}$ be so that $D_0 \in \mathcal{Z}_f$, $D_{j-1} \to D_j$ for $j \in \{1,2,\ldots,q-1\}$ and ${T_f}^j 1 \in \overline{D_j}$. Since $T_f 0 = 0$ we have $\inf D \in \{0,c\}$ for every $D \in \mathcal{D}$. Furthermore ${T_f}^{q-1} 1 = c$. Hence $D_{q-1} = (0,c)$, and we get $D_{q-1} \to D_0 = (c,1)$ and $D_{q-1} \to (0,c)$. Next we define a map $\varphi : \mathcal{G}' \to \mathcal{D}$. Set $\varphi(0) := \varphi(c^-) := \varphi(c^+) := (0,c) = D_{q-1}$. If $a \in \mathcal{G}' \setminus \{0,c^-,c^+\}$, then there exists a $j \in \{0,1,\ldots,q-2\}$ with $a = {T_f}^j 1$, and we define $\varphi(a) := D_j$. For $a,b \in \mathcal{G}'$ we have that $a \to b$ in $(\mathcal{G}',\to)$ implies $\varphi(a) \to \varphi(b)$ in $(\mathcal{D},\to)$. Fix an $a \in \mathcal{G}'$. Let $b_0 \to b_1 \to \cdots \to b_n$ be a path of length n in $(\mathcal{G}',\to)$ with $b_0 = a$, and let $b_{n+1} \in \mathcal{G}'$ with $b_n \to b_{n+1}$. Define $\varphi_n(b_0 \to b_1 \to \cdots \to b_n) := \varphi(b_0) \to \varphi(b_1) \to \cdots \to \varphi(b_{n+1})$. Observe that φ_n is an injective map from the set of all paths $b_0 \to b_1 \to \cdots \to b_n$ of length n in $(\mathcal{G}',\to)$ with $b_0 = a$ to the set of all paths $C_0 \to C_1 \to \cdots \to C_{n+1}$ of length $n+1$ in $(\mathcal{D},\to)$ with $C_0 = \varphi(a)$. Therefore (1.5) and (2.4) give $\|G'(0)^n\| \le \|F(0)^{n+1}\|$ for all $n \in \mathbb{N}$. Hence by (1.6), (1.7) and (2.5) we obtain $\log r\big(G'(0)\big) \le h_{\mathrm{top}}(T_f)$. Now (1) of Theorem 1 contradicts (1). $\square$

The example described in (3.1) shows that the entropy need not be continuous in our situation, if it is continuous in the R^0-topology. From Theorem 1 and Theorem 4 it is easy to deduce the following result.

Corollary 4.1. *Let $f : [0,1] \to \mathbb{R}$ be a strictly increasing and continuous function. If $h_{\mathrm{top}}(T_f) > 0$, then $\lim_{\|\tilde{f}-f\|_\infty \to 0} h_{\mathrm{top}}(T_{\tilde{f}}) = h_{\mathrm{top}}(T_f)$.*

By Theorem 1 in [9] and Corollary 4.1 we obtain that $h_{\mathrm{top}}(T_f) > 0$ implies the continuity of the topological entropy both in our situation and with respect to the R^0-topology.

References

[1] Alsedà Ll., Mañosas F., Mumbrú P., *Continuity of entropy for bimodal maps*, J. London Math. Soc. **52** (1995), 547–567.

[2] Hofbauer F., *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. Math. **34** (1979), 213–237; *Part 2*, Israel J. Math. **38** (1981), 107–115.

[3] Hofbauer F., *Monotonic mod one transformations*, Studia Math. **80** (1984), 17–40.

[4] Misiurewicz M., *Jumps of entropy in one dimension*, Fund. Math. **132** (1989), 215–226.

[5] Misiurewicz M., Shlyachkov S.V., *Entropy of piecewise continuous interval maps*, European Conference on Iteration Theory (ECIT 89), Batschuns, 1989 (Ch. Mira, N. Netzer, C. Simó, Gy. Targoński, eds.), World Scientific, Singapore, 1991, pp. 239–245.

[6] Raith P., *Hausdorff dimension for piecewise monotonic maps*, Studia Math. **94** (1989), 17–33.

[7] Raith P., *Continuity of the Hausdorff dimension for piecewise monotonic maps*, Israel J. Math. **80** (1992), 97–133.

[8] Raith P., *Continuity of the Hausdorff dimension for invariant subsets of interval maps*, Acta Math. Univ. Comenian. **63** (1994), 39–53.

[9] Raith P., *Continuity of the entropy for monotonic mod one transformations*, Acta Math. Hungar. **77** (1997), 247–262.
[10] Raith P., *Stability of the maximal measure for piecewise monotonic interval maps*, Ergodic Theory Dynam. Systems **17** (1997), 1419–1436.
[11] Raith P., *The dynamics of piecewise monotonic maps under small perturbations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **24** (1997), 783–811.
[12] Raith P., *Perturbations of a topologically transitive piecewise monotonic map on the interval*, Proceedings of the European Conference on Iteration Theory (ECIT 96), Urbino, 1996 (L. Gardini et al., eds.), Grazer Math. Ber. **339** (1999), 301–312.
[13] Raith P., *Discontinuities of the pressure for piecewise monotonic interval maps*, preprint, Wien, 1997, Ergodic Theory Dynam. Systems, to appear.
[14] Walters P., *An introduction to ergodic theory*, Graduate Texts in Mathematics 79, Springer, New York, 1982.

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