Algebraic aspects of web geometry

Maks A. Akivis, Vladislav V. Goldberg

On the occasion of 70 years of web theory

Abstract. Algebraic aspects of web geometry, namely its connections with the quasigroup and loop theory, the theory of local differential quasigroups and loops, and the theory of local algebras are discussed.

Keywords: quasigroup, loop, web, group, local quasigroup, local loop, Akivis algebra, n-quasigroup

Classification: 53A60, 17D99, 20N05, 22A30

0. Introduction

Web theory, which started more than 70 years ago in the works of Blaschke and his collaborators, was successfully developed during all years after its foundation and appeared to be connected with many branches of mathematics. Different aspects of this theory set forth in the monographs [BB 38], [Bl 55], [Be 67], [Be 71], [Be 72], [P 75], [G 88], [AS 92], and also in the surveys and papers [Ac 65], [BS 83], [Ba 90], [HS 90], [G 90], [MS 90], [NS 94], [AG 00].

In the current paper we consider connections of web geometry with the theory of smooth local differentiable quasigroups and loops as well as with the theory of binary-ternary algebras. We show that for webs and quasigroups one can construct local algebras following the lines of construction of the Lie algebras for the Lie groups. We study different special classes of webs and quasigroups defined by certain closure conditions and prove for these classes that the theorems analogous to the converse part of Lie's Third Fundamental Theorem (see [Lie 93]) hold for them.

Note that because of the length restriction, it is impossible to present all algebraic aspects in detail. Thus we discuss only the most important interactions of web geometry and algebra. For example, we do not consider canonical expansions of equations of a quasigroup, algebraic constructions in n-loops generalizing the Akivis algebra, a relationship between subwebs and subquasigroups, and consider only a few special classes of $(n+1)$-webs and the corresponding n-quasigroups. We also do not give rigorous proofs. The reader can find them in the books [AS 92], [G 88], and in the review papers [G 90] and [AG 00].

1. Two-dimensional 3-webs

Consider a regular family of smooth curves in a two-dimensional domain D. By means of an appropriate differentiable transformation, the domain D can be transferred into a domain $\widetilde{D}$ of an affine plane A^2 in such a way that the family of lines given in D will be transferred into a family of parallel lines of $\widetilde{D}$. This shows that a family of smooth curves in D does not have local invariants. Two regular families of smooth curves, that are in general position in D, also do not have local invariants since one can always find a diffeomorphism that transfers them into two families of parallel lines of a domain $\widetilde{D}$ of an affine plane A^2. Thus, the structures, defined in D by one or two families of curves, are locally trivial.

Consider now three regular families of smooth curves in D, that are in general position in D. We will say that they form a 3-*web* in the domain D. Two 3-webs are *equivalent* one to another if there exists a local diffeomorphism which maps the families of one web into the families of another one.

Blaschke and Thomsen (see [Bl 28] and [T 27]) started to study 3-webs after they realized that the configuration of three foliations of curves in the plane has local invariants. Such a structure is no longer locally trivial. In fact, in a neighborhood of each point p of the domain D, where the web is given, one can construct a family of hexagonal figures as shown in Figure 1.

In this figure the lines P_1P_4, P_2P_3, P_5P_6 belong to the first family, the lines P_2P_5, P_3P_4, P_6P_7 belong to the second family, and the lines P_3P_6, P_4P_5, P_2P_1 belong to the third family. In the general case, the points P_6 and P_7 of these figures do not coincide, i.e., hexagonal figures (H) are not closed. However, there exist 3-webs, on which all hexagonal figures (H) are closed, i.e., on these webs the points P_6 and P_7 coincide. Such 3-webs are called *hexagonal*. For example, the web formed by three families of parallel lines in an affine plane A^2 is hexagonal (see Figure 2). It is remarkable that the converse theorem also holds: any hexagonal 3-web in a plane admits a differentiable mapping on a 3-web, formed by three families of parallel lines, i.e., such a web is parallelizable. This theorem was first proved by Thomsen in the above mentioned paper [T 27] (see also the book [Bl 55]).

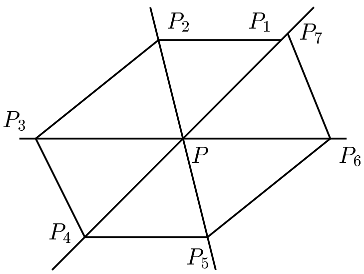

FIGURE 1

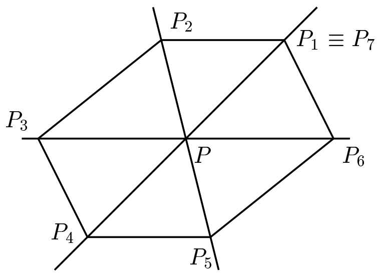

FIGURE 2

If a two-dimensional web is formed by the level sets of variables x, y, and z that are connected by the equation $z = f(x, y)$, then its curvature b can be written in the following elegant form:

$$b = -\frac{1}{f_x f_y}\frac{\partial^2}{\partial x \partial y}\left(\ln \frac{f_x}{f_y}\right) \tag{1}$$

(see [Bl 55, §9] or [AS 92, p. 43]).

The web curvature b is a measure of nonclosure of hexagonal figures:

$$b = 0 \Longleftrightarrow (H). \tag{2}$$

Next consider a 3-web formed by three families of parallel straight lines in an affine plane A^2. Such a 3-web is called *parallel* (see Figure 2). A 3-web which is equivalent to a parallel web is called *parallelizable*.

Of course, we have $(P) \rightarrow (H)$. It appeared that for two-dimensional 3-webs, condition (H) implies condition (P). Thus, we have

$$(H) \Longleftrightarrow (P),$$

that is, these two conditions are equivalent.

2. Groups and webs

1. Consider a group G and the direct product $M = G \times G$. Three curves $x = a$, $y = b$ and $x \cdot y = a \cdot b$ pass through a point $(a, b) \in M$. Thus the three families $x = \text{const}$, $y = \text{const}$, and $x \cdot y = \text{const}$ form a 3-web on M. Such a 3-web is called the *group* 3*-web*. Figure 3 shows a geometric scheme of this multiplication. The group 3-webs were introduced by Reidemeister [Re 28] and Knesser [Kn 32].

2. Now consider Thomsen's figure (T) (see Figure 4). It is easy to see from Figure 4 that if all Thomsen's figures (T) are closed, then in G we have

$$(T) \Longleftrightarrow u