Universal objects in qasiconstructs

R. Rother

Abstract. The general theory of Jónsson-classes is generalized to strongly smooth quasiconstructs in such a way that it also allows the construction of universal categories. One example of the theory is the existence of a concrete universal category over every base category. Properties are given which are (under certain conditions) equivalent to the existence of homogeneous universal objects. Thereby, we disprove the existence of a homogeneous C-universal category. The notion of homogeneity is strengthened to extremal homogeneity. Extremally homogeneous universal objects, for which additionally every morphism between smaller subobjects is extendable to an endomorphism, are constructed in so called extremally smooth quasiconstructs.

Keywords: universal object, universal category, smooth category, homogeneous, Jónsson class, special structure

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1. Homogeneous universal objects

A theory for homogeneous universal objects was introduced by B. Jónsson in [4]. For a detailed bibliography in this context consult [2]. Where Jónsson dealt with relational systems, we will use the more abstract setting of concrete categories of structured sets, constructs for short. A construct is a pair $(\mathcal{A}, U)$ consisting of a category $\mathcal{A}$ and a faithful functor $U : \mathcal{A} \to \mathbf{Set}$. A theory of concrete categories is given in [1], where one can find all the technical notions we will use. We will introduce strongly smooth quasiconstructs (where a quasiconstruct is a quasicategory $\mathcal{A}$ together with a faithful (quasi-)functor $U : \mathcal{A} \to \mathbf{CLS}$ into the quasicategory of classes and maps) and show in Section 3 by an example that this notion is more general than the concept of Jónsson-classes. For a set-theory which allows the formulation of quasicategories consult the appendix of [3] and bibliographical remarks there. We will use Ω to denote the (quasi-)cardinality of proper classes (we assume the axiom of choice for classes, so all proper classes are equipollent) and we remark that Ω is strongly inaccessible, especially $\Omega^{<\Omega} = \Omega$ (where $\kappa^{<\alpha} := \sum_{\gamma<\alpha} \kappa^{\gamma}$). Moreover we have $\omega^{<\omega} = \omega$, but $\kappa^{<\kappa} = \kappa$ in ZF is not provable for any other cardinal. For $A \in \mathcal{A}$ we define $|A| := |UA|$. We denote by $\mathcal{A}_{<\kappa}$ ($\mathcal{A}_{\leq\kappa}$) the full subcategory of $\mathcal{A}$ with objects $|A| < \kappa$ ($|A| \leq \kappa$). For $f \in \mathcal{A}$ we often write f instead of Uf.

Recall that a morphism $f : A \to B$ is initial if for every map $g : UC \to UA$ such that $f \circ g : C \to B$ is a morphism, $g : C \to A$ is a morphism as well. An initial injective morphism we call an embedding. It has turned out that, in most cases, this notion describes the usual choice of embeddings. We fix a subcategory $\mathcal{M}$ of $\mathcal{A}_{\leq\kappa}$ which consists of embeddings only and is hereditary, i.e. for $f, g \in \mathcal{M}$ and $f = g \circ h$, h also belongs to $\mathcal{M}$. The heredity yields that $\mathcal{M} \hookrightarrow \mathcal{A}$ reflects isos, because $\mathcal{M}$-morphisms are also embeddings in $\mathcal{M}$. Think of $\mathcal{M}$ as carrying κ as an invisible index. An $\mathcal{M}$-chain is a diagram $(a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta<\gamma}$ of $\mathcal{M}$-morphisms for an ordinal $\gamma \leq \kappa$ such that always $a_{\delta,\beta} \circ a_{\alpha,\delta} = a_{\alpha,\beta}$ and $|A_\alpha| < \kappa$ (more conveniently, an $\mathcal{M}$-chain is simply a functor $D : \gamma \to \mathcal{M}_{<\kappa}$). If $(a_\alpha : A_\alpha \to A)_{\alpha<\gamma}$ in $\mathcal{M}$ is a compatible cocone for an $\mathcal{M}$-chain $\mathbf{A} = (a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta<\gamma}$ which covers UA then we say that $\mathbf{A}$ converges to A (by the $(a_\alpha)_{\alpha<\gamma}$). Compatibility of the cocone means $a_\alpha = a_\beta \circ a_{\alpha,\beta}$ for all $\alpha < \beta < \gamma$.

Definition 1.1. A quasiconstruct $(\mathcal{A}, U)$ is called $\mathcal{M}$-smooth if:

(F1) (Downward Löwenheim Skolem Property)
For each $A \in \mathcal{A}_{\leq\kappa}$ and each $X \subseteq UA$ such that $|X| < \kappa$ there is an $m : B \to A$ in $\mathcal{M}$ such that $X \subseteq \mathrm{im}(m)$ and $|B| < \kappa$.

(F2) If $(a_\alpha : A_\alpha \to A)_{\alpha<\gamma}$ in $\mathcal{M}$ is a compatible cocone for the $\mathcal{M}$-chain $\mathbf{A} = (a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta<\gamma}$ then all the a_α factor through a fixed $m : B \to A$ in $\mathcal{M}$ by some $b_\alpha : A_\alpha \to B$ in $\mathcal{M}$ such that $\mathbf{A}$ converges to B by the $(b_\alpha)_{\alpha<\gamma}$.

(F3) Let $(a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta<\gamma}$ converge to A by $(a_\alpha : A_\alpha \to A)_{\alpha<\gamma}$, let $(b_{\alpha,\beta} : B_\alpha \to B_\beta)_{\alpha<\beta<\gamma}$ converge to B by $(b_\alpha : B_\alpha \to B)_{\alpha<\gamma}$, and let $(e_\alpha : A_\alpha \to B_\alpha)_{\alpha<\gamma}$ be a family in $\mathcal{M}$ such that always $e_\beta \circ a_{\alpha,\beta} = b_{\alpha,\beta} \circ e_\alpha$. Then there exists an $e : A \to B$ in $\mathcal{M}$ such that always $e \circ a_\alpha = b_\alpha \circ e_\alpha$.

If $\mathcal{M}$ denotes all embeddings of $\mathcal{A}_{\leq\kappa}$ we call $(\mathcal{A}, U)$ κ-smooth.

If $\mathcal{M}$ denotes all embeddings then in topological constructs we just have to check (F3), whereas in varieties, it suffices to check (F1). The idea behind (F2) is that one can restrict a sink of $\mathcal{M}$-morphisms to its image. The idea behind (F3) is that converging chains should be directed colimits, preserving $\mathcal{M}$-morphisms. (F3) fails to hold in the category **Top** of topological spaces, for example, and (F1) often fails to hold in algebra for $\kappa = \omega$. Obviously Jónsson-classes (as [linearly] ordered sets or non-trivial Boolean algebras) are κ-smooth for infinite κ. If $\mathcal{M}$ denotes all full embeddings we get that **CAT**, the quasicategory of all categories, and $\mathbf{CAT}(\mathcal{X})$, the quasicategory of concrete categories over the base category $\mathcal{X}$ together with concrete functors, are $\mathcal{M}$-smooth. The forgetful functors are chosen in the natural way, since functors between categories are maps between structured classes. If nothing else is said, we always choose $\kappa = \Omega$.

Obviously a subconstruct of an $\mathcal{M}$-smooth quasiconstruct which is closed w.r. to $\mathcal{M}$-subobjects is $\mathcal{M}$-smooth. Since the quasicategory of preordered classes is κ-smooth for infinite κ, also [linearly] [pre-]ordered classes are κ-smooth for infinite κ. Because **CAT** is $\mathcal{M}$-smooth for $\mathcal{M}$ the full embeddings, we also get that $\mathbf{C}-\mathbf{CAT}$, the quasicategory of all concretizable categories, is $\mathcal{M}$-smooth.

We say that $W \in \mathcal{A}$ is $\mathcal{M}$-universal if $|W| = \kappa$ and for each $A \in \mathcal{A}_{\leq\kappa}$ there is an $m : A \to W$ in $\mathcal{M}$. We say $W \in \mathcal{A}$ is weakly $\mathcal{M}$-universal if $|W| = \kappa$ and for each $A \in \mathcal{A}_{<\kappa}$ there is an $m : A \to W$ in $\mathcal{M}$. We say that $W \in \mathcal{A}$ is $\mathcal{M}$-homogeneous if for all $m_1 : A \to W$ and $m_2 : A \to W$ in $\mathcal{M}$ such that $|A| < \kappa$ there is an iso $h : W \to W$ in $\mathcal{M}$ such that $h \circ m_1 = m_2$. When $\mathcal{M}$ consists of all embeddings in $\mathcal{A}_{\leq\kappa}$, we also use the terms homogeneous resp. [weakly] κ-universal.

Given some A with $|A| = \kappa \geq \omega$ in an $\mathcal{M}$-smooth $(\mathcal{A}, U)$, one can construct a chain in it by transfinite recursion. Let $\{X_\alpha\}_{\alpha<\mathrm{cof}(\kappa)}$ be a chain of sets in $\mathbf{Set}_{<\kappa}$ with union UA. By (F1) we obtain an $m_0 : A_0 \to A$ (with $\emptyset \subseteq \mathrm{im}(m_0)$). In the isolated step, there is an $m_{\alpha+1} : A_{\alpha+1} \to A$ such that $\mathrm{im}(m_\alpha) \cup X_\alpha \subseteq \mathrm{im}(m_{\alpha+1})$. We obtain an $a_{\alpha,\alpha+1} : A_\alpha \to A_{\alpha+1}$, because $m_{\alpha+1}$ is an embedding and because $\mathcal{M}$ is hereditary. For the limit step we use (F2) in the obvious way. Because in each limit step λ, the chain converges to A_λ, we call it continuous (or smooth).

Lemma 1.2. *Let W be $\mathcal{M}$-homogeneous and weakly $\mathcal{M}$-universal in an $\mathcal{M}$-smooth $(\mathcal{A}, U)$ for $\kappa \geq \omega$. Then*

(1) *W is $\mathcal{M}$-universal. Moreover if $m_0 : A_0 \to W$ is in $\mathcal{M}$, where $|A_0| < \kappa$, and there is an $a_0 : A_0 \to A$ in $\mathcal{M}$ then m_0 factors through a_0 in $\mathcal{M}$.*

(2) *If V is also $\mathcal{M}$-homogeneous and $\mathcal{M}$-universal then $V \cong W$.*

Proof: (1) Let the continuous $\mathcal{M}$-chain $(a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta<\mathrm{cof}(\kappa)}$ converge to A by $(a_\alpha : A_\alpha \to A)_{\alpha<\mathrm{cof}(\kappa)}$. Let $(m_\alpha : A_\alpha \to W)_{\alpha\leq\gamma}$ be given as a compatible cocone for $(a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta\leq\gamma}$. Let $\bar{m}_{\gamma+1} : A_{\gamma+1} \to W$ be an $\mathcal{M}$-morphism. Because of the $\mathcal{M}$-homogeneity of W, there is an iso φ on W such that $\varphi \circ (\bar{m}_{\gamma+1} \circ a_{\gamma,\gamma+1}) = m_\gamma$. Now we choose $m_{\gamma+1} := \varphi \circ \bar{m}_{\gamma+1}$.

In the limit step we have a compatible cocone $(m_\alpha : A_\alpha \to W)_{\alpha<\lambda}$ for $(a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta<\lambda}$. Because the chain is continuous, $(a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta<\lambda}$ converges to A_λ by $(a_{\alpha,\lambda} : A_\alpha \to A_\lambda)_{\alpha<\lambda}$. Hence by $\mathcal{M}$-smoothness, there exists $m_\lambda : A_\lambda \to W$ such that always $m_\lambda \circ a_{\alpha,\lambda} = m_\alpha$ (by (F1) and (F2), one constructs an $\mathcal{M}$-chain $(\tilde{m}_\alpha : A_\alpha \to B_\alpha)_{\alpha<\lambda}$ converging to some B_λ generated by the images of the m_α in W; using (F3), one obtains m_λ).

(2) Let $(a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta<\mathrm{cof}(\kappa)}$ (resp. $(b_{\alpha,\beta} : B_\alpha \to B_\beta)_{\alpha<\beta<\mathrm{cof}(\kappa)}$) be a continuous $\mathcal{M}$-chain converging to W (resp. V) by $(a_\alpha : A_\alpha \to W)_{\alpha<\mathrm{cof}(\kappa)}$ (resp. $(b_\alpha : B_\alpha \to V)_{\alpha<\mathrm{cof}(\kappa)}$). Define $V_0 := B_0$ and $v_0 := b_0$. Choose $m : V_0 \to W$ in $\mathcal{M}$. By (F1) there are $|W_0| < \kappa$ and $m_0 : V_0 \to W_0$, $w_0 : W_0 \to W$ in $\mathcal{M}$ such that $w_0 \circ m_0 = m$.

By transfinite recursion, we construct $(m_\alpha : V_\alpha \to W_\alpha)_{\alpha<\mathrm{cof}(\kappa)}$ in $\mathcal{M}$ such that $|V_\alpha|, |W_\alpha| < \kappa$ for each $\alpha < \mathrm{cof}(\kappa)$, $(v_\alpha : V_\alpha \to V)_{\alpha<\mathrm{cof}(\kappa)}$, $(w_\alpha : W_\alpha \to W)_{\alpha<\mathrm{cof}(\kappa)}$, and $(v_{\alpha,\beta} : V_\alpha \to V_\beta)_{\alpha<\beta<\mathrm{cof}(\kappa)}$, $(w_{\alpha,\beta} : W_\alpha \to W_\beta)_{\alpha<\beta<\mathrm{cof}(\kappa)}$ also in $\mathcal{M}$ such that always $m_\beta \circ v_{\alpha,\beta} = w_{\alpha,\beta} \circ m_\alpha$ and $v_\beta \circ v_{\alpha,\beta} = v_\alpha$ and $w_\beta \circ w_{\alpha,\beta} = w_\alpha$.

In the isolated step, we obtain by (F1) $|\bar{W}_\alpha| < \kappa$ and $\bar{w}_\alpha : \bar{W}_\alpha \to W$ in $\mathcal{M}$ such that $a_\alpha[UA_\alpha] \cup w_\alpha[UW_\alpha] \subseteq \bar{w}_\alpha[U\bar{W}_\alpha]$. Because $\bar{w}_\alpha$ is an embedding we obtain $\bar{w}_{\beta,\alpha} : W_\beta \to \bar{W}_\alpha$ such that $\bar{w}_\alpha \circ \bar{w}_{\beta,\alpha} = w_\beta$ for each $\beta \leq \alpha$. Now by (1), there is an embedding $m : \bar{W}_\alpha \to V$ such that $m \circ (\bar{w}_{\alpha,\alpha} \circ m_\alpha) = v_\alpha$.

Now we choose $|V_{\alpha+1}| < \kappa$ and $v_{\alpha+1} : V_{\alpha+1} \to V$ in $\mathcal{M}$ such that $b_\alpha[UB_\alpha] \cup v_\alpha[UV_\alpha] \cup m[U\bar{W}_\alpha] \subseteq v_{\alpha+1}[UV_{\alpha+1}]$. Because $v_{\alpha+1}$ is an embedding we obtain

$v_{\beta,\alpha+1} : V_\beta \to V_{\alpha+1}$ such that $v_{\alpha+1} \circ v_{\beta,\alpha+1} = v_\beta$ for each $\beta \leq \alpha$, and $\bar{m} : \bar{W}_\alpha \to V_{\alpha+1}$ such that $v_{\alpha+1} \circ \bar{m} = m$. By (1), there is an embedding $\tilde{m}_{\alpha+1} : V_{\alpha+1} \to W$ such that $\tilde{m}_{\alpha+1} \circ \bar{m} = \bar{w}_\alpha$. By (F1), there is $|W_{\alpha+1}| < \kappa$ and $w_{\alpha+1} : W_{\alpha+1} \to W$ such that $\mathrm{im}(\tilde{m}_{\alpha+1}) \subseteq \mathrm{im}(w_{\alpha+1})$. Because $w_{\alpha+1}$ is an embedding, there is $m_{\alpha+1} : V_{\alpha+1} \to W_{\alpha+1}$ such that $\tilde{m}_{\alpha+1} = w_{\alpha+1} \circ m_{\alpha+1}$ and $w_{\beta,\alpha+1} : W_\beta \to W_{\alpha+1}$ for each $\beta \leq \alpha$ with the appropriate properties.

In the limit step $\mathcal{M}$-smoothness yields v_λ, w_λ and $m_\lambda : V_\lambda \to W_\lambda$ as desired. In the same way one obtains $m : V \to W$ such that always $m \circ v_\alpha = w_\alpha \circ m_\alpha$. Now m is a surjective embedding, hence an iso. □

Definition 1.3. An $\mathcal{M}$-smooth $(\mathcal{A}, U)$ is called strongly $\mathcal{M}$-smooth if

- (SF1) (a) $\mathcal{A}_{<\kappa} \neq \emptyset$ has at most $\kappa^{<\kappa}$ objects modulo isomorphy;
  (b) for each $A \in \mathcal{A}_{<\kappa}$, there is a non-surjective $m : A \to B$ in $\mathcal{M}$.
- (SF2) Every $\mathcal{M}$-chain converges to some object.
- (SF3) For all $A, B \in \mathcal{A}_{<\kappa}$ there is an object $A + B$ and $m_A : A \to (A + B)$, $m_B : B \to (A + B)$ in $\mathcal{M}$.
- (SF4) For all $m_A : D \to A$ and $m_B : D \to B$ in $\mathcal{M}_{<\kappa}$ there are $p_A : A \to P$ and $p_B : B \to P$ in $\mathcal{M}$ such that $p_A \circ m_A = p_B \circ m_B$. (p_A, p_B) is called an amalgamation of (m_A, m_B).

By (F1) one can choose $|A + B|, |P| < \kappa$.

The conditions (SF2) to (SF4) have some colimit-like flavour. If in a strongly $\mathcal{M}$-smooth $\mathcal{A}_{\leq\kappa}$, the appropriate colimits exist then they are easily seen to consist of embeddings. So if $\mathcal{M}$ contains all embeddings, the colimits are in fact possible choices for the constructions in (SF2) to (SF4). Thus, to check strong $\mathcal{M}$-smoothness in cocomplete (quasi-)constructs, we usually have to check that certain colimits preserve $\mathcal{M}$-morphisms in the following sense.

Definition 1.4. We say that a collection of colimits preserves $\mathcal{M}$-morphisms if, whenever the diagram is in $\mathcal{M}$, then the colimit cocone is in $\mathcal{M}$.

$(\mathcal{A}, U)$ is called transportable if for each structured bijection $b : X \to UA$, there exists an object B with $UB = X$ such that $b : B \to A$ is an iso. Obviously the transportable hull of a strongly $\mathcal{M}$-smooth category is strongly $\mathcal{M}$-smooth. If B is $\mathcal{M}$-homogeneous and $\mathcal{M}$-universal in the transportable hull of $(\mathcal{A}, U)$ then so is some $W \cong B$ in $\mathcal{A}$. To be more precise, when talking about the transportable hull of $\mathcal{A}$ we also consider the transportable hull of $\mathcal{M}$.

Theorem 1.5. *Let $(\mathcal{A}, U)$ be $\mathcal{M}$-smooth and $\kappa^{<\kappa} = \kappa$. E.a.*

1. *$(\mathcal{A}, U)$ is strongly $\mathcal{M}$-smooth.*
2. *There is an $\mathcal{M}$-homogeneous $\mathcal{M}$-universal object.*

Proof: (1) $\Longrightarrow$ (2). This direction is the special case $\mu = \kappa$ of Proposition 1.6.

We prove (2) $\Longrightarrow$ (1). Obviously (SF1) is necessary for the existence of an $\mathcal{M}$-universal object where $\mathcal{A}_{<\kappa} \neq \emptyset$ follows by (F1), because $\emptyset \subseteq UW$ for the $\mathcal{M}$-homogeneous $\mathcal{M}$-universal object W.

For (SF2), one $\mathcal{M}$-embeds the $\mathcal{M}$-chain into W recursively by Lemma 1.2; by (F2) it converges to some B.

For (SF3), one $\mathcal{M}$-embeds A and B into W.

For (SF4), let $m_A : D \to A$ and $m_B : D \to B$ be $\mathcal{M}$-morphisms in $\mathcal{A}_{<\kappa}$. There are $\mathcal{M}$-morphisms $p_A : A \to W$ and $\bar{p}_B : B \to W$. By $\mathcal{M}$-homogeneity, there is an iso $\varphi : W \to W$ such that $\varphi \circ \bar{p}_B \circ m_B = p_A \circ m_A$. Choose $p_B := \varphi \circ \bar{p}_B$. □

For technical reasons we introduce the notion of a (μ, κ)-$\mathcal{M}$-homogeneous-universal object which itself is of cardinality κ but is universal just for objects of at most the cardinality μ and homogeneous just for objects in $\mathcal{A}_{<\mu}$.

Proposition 1.6. *Let $\omega \leq \mu \leq \kappa = \kappa^{<\mu}$ for some regular κ and $(\mathcal{A}, U)$ be strongly $\mathcal{M}$-smooth. If $\mathcal{A}_{<\mu}$ contains at most κ objects modulo isomorphy and:*

(F1)$_{\mu,\kappa}$ *For $A \in \mathcal{A}_{\leq\kappa}$ and $X \subseteq UA$ such that $|X| < \mu$ there is a $B \in \mathcal{A}_{<\mu}$ and an $m : B \to A$ in $\mathcal{M}$ such that $X \subseteq m[UB]$.*

Then for each $C \in \mathcal{A}_{\leq\kappa}$, there is a (μ, κ)-$\mathcal{M}$-homogeneous-universal object containing C as an $\mathcal{M}$-subobject.

Proof: Without loss of generality, we assume that $(\mathcal{A}, U)$ and $\mathcal{M}$ are transportable. Because of $(\mathrm{F1})_{\mu,\kappa}$ and (SF1), there is some $E \in \mathcal{A}_{<\mu}$ such that $UE \neq \emptyset$. Let $(E_\alpha)_{\alpha<\kappa}$ be the family of all non-empty objects of $\mathcal{A}_{<\mu}$ modulo isomorphy. Let $(C_\alpha)_{\alpha<\kappa}$ be a continuous $\mathcal{M}$-chain in C converging to C by $\mathcal{M}$-inclusions. We define an $\mathcal{M}$-chain $(w_{\alpha,\beta} : W_\alpha \to W_\beta)_{\alpha<\beta<\kappa}$ of inclusions and, for each $\alpha < \kappa$, an $\mathcal{M}$-morphism $e_\alpha : E_\alpha \to W_{\alpha+1}$. We choose W such that $(w_{\alpha,\beta} : W_\alpha \to W_\beta)_{\alpha<\beta<\kappa}$ converges to W. Then W is weakly (μ, κ)-$\mathcal{M}$-universal.

During the transfinite recursion we define, step by step, the family $(\varphi_{\alpha,\beta} : A_{\alpha,\beta} \to B_{\alpha,\beta})_{\alpha,\beta<\kappa}$ of all isos between concrete $\mathcal{M}$-subobjects of W with smaller cardinality than μ and some yet to be specified extensions thereof. A concrete $\mathcal{M}$-subobject is an $\mathcal{M}$-subobject for which the $\mathcal{M}$-morphism is an inclusion. Let $\rho : \kappa \to \kappa^2$ count every $(\alpha, \beta) \in \kappa^2$ exactly κ times; then for $\rho(\alpha) = (\rho_1(\alpha), \rho_2(\alpha))$ we choose $\sigma(\alpha) := (\min(\alpha, \rho_1(\alpha)), \min(\alpha, \rho_2(\alpha)))$.

Let $W_0 := E_0$ and $\mathbf{A}_0(\varphi) := \{\varphi\}$ for each iso φ between concrete $\mathcal{M}$-subobjects of W_0. Now assume that the $\mathcal{M}$-chain $(w_{\alpha,\beta} : W_\alpha \to W_\beta)_{\alpha<\beta\leq\gamma}$ and the family $(\varphi_{\alpha,\beta} : A_{\alpha,\beta} \to B_{\alpha,\beta})_{\alpha<\gamma,\beta<\kappa}$ of isos are already defined, and W_γ contains C_γ. Let $(\varphi_{\gamma,\beta} : A_{\gamma,\beta} \to B_{\gamma,\beta})_{\beta<\kappa}$ be the family of all isos φ between concrete $\mathcal{M}$-subobjects of W_γ in $\mathcal{A}_{<\mu}$ and, additionally, of all elements of $\mathbf{A}_\gamma(\varphi)$. For each φ that has not yet appeared as a $\varphi_{\alpha,\beta}$ for $\alpha < \gamma$, we set $\mathbf{A}_\gamma(\varphi) := \{\varphi\}$. We construct $W_{\gamma+1}$ in two steps.

Step 1. Let W^1_γ be an amalgamation of $C_\gamma \to W_\gamma$ and $C_\gamma \to (C_{\gamma+1} + E_\gamma)$ containing W_γ as a concrete $\mathcal{M}$-subobject. By (SF1), we can assume that the $\mathcal{M}$-inclusion $m_\gamma : W_\gamma \to W^1_\gamma$ is non-surjective. Obviously, there is an $\mathcal{M}$-morphism $e_\gamma : E_\gamma \to W^1_\gamma$.

Step 2. We denote by $i_A : A_{\sigma(\gamma)} \to W^1_\gamma$ and $i_B : B_{\sigma(\gamma)} \to W^1_\gamma$ the inclusions. By (SF4), there are $p_A : W^1_\gamma \to W_{\gamma+1}$, $p_B : W^1_\gamma \to W_{\gamma+1}$ as an amalgamation

$p_A \circ i_A = p_B \circ i_B \circ \varphi_{\sigma(\gamma)}$. We can choose p_B as an inclusion. Hence W_γ^1 is a concrete $\mathcal{M}$-subobject of $W_{\gamma+1}$, as is (by transportability) the image of p_A denoted by $\mathrm{im}(p_A)$. $\varphi := p_A|^{\mathrm{im}(p_A)} : W_\gamma^1 \to \mathrm{im}(p_A)$ is an iso extending $\varphi_{\sigma(\gamma)} : A_{\sigma(\gamma)} \to B_{\sigma(\gamma)}$. For each ψ such that $\varphi_{\sigma(\gamma)}$ is the greatest element of the chain $\mathbf{A}_\gamma(\psi)$, set $\mathbf{A}_{\gamma+1}(\psi) := \mathbf{A}_\gamma(\psi) \cup \{\varphi\}$. We have $|\mathbf{A}_{\gamma+1}(\psi)| < \kappa$.

By (SF2), the $\mathcal{M}$-chain $(w_{\alpha,\beta} : W_\alpha \to W_\beta)_{\alpha<\beta<\lambda}$ of inclusions converges to some W_λ by some $(w_\alpha : W_\alpha \to W_\lambda)_{\alpha<\lambda}$. By transportability we can choose W_λ such that the w_α are inclusions, too. For each iso ψ between concrete $\mathcal{M}$-subobjects of W_λ, define $\mathbf{A}_\lambda(\psi) := \bigcup_{\alpha<\lambda} \mathbf{A}_\alpha(\psi) \cup \{\bar\psi\}$, where $\bar\psi$ is the extension of all elements of $\bigcup_{\alpha<\lambda} \mathbf{A}_\alpha(\psi)$, which exists by $\mathcal{M}$-smoothness. For $\lambda < \kappa$, we have $|\mathbf{A}_\lambda(\psi)| < \kappa$ because κ is regular.

It remains to show the $\mathcal{M}$-homogeneity of W. Given some iso $\varphi : A \to B$ in $\mathcal{A}_{<\mu}$ of concrete $\mathcal{M}$-subobjects of W, it is enough to construct an automorphism $\bar\varphi : W \to W$ extending φ. Because of $|A|, |B| < \mathrm{cof}(\kappa)$, we get that A, B appear as concrete $\mathcal{M}$-subobjects of some W_α for some $\alpha < \kappa$. So there is some $\alpha < \mu$ such that $\varphi = \varphi_{\sigma(\alpha)} =: \psi_0$. Let $D_0 := A_{\sigma(\alpha)}$ and $C_0 := B_{\sigma(\alpha)}$. If $\psi_\gamma : D_\gamma \to C_\gamma$ is already defined then $\psi^{-1} : C_\gamma \to D_\gamma$ also appears as $\varphi_{\sigma(\alpha)} : A_{\sigma(\alpha)} \to B_{\sigma(\alpha)}$ for some $\alpha > \gamma$ and can be extended to an iso $\bar\psi_\gamma : \bar C_\gamma \to \bar D_\gamma$ such that W_γ a concrete $\mathcal{M}$-subobject of $\bar C_\gamma$, which follows by the construction of W. Now $\bar\psi_\gamma^{-1}$ is again $\varphi_{\sigma(\alpha)}$ for some $\alpha > \gamma$ and there is an extension $\psi_{\gamma+1} : D_{\gamma+1} \to C_{\gamma+1}$ between concrete $\mathcal{M}$-subobjects of W such that W_γ is a concrete $\mathcal{M}$-subobject of both $D_{\gamma+1}$ and $C_{\gamma+1}$. In the limit step, we choose the iso $\psi_\lambda : D_\lambda \to C_\lambda$ as the union of all isos already defined, which exists by (F3) and (F2). Choosing $\bar\varphi$ as ψ_κ gives us an automorphism on W extending $\varphi : A \to B$. □

We introduce a stronger notion of a homogeneous object, which in most cases can be constructed analogously. We call an $\mathcal{M}$-homogeneous object W extremally $\mathcal{M}$-homogeneous if for each $A \in \mathcal{A}_{<\kappa}$, each $\mathcal{M}$-morphism $m : A \to W$, and each morphism $f : A \to W$ there is some endomorphism $\bar f : W \to W$ such that $\bar f \circ m = f$. We call a strongly $\mathcal{M}$-smooth quasiconstruct $(\mathcal{A}, U)$ extremally $\mathcal{M}$-smooth if:

- (E1) Every $\mathcal{M}$-chain has a covering colimit consisting of $\mathcal{M}$-morphisms.
- (E2) $\mathcal{A}_{<\kappa}$ has $\mathcal{M}$-extensions, i.e. for each $m : A \to B$ in $\mathcal{M}_{<\kappa}$ and each $f : A \to C$ in $\mathcal{A}_{<\kappa}$, there is some $\bar m : C \to P$ in $\mathcal{M}_{<\kappa}$ and $\bar f : B \to P$ in $\mathcal{A}_{<\kappa}$ such that $\bar m \circ f = \bar f \circ m$.

By the dual of the pullback lemma in an $(\mathcal{E}, \mathcal{M})$-factorizable category $(\mathcal{A}, U)$, where $\mathcal{E} \subseteq \mathrm{Epi}$, $\mathcal{M}$-morphisms are pushout-stable iff they are pushout-stable along $\mathcal{M}$- and $\mathcal{E}$-morphisms.

Theorem 1.7. (1) *If $(\mathcal{A}, U)$ is extremally $\mathcal{M}$-smooth for some $\kappa^{<\kappa} = \kappa$ then there is an extremally $\mathcal{M}$-homogeneous $\mathcal{M}$-universal object* (*which necessarily is the $\mathcal{M}$-homogeneous $\mathcal{M}$-universal one*).

(2) *Let $\omega \leq \mu \leq \kappa = \kappa^{<\mu}$ for some regular κ and $(\mathcal{A}, U)$ be extremally $\mathcal{M}$-smooth. If $\mathcal{A}_{<\mu}$ contains at most κ objects modulo isomorphy then under* $(\mathrm{F1})_{\mu,\kappa}$ *there is an extremally (μ,κ)-$\mathcal{M}$-homogeneous-universal object.*

Proof: We give instructions on how to modify the proof of Proposition 1.6. Besides the isos $\varphi_{\alpha,\beta}$, we construct the family $(f_{\alpha,\beta} : \tilde{A}_{\alpha,\beta} \to \tilde{B}_{\alpha,\beta})_{\alpha,\beta<\kappa}$ of all morphisms in $\mathcal{A}_{<\mu}$ between concrete $\mathcal{M}$-subobjects of W. If $(f_{\alpha,\beta} : \tilde{A}_{\alpha,\beta} \to \tilde{B}_{\alpha,\beta})_{\alpha<\gamma,\beta<\kappa}$ and W_γ are already defined, let $(f_{\gamma,\beta} : \tilde{A}_{\gamma,\beta} \to \tilde{B}_{\gamma,\beta})_{\beta<\kappa}$ be the family of all morphisms f between concrete $\mathcal{M}$-subobjects of W_γ in $\mathcal{A}_{<\mu}$ and morphisms in $\mathbf{B}_\gamma(f)$, where $\mathbf{B}_\gamma(f) := \{f\}$ for recently involved f.

Replace $W_{\gamma+1}$ by W_γ^2 in Step 2 and insert:

Step 3. We denote by $i_A : \tilde{A}_{\sigma(\gamma)} \to W_\gamma^2$, $i_B : \tilde{B}_{\sigma(\gamma)} \to W_\gamma^2$ the $\mathcal{M}$-inclusions. By (E2) we take the $\mathcal{M}$-extension $\tilde{f}_{\sigma(\gamma)} : W_\gamma^2 \to W_{\gamma+1}$, $i : W_\gamma^2 \to W_{\gamma+1}$ of i_A and $i_B \circ f_{\sigma(\gamma)}$. For each f such that $f_{\sigma(\gamma)}$ is the greatest element of $\mathbf{B}_\gamma(f)$, let $\mathbf{B}_{\gamma+1}(f) := \mathbf{B}_\gamma(f) \cup \{\tilde{f}_{\sigma(\gamma)}\}$.

For a limit ordinal λ and each morphism f between concrete $\mathcal{M}$-subobjects of W_λ, let $\mathbf{B}_\lambda(f) := \bigcup_{\alpha<\lambda} \mathbf{B}_\alpha(f) \cup \{\bar{f}\}$, where $\bar{f}$ denotes the extension of all morphisms in $\bigcup_{\alpha<\lambda} \mathbf{B}_\alpha(f)$ according to (E1).

Now we have to check that W is extremally $\mathcal{M}$-homogeneous. To this end, let $f : A \to W$ be a morphism, and let $m : A \to W$ be an $\mathcal{M}$-morphism such that $|A| < \mu$. Without loss of generality, we assume that m is the inclusion. Let $g_0 := f$. Assume $g_\alpha : \tilde{D}_\alpha \to \tilde{C}_\alpha$ is already constructed. Because κ is regular, there is some $\gamma > \alpha$ such that $g_\alpha = f_{\sigma(\gamma)}$. Define $g_{\alpha+1} := \tilde{f}_{\sigma(\gamma)}$; hence W_γ, and thus W_α, is a concrete $\mathcal{M}$-subobject of $\tilde{D}_{\alpha+1}$. In the limit step one uses property (E1). □

If one drops the universal property of the colimits in (E1) then under some extendability of morphisms in a sense related to (F3), the properties (E1), (E2) become necessary for the existence of extremally $\mathcal{M}$-homogeneous $\mathcal{M}$-universal objects.

Example 1.8. (1) Denote by $\mathbb{B}$ the Jónsson-class of proper Boolean algebras (i.e. all Boolean algebras except for the singleton). $\mathbb{B}$ is not closed w.r. to pushouts in the variety of Boolean algebras. Take, for example, the 4-element Boolean algebra and project it in the two possible ways onto the two element chain. The pushout is the one-element Boolean algebra. But for $\mathcal{M}$ all embeddings, $\mathbb{B}$ is extremally $\mathcal{M}$-smooth, and thus there is an extremally $\mathcal{M}$-homogeneous $\mathcal{M}$-universal Boolean algebra for each $\kappa = \kappa^{<\kappa}$. Because homogeneous κ-universal objects are unique, this is the $\mathcal{M}$-homogeneous $\mathcal{M}$-universal Boolean algebra constructed in [6], where it is also shown that $\kappa = \kappa^{<\kappa}$ is necessary for its existence.

Proof: It remains to check (E2) to prove extremal $\mathcal{M}$-smoothness. But in Stone spaces, epis are pullback-stable, because pullbacks are concrete, and thus monos are pushout-stable in $\mathbb{B}$. Hence, if any mono is part of the diagram in $\mathbb{B}$, the pushout will not be trivial. $\square$

(2) **PROCL** (preordered classes), **POCL** (partially ordered classes), **GRA** (classes with one binary relation), **RERE** (classes with a reflexive relation), **SYM** (classes with a symmetric relation) and many similar quasiconstructs contain an extremally homogeneous κ-universal object for each $\kappa = \kappa^{<\kappa}$.

PROOF: We first look at **GRA**. Let $e : (A, \rho) \hookrightarrow (B, \sigma)$ be an initial inclusion and $f : (A, \rho) \to (C, \tau)$ be a monotone map such that $C \cap B = \emptyset$. We set $P := (B \backslash A) \cup C$. Then (P, π) is the pushout by the initial inclusion $\bar{e} : (C, \tau) \hookrightarrow (P, \pi)$ and the monotone map $\bar{f} : (B, \sigma) \to (P, \pi)$ with $\bar{f}(b) = b$ for all $b \in B \backslash A$ and $\bar{f}(a) = f(a)$ for all $a \in A$, where $\pi = \tau \cup \bar{f}^2[\sigma]$. **RERE** and **SYM** are finally closed in **GRA** and thus closed under pushouts. In **PROCL** one has to take the transitive hull $<\!\pi\!>$ of π to get a preordering. One checks easily that this does not destroy the initiality of $\bar{e}$:

Assume $c\,\pi\, b\, \pi\, b'\, \pi\, c'$ for $c, c' \in C$ and $b, b' \in B \backslash A$. Then by definition of π, there are a, a' in A such that $f(a) = c$, $f(a') = c'$ and $a\,\sigma\, b\,\sigma\, b'\,\sigma\, a'$. Hence $a\,\sigma\, a'$ and thus $c\,\pi\, c'$.

For **POCL** one takes the antisymmetric reflection of $(P, <\!\pi\!>)$. Then $\bar{e}$ remains an embedding, because (C, τ) is in **POCL**. $\square$

(3) In **LOCL**, the quasiconstruct of linearly ordered classes and monotone maps, the homogeneous κ-universal objects (e.g. the rationals) are extremally homogeneous.

That is because, for each monotone map $f : A \to C$ and each embedding $e : A \to B$, there is a monotone map $\bar{f} : B \to P$ and an embedding $\bar{e} : C \to P$ such that $\bar{e}f = \bar{f}e$. Note that the pushout need not exist!

PROOF: We assume that f is surjective, because we can factorize f through its image and, as is proved in [2], for example, there are amalgamations in **LOCL**. But if f is surjective then $(P, <\!\pi\!>)$ from (2) is easily seen to be linearly ordered. $\square$

2. Special objects

In model theory and the theory of universal objects special objects are introduced, because they behave similar to homogeneous universal ones, but can be constructed without using the generalized continuum hypothesis. An object W with $|W| = \kappa$ is called $\mathcal{M}$-special if there is some family $(w_{\kappa_1,\kappa_2} : W_{\kappa_1} \to W_{\kappa_2})_{\kappa_1 < \kappa_2 < \kappa}$ of $\mathcal{M}$-morphisms converging to W by $(w_{\kappa'} : W_{\kappa'} \to W)_{\kappa' < \kappa}$ such that for each $\kappa_1 < \kappa$, W_{κ_1} is (κ_1^+, κ_2)-$\mathcal{M}$-homogeneous-universal for some $\kappa_2 \leq \kappa$.

Lemma 2.1. *Let W be $\mathcal{M}$-special in a strongly $\mathcal{M}$-smooth $(\mathcal{A}, U)$ for infinite $\kappa = |W|$. Then:*

(1) *W is $\mathcal{M}$-universal and $(\mathrm{cof}(\kappa), \kappa)$-$\mathcal{M}$-homogeneous-universal.*

(2) *$V \cong W$ holds for each $\mathcal{M}$-special object V.*

PROOF: For proving the $\mathcal{M}$-universality let $(a_{\alpha,\beta} : A_\alpha \to A_\beta)_{\alpha<\beta<\mathrm{cof}(\kappa)}$ be an $\mathcal{M}$-chain converging to an $A \in \mathcal{A}$ by $(a_\alpha : A_\alpha \to A)_{\alpha<\mathrm{cof}(\kappa)}$. We construct an $\mathcal{M}$-morphism $m_{\mathrm{cof}(\kappa)} : A \to W$. There is an $\mathcal{M}$-morphism $m_0 : A_0 \to W_{|A_0|}$. If the $\mathcal{M}$-morphism $m_\xi : A_\xi \to W_{|A_\xi|}$ is already defined then there is an $m : A_{\xi+1} \to W_{|A_{\xi+1}|}$ in $\mathcal{M}$. Let $\bar{m} := m \circ w_{|A_\xi|,|A_{\xi+1}|}$ (where $w_{\alpha,\alpha} := \mathrm{id}$). Now there is some iso h with $h \circ \bar{m} = m_\xi$. We define $m_{\xi+1} := h \circ m$. For limit ordinals $\lambda \le \mathrm{cof}(\kappa)$ let $m_\lambda : A_\lambda \to W_{|A_\lambda|}$ be the “union” of the $(m_\xi)_{\xi<\lambda}$, which exists by $\mathcal{M}$-smoothness.

For an $\mathcal{A}$-object is (α,κ)-$\mathcal{M}$-homogeneous-universal if it is so in the transportable hull, we assume transportability. We assume, without loss of generality, that all the w_{κ_1,κ_2} and w_{κ_1} are inclusions. It is easy to see that there is a continuous $\mathcal{M}$-chain $(a_{\alpha,\beta} : \tilde{W}_\alpha \to \tilde{W}_\beta)_{\alpha<\beta<\mathrm{cof}(\kappa)}$ converging to W by $(a_\alpha : \tilde{W}_\alpha \to W)_{\alpha<\mathrm{cof}(\kappa)}$, such that all involved $\mathcal{M}$-morphisms are inclusions and for each $\tilde{W}_\alpha$, there is a $\kappa(\alpha) < \kappa$ such that $U\tilde{W}_\alpha \subseteq UW_{\kappa(\alpha)}$. Let $m_1 : A \to W$ and $m_2 : A \to W$ be $\mathcal{M}$-morphisms such that $|A| < \mathrm{cof}(\kappa)$. There is an $\alpha < \kappa$ such that $\mathrm{im}(m_1), \mathrm{im}(m_2) \subseteq U\tilde{W}_\alpha \subseteq UW_{\kappa(\alpha)}$. Now there is an iso $\varphi : W_{\kappa(\alpha)} \to W_{\kappa(\alpha)}$ such that $\varphi \circ m_1 = m_2$. Hence $\bar{\varphi} \circ m_1 = m_2$ holds for the restriction $\bar{\varphi} : \tilde{W}_\alpha \to \mathrm{im}(\varphi)$ of φ.

It suffices to show that each iso $\varphi : A \to B$ in $\mathcal{A}_{<\kappa}$ between concrete subobjects of some $W_{\kappa'}$ can be extended to an iso on W. For that one constructs a chain of isos $(\varphi_\alpha : A_\alpha \to B_\alpha)_{\alpha<\mathrm{cof}(\kappa)}$ extending φ such that $U\tilde{W}_\alpha \subseteq UA_{\alpha+1} \cap UB_{\alpha+1}$ holds. Hence the union of the isos is an automorphism on W. We just look at the isolated step. Suppose A_α, B_α are concrete $\mathcal{M}$-subobjects of W_{κ_1}, where W_{κ_1} is $(\max(|A_\alpha|,|B_\alpha|)^+, \kappa_1')$-$\mathcal{M}$-homogeneous-universal. Then φ_α^{-1} can be extended to an iso $\psi_\alpha : \tilde{B}_\alpha \to \tilde{A}_\alpha$ between concrete $\mathcal{M}$-subobjects of some W_{κ_2}, which is $(\max(|\tilde{A}_\alpha|,|\tilde{B}_\alpha|)^+, \kappa_2')$-$\mathcal{M}$-homogeneous-universal, where $U\tilde{B}_\alpha$ contains $U\tilde{W}_\alpha$ (and $U\tilde{W}_\alpha$ is contained in $UW_{\kappa(\alpha)}$). In the same way we can extend ψ_α^{-1} to an automorphism $\varphi_{\alpha+1} : A_{\alpha+1} \to B_{\alpha+1}$ between concrete subobjects of some W_{κ_3}, where $UA_{\alpha+1}$ contains $U\tilde{W}_\alpha$. The construction of an iso between two $\mathcal{M}$-special objects is similar. □

A limit cardinal λ is called strong if $2^\kappa < \lambda$ holds for each cardinal $\kappa < \lambda$. It is easy to see that there is a proper class of strong limit cardinals. The first one is ω.

Remark 2.2. (1) If $\alpha = 2^{<\alpha}$ (e.g. for strong limits) then $\alpha = \alpha^{<\mathrm{cof}(\alpha)}$.

(2) If $\kappa = \kappa^{<\alpha}$ then $\kappa^+ = \kappa^{+<\alpha}$ for infinite κ.

PROOF: (1) Let $0 < \beta < \mathrm{cof}(\alpha)$. Then

$$\alpha^\beta = |\{\beta \to \alpha\}| = |\bigcup_{s<\alpha} \{\beta \to s\}| \le \sum_{s<\alpha} s^\beta \le \sum_{s<\alpha} 2^{\max(\beta,s)} \le \alpha^2 = \alpha.$$

(2) Let $\beta < \alpha$. Then $\beta < \operatorname{cof}(\kappa^+)$ and thus

$$\kappa^{+\beta} \leq \sum_{s<\kappa^+} s^\beta \leq \sum_{s<\kappa^+} \kappa^{<\alpha} = \sum_{s<\kappa^+} \kappa = \kappa \cdot \kappa^+ = \kappa^+$$

and

$$\kappa^{+<\alpha} = \sum_{\beta<\alpha} \kappa^{+\beta} = \sum_{\beta<\alpha} \kappa^+ \leq \alpha \cdot \kappa^+ = \kappa^+.$$

□

If $(\mathcal{A}, U)$ is strongly $\mathcal{M}$-smooth for $\kappa = \omega$ then there is an $\mathcal{M}$-special object by Theorem 1.5. We remember that $(2^\kappa)^{<\kappa^+} = 2^\kappa$ for infinite κ.

Theorem 2.3. *Let $(\mathcal{A}, U)$ be strongly $\mathcal{M}$-smooth for some strong limit cardinal κ with the additional property that for each cardinal $\alpha < \kappa$, there are less than κ objects modulo isomorphy in $\mathcal{A}_{\leq\alpha}$. If $(\mathrm{F1})_{\alpha,\kappa}$ holds for some $\omega \leq \alpha < \kappa$ then there is an $\mathcal{M}$-special object of cardinality κ.*

Proof: Again we assume transportability. One observes (using transfinite induction) that $(F1)_{\alpha',\kappa'}$ holds for all $\alpha \leq \alpha' \leq \kappa' \leq \kappa$ (analogous to the case of relational structures in [2], for example). There are at most $\kappa(\alpha') < \kappa$ structures on sets with at most α' elements for some $\kappa(\alpha')$. We can choose a regular $\kappa(\alpha') \geq \alpha'$ such that $\kappa(\alpha')^{<\alpha'} = \kappa(\alpha')$ (e.g. one can redefine it as $(2^{\kappa(\alpha')+\alpha'})^+$) and such that $\kappa(\_) : \kappa \to \kappa$ is a monotone map. So by Proposition 1.6, there exists an $(\alpha^+, \kappa(\alpha^+))$-$\mathcal{M}$-homogeneous-universal object W_α. Let $W_\xi := W_\alpha$ for each cardinal $\xi \leq \alpha$. Let $\sigma' : \operatorname{cof}(\kappa) \to \kappa$ be some unbounded monotone map. Now let $\sigma(\gamma) := |\sigma'(\gamma)|$ and assume $\sigma(0) = \alpha$. Let $W_{\sigma(\xi)}$ be already defined for each cardinal $\sigma(\xi)$, where $\xi < \gamma < \operatorname{cof}(\kappa)$. Because of Proposition 1.6, there is some $(\sigma(\gamma)^+, \kappa(\sigma(\gamma)^+))$-$\mathcal{M}$-homogeneous-universal object $W_{\sigma(\gamma)}$ containing the union of all $W_{\sigma(\xi)}$, where $\xi < \gamma$, as an $\mathcal{M}$-subobject. Define $W_\zeta := W_{\sigma(\gamma)}$ for each cardinal ζ greater than all the $\sigma(\xi)$, where $\xi < \gamma$, and less than $\sigma(\gamma)$.

Let W be the union of all $W_{\sigma(\gamma)}$, where $\gamma < \operatorname{cof}(\kappa)$, according to (SF2). Hence W is $\mathcal{M}$-special. □

We say that W is extremally $\mathcal{M}$-special if there is a family $(w_{\kappa_1,\kappa_2} : W_{\kappa_1} \to W_{\kappa_2})_{\kappa_1<\kappa_2<\kappa}$ of $\mathcal{M}$-morphisms converging to W by $(w_{\kappa'} : W_{\kappa'} \to W)_{\kappa'<\kappa}$ such that, for each $\kappa_1 < \kappa$, W_{κ_1} is extremally (κ_1^+, κ_2)-$\mathcal{M}$-homogeneous-universal for some $\kappa_2 \leq \kappa$.

Theorem 2.4. *Let $(\mathcal{A}, U)$ be extremally $\mathcal{M}$-smooth for some strong limit cardinal κ with the additional property that for each cardinal $\alpha < \kappa$ there are less than κ objects modulo isomorphy in $\mathcal{A}_{\leq\alpha}$. If $(\mathrm{F1})_{\alpha,\kappa}$ holds for some $\omega \leq \alpha < \kappa$ then there is an extremally $\mathcal{M}$-special W such that $|W| = \kappa$.*

Proof: Using Theorem 1.7, the proof is analogous to the proof of Theorem 2.3. □

3. Homogeneous universal categories

In the case of categories as objects of quasiconstructs, we always choose $\mathcal{M}$ as the conglomerate of full embeddings. The existence of a universal category was proved by Trnková in [9]. Let $\mathbf{CAT}(\mathcal{X})$ be the quasicategory of concrete categories over the basecategory $\mathcal{X}$ (i.e. pairs $(\mathcal{A}, U)$ consisting of a category $\mathcal{A}$ and a faithful functor $U : \mathcal{A} \to \mathcal{X}$) and concrete functors $F : (\mathcal{A}, U) \to (\mathcal{B}, V)$ (i.e. $VF = U$). The existence of a concrete universal construct (that is the special case $\mathcal{X} = \mathbf{Set}$) was proved by Kučera in [5], and with slight changes in the proof one can verify the existence of a concrete universal category over every concretizable basecategory, as is done in [7]. The concrete universal categories $S(\mathcal{P} \circ Q_2 \circ G, \mathcal{U})$ involved therein are not homogeneous (except for $\mathcal{X} = \emptyset$).

Example 3.1. We define full embeddings $E_1 : \mathcal{A} \to \mathcal{A}_1$ and $E_2 : \mathcal{A} \to \mathcal{A}_2$ (between concretizable categories) for which there are no functors $V : \mathcal{A}_1 \to \mathcal{P}$ and $F : \mathcal{A}_2 \to \mathcal{P}$ such that V is faithful and $FE_2 = VE_1$.

So neither in $\mathbf{CAT}$ nor in $\mathbf{C-CAT}$, there are big amalgamations as is necessary for a Jónsson-class. Furthermore, the extension lemma formulated by Trnková in [10] cannot be generalized to arbitrary functors between concretizable categories. That is, not all such functors can be extended along a realization.

Proof: $\mathcal{A}$ has as objects $A_{\{\alpha,\beta\}}$ for ordinals α, β. For all ordinals $\alpha < \beta$ and γ there is a morphism $a_{\gamma,\alpha,\beta} : A_{\{\gamma\}} \to A_{\{\alpha,\beta\}}$. $\mathcal{A}_2$ has B as an additional object and additional morphisms $b_\alpha : A_{\{\alpha\}} \to B$ and $k_{\{\alpha,\beta\}} : B \to A_{\{\alpha,\beta\}}$ for all pairs of ordinals $\alpha < \beta$, such that $a_{\gamma,\alpha,\beta} = k_{\{\alpha,\beta\}} \circ b_\gamma$. $\mathcal{A}_1$ has A as an additional object. Additional morphisms are $h_{\alpha,\beta}, h_{\beta,\alpha} : A \to A_{\{\alpha,\beta\}}$ for all ordinals α, β; for $\alpha < \beta$ we declare $a_{\gamma,\alpha,\beta} \circ h_{\gamma,\gamma}$ to be $h_{\alpha,\beta}$, whenever $\gamma \leq \alpha$, and $h_{\beta,\alpha}$ otherwise. We claim $h_{\alpha,\beta} \neq h_{\beta,\alpha}$ for $\alpha \neq \beta$.

$\mathcal{A}_2$ is concretizable because it is thin. We define a forgetful functor $|...| : \mathcal{A}_1 \to \mathbf{Set}$ by $|A_{\{\alpha,\beta\}}| = |A| = \{0, 1\}$, $h_{\alpha,\alpha} :\equiv 0$, and, for $\alpha < \beta$, $h_{\alpha,\beta} :\equiv 0$, $h_{\beta,\alpha} :\equiv 1$, $a_{\gamma,\alpha,\beta} :\equiv 0$ if $\gamma \leq \alpha$, and $a_{\gamma,\alpha,\beta} :\equiv 1$ otherwise. E_1 and E_2 are the inclusions.

Suppose there is a faithful functor $V : \mathcal{A}_1 \to \mathcal{P}$ and a functor $F : \mathcal{A}_2 \to \mathcal{P}$ such that $FE_2 = VE_1$. Then we show that $\hom_{\mathcal{P}}(A, B)$ is a proper class. For $\alpha < \beta$, one has $Fb_\alpha \circ Vh_{\alpha,\alpha} \neq Fb_\beta \circ Vh_{\beta,\beta}$ because

$$\begin{aligned} Fk_{\{\alpha,\beta\}} \circ Fb_\alpha \circ Vh_{\alpha,\alpha} &= Fa_{\alpha,\alpha,\beta} \circ Vh_{\alpha,\alpha} = Va_{\alpha,\alpha,\beta} \circ Vh_{\alpha,\alpha} = Vh_{\alpha,\beta} \\ &\neq Vh_{\beta,\alpha} = Fa_{\beta,\alpha,\beta} \circ Vh_{\beta,\beta} = Fk_{\{\alpha,\beta\}} \circ Fb_\beta \circ Vh_{\beta,\beta}. \end{aligned}$$

□

Lemma 3.2. *In $\mathbf{CAT}(\mathcal{X})$ full concrete embeddings are pushout-stable.*

Proof: First we assume a situation where $\mathcal{A} = \mathcal{A}' \cap \mathcal{B}$ and $\mathcal{A}$ is a full concrete subcategory of both $\mathcal{A}'$ and $\mathcal{B}$. Now take the union of $\mathcal{A}'$ and $\mathcal{B}$ and add all compositions of $\mathcal{X}$-morphisms in $\mathcal{A}' \cup \mathcal{B}$ to get the pushout. As one can easily verify, this does not destroy the fullness of the embeddings.

Now we consider a concrete functor $F : \mathcal{A} \to \mathcal{B}$ and a full concrete embedding $E : \mathcal{A} \to \mathcal{A}'$. We assume that E is an inclusion, F is surjective on objects (factorize it through the full subcategory generated by its image), and $\mathcal{B}$ and $\mathcal{A}'$ are disjoint. Now we define $\mathbf{ob}\mathcal{P} := \mathbf{ob}(\mathcal{A}'\backslash\mathcal{A}) \cup \mathbf{ob}\mathcal{B}$. The morphisms of $\mathcal{P}$ contain the ones of $\mathcal{A}'\backslash\mathcal{A}$ and $\mathcal{B}$; additionally we claim for $B \in \mathcal{B}$ and $A' \in \mathcal{A}'\backslash\mathcal{A}$

$B \xrightarrow{f} A' \in \mathcal{P}$ if $\exists A \in \mathcal{A} : (B = FA \ \wedge \ A \xrightarrow{f} A' \in \mathcal{A}')$;

$A' \xrightarrow{f} B \in \mathcal{P}$ if $\exists A \ \in \mathcal{A} : (B = FA \ \wedge \ A' \xrightarrow{f} A \in \mathcal{A}')$.

Now we close $\mathcal{P}$ w.r. to composition.

Let $\bar{E}B := B$ for $B \in \mathcal{B}$; let $\bar{F}A' = A'$ for $A' \in \mathcal{A}'\backslash\mathcal{A}$ and $\bar{F}A = FA$ for $A \in \mathcal{A}$. It suffices to show that $\bar{E}$ is a full embedding. All that remains to check is the fullness. Let $f = B_1 \xrightarrow{g} A' \xrightarrow{h} B_2$ with $B_1 = FA_1$, $B_2 = FA_2$ and $A_1 \xrightarrow{g} A', A' \xrightarrow{h} A_2 \in \mathcal{A}'$. Thus $f : A_1 \to A_2 \in \mathcal{A}$, because E is full and hence $f : FA_1 \to FA_2$ is in $\mathcal{B}$ which is a subcategory of $\mathcal{P}$. $\square$

Lemma 3.3. *In* **Cat** *full embeddings are pushout-stable.*

Proof: Suppose there is a functor $F : \mathcal{A} \to \mathcal{B}$ and a full embedding $E : \mathcal{A} \to \mathcal{A}'$ in **Cat** the category of small categories.

(1) We assume that F is onto on objects. Otherwise factor F through the full subcategory generated by its image. If there is an extension for the first part of the factorization then by the amalgamation lemma in [8], or a bit more handsome in [11], there is also an extension of the second part. The extension of the second part appears as a pushout.

(2) We assume that F is one to one on objects. Otherwise factor F through some $\tilde{\mathcal{B}}$ in which one adds isomorphic copies of objects which appear more than once as images of $\mathcal{A}$-objects. Now one defines $\tilde{F} : \mathcal{A} \to \tilde{\mathcal{B}}$ in the obvious way such that the retraction $R : \tilde{\mathcal{B}} \hookrightarrow \mathcal{B}$ which identifies the artificially defined isomorphic copies composes with $\tilde{F}$ to yield F. It is easy to see that the pushout of a full embedding along a surjective equivalence is a full embedding.

(3) Now F is bijective on objects. Let us assume that E is an inclusion. We define forgetful functors $U : \mathcal{A}' \to \mathbf{Set}$ and $V : \mathcal{B} \to \mathbf{Set}$, such that VF is a retract of $U : \mathcal{A} \to \mathbf{Set}$ (as the restriction of U). Under these conditions, an extension-lemma which states that there exist a functor $\tilde{F} : \mathcal{A}' \to \mathcal{C}$ and a full embedding $\tilde{E} : \mathcal{B} \to \mathcal{C}$ such that $\tilde{F} \circ E = \tilde{E} \circ F$ is shown in [10].

Let us first define an equivalence relation between pairs in $\mathcal{B} \times \mathcal{A}'$ of the form $(B \xrightarrow{g} FA, A \xrightarrow{h} A')$. We write

$$(B \xrightarrow{g} FA, A \xrightarrow{h} A') \sim (B \xrightarrow{\tilde{g}} F\tilde{A}, \tilde{A} \xrightarrow{\tilde{h}} A')$$

iff $F(kh) \circ g = F(k\tilde{h}) \circ \tilde{g}$ for each $A' \xrightarrow{k} \bar{A}$ with $\bar{A} \in \mathcal{A}$. For $A' \in \mathcal{A}'$ we define

$$UA' := \{[(B \xrightarrow{g} FA, A \xrightarrow{h} A')]_{\sim} \mid (g, h) \in \mathcal{B} \times \mathcal{A}'\} \amalg \{0\} \times \bigcup_{A'' \in \mathcal{A}'} \hom(A'', A') \amalg \{0\}.$$

For $A'_1 \xrightarrow{f} A'_2$ we define $Uf([(g,h)],g') := ([(g,fh)],fg')$ and $Uf(0,g') := (0,fg')$, where $f0 := 0$.

Now for $B' \in \mathcal{B}$ we define $VB' := \{[(B \xrightarrow{g} FA, A \xrightarrow{h} A')] \mid FA' = B'\} \amalg \{0\}$ and $V(FA' \xrightarrow{b} FA'')([(B \xrightarrow{g} FA, A \xrightarrow{h} A')]) := [(B \xrightarrow{b\circ Fh\circ g} FA'', \mathrm{id}_{A''})]$, resp. $Vb(0) := 0$.

Moreover we define natural transformations $\pi : U \to VF$ and $\mu : VF \to U$ where π is pointwise the first projection and $\mu_{A'}([(g,h)]) := ([(g,h)],0)$, resp. $\mu_{A'}(0) := (0,0)$.

- U is well-defined by the definition of $\sim$.
- U is faithful. Suppose that $f \neq g$ are parallel morphisms with common domain A. Then $Uf(0,\mathrm{id}_A) = (0,f) \neq (0,g) = Ug(0,\mathrm{id}_A)$.
- V is well-defined. Suppose that $(g,h) \sim (\tilde{g},\tilde{h})$, where $\mathrm{cod}(h) = A \in \mathcal{A}$. Now for $FA \xrightarrow{b} FA'$, we get $(b \circ Fh \circ g, \mathrm{id}_{A'}) = (b \circ F\tilde{h} \circ \tilde{g}, \mathrm{id}_{A'})$ simply because $F(\mathrm{id}_A \circ h) \circ g = F(\mathrm{id}_A \circ \tilde{h}) \circ \tilde{g}$.
- V is faithful. Let $b_1, b_2 : FA_1 \to FA_2$ be distinct, then for $k = \mathrm{id}_{A_2}$ one has $F(k \circ \mathrm{id}_{A_2}) \circ b_1 = b_1 \neq b_2 = F(k \circ \mathrm{id}_{A_2}) \circ b_2$ and thus $[(b_1, \mathrm{id}_{A_2})] \neq [(b_2, \mathrm{id}_{A_2})]$. Hence $Vb_1[(\mathrm{id}_{FA_1}, \mathrm{id}_{A_1})] = [(b_1, \mathrm{id}_{A_2})] \neq [(b_2, \mathrm{id}_{A_2})] = Vb_2[(\mathrm{id}_{FA_1}, \mathrm{id}_{A_1})]$.

One easily checks the naturality of π and μ.

(4) As in (3), we assume that F is bijective on objects and E is an inclusion. Because **Cat** is cocomplete, the pushout $\bar{F} \circ E = \bar{E} \circ F$ exists; let $\mathcal{P}$ denote its codomain. Because of (3), $\bar{E}$ is an embedding. It suffices to check fullness. Recall the construction of the pushout: On objects it is the disjoint union of $\mathcal{A}' \backslash \mathcal{A}$ and $\mathcal{B}$, because the functor $\mathbf{Cat} \to \mathbf{Set}$ that forgets the morphisms is left adjoint. To build a category out of this we have to add certain compositions as new morphisms (as free as possible) and identify certain parallel morphisms. The latter does not destroy fullness, so we do not really have to understand this equivalence relation. In general, new morphisms appear in $\mathcal{A}'$ but not in the $\mathcal{B}$-part of $\mathcal{P}$. Let $f_1, \ldots, f_n$ be morphisms in $(\mathcal{A}' \backslash \mathcal{A}) \cup \mathcal{B}$ that are composable in $\mathcal{P}$ (note that the objects of $\mathcal{B}$ correspond to the objects of $\mathcal{A}$). If $\mathrm{dom}(f_1)$ and $\mathrm{cod}(f_n)$ are in $\mathcal{B}$ then we show that the composition has to be defined as a morphism that already exists in $\mathcal{B}$. Suppose $f : B_1 \to A'$ and $g : A' \to B_2$ are in $\mathcal{A}'$ with $B_1, B_2 \in \mathbf{ob}\mathcal{B} (\approx \mathbf{ob}\mathcal{A})$. Then the composition $h = g \circ f$ is defined in $\mathcal{A}$. So we have to define $[g] \circ [f] := [Fh]$, where $Fh \in \mathcal{B}$. All other cases are easy or follow from this one.

With a little bit more effort one can describe the pushout explicitly in a way analogous to the construction for Lemma 3 in [8]. □

Theorem 3.4. (1) *There is an extremally homogeneous universal category.*

(2) *There is an extremally homogeneous concrete universal category over each basecategory $\mathcal{X}$.*

(3) *There is an extremally homogeneous universal object in each comma category* $\mathbf{CAT} \downarrow \mathcal{X}$.

Proof: The only difficult property to check is (E2), the existence of small extensions. In all cases we construct the extension as a pushout. (E2) in cases (1) and (2) follows from the Lemmas 3.3 and 3.2. In the case of (3), given objects A, A', B in $\mathbf{Cat} \downarrow \mathcal{X}$, a functor $F : A \to B$ and a full embedding $E : A \to A'$, there is a pushout of E and F in $\mathbf{Cat}$, say $\bar{F} \circ E = \bar{E} \circ F$. This yields a functor P to $\mathcal{X}$ such that $\bar{E} : B \to P$ and $\bar{F} : A' \to P$ form the pushout of F and E in $\mathbf{Cat} \downarrow \mathcal{X}$. □

Remark 3.5. There is no homogeneous C-universal category in $\mathbf{C} - \mathbf{CAT}$: Let $\mathcal{A}_\alpha$ be an $\mathcal{M}$-chain in a non-concretizable category $\mathcal{A}$. Since every small category is concretizable, so are the $\mathcal{A}_\alpha$. Hence (SF2) is not fulfilled. On the other hand, $\mathbf{C} - \mathbf{CAT}$ is obviously $\mathcal{M}$-smooth.

Proposition 3.6. (1) *There is an extremally homogeneous κ-universal category in $\mathbf{Cat}_{(\kappa)}$ for each $\kappa = \kappa^{<\kappa}$. Objects of $\mathbf{Cat}_{(\kappa)}$ are categories $\mathcal{A}$ such that $|\mathbf{ob}\mathcal{A}| \leq \kappa$ and $|\hom_{\mathcal{A}}(A, B)| < \kappa$, for all $A, B \in \mathbf{ob}\mathcal{A}$.*

(2) *There is an extremally homogeneous concretely κ-universal category in $\mathbf{Cat}_{(\kappa)}(\mathcal{X})$ for each $\mathcal{X} \in \mathbf{Cat}_{(\kappa)}$, where $\kappa = \kappa^{<\kappa}$.*

Proof: One observes that the extensions in $\mathbf{Cat}$ and $\mathbf{CAT}(\mathcal{X})$ do not get too big. For (F1) let $\mathcal{A} \in \mathbf{Cat}_{(\kappa)}$ and $X \subseteq \mathbf{ob}\mathcal{A}$ with $|X| < \kappa$. Now $\sum_{A,B \in X} |\hom(A, B)| < \kappa$ because of the regularity of κ. □

For non-regular κ, the downward Löwenheim Skolem property fails to hold in $\mathbf{Cat}_{(\kappa)}$. But nevertheless one can construct $\mathcal{M}$-special categories for strong limit cardinals κ. To this end, we define $\mathbf{Cat}_{(\mu,\kappa)}$ as the category of all categories of cardinality at most κ with hom-sets smaller than μ.

Proposition 3.7. *Let κ be some strong limit cardinal.*

(1) *There is an extremally $\mathcal{M}$-special category in $\mathbf{Cat}_{(\kappa)}$.*

(2) *There is an extremally $\mathcal{M}$-special concrete category in $\mathbf{Cat}_{(\kappa)}(\mathcal{X})$ for each $\mathcal{X} \in \mathbf{Cat}_{(\kappa)}$.*

Proof: By Theorem 1.7, there exist extremally (μ, ρ)-homogeneous-universal categories in $\mathbf{Cat}_{(\mu,\rho)}$,where μ and ρ are regular cardinals such that $\omega \leq \mu \leq \rho = \rho^{<\mu}$. Now one obtains an extremally $\mathcal{M}$-special object in $\mathbf{Cat}_{(\kappa)}$ in a way similar to the construction for Theorem 2.3. The case $\mathbf{Cat}_{(\kappa)}(\mathcal{X})$ is analogous. □

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University of Bremen, Bibliothekstr. 1, 28359 Bremen, Germany

E-mail: rother@uni-bremen.de

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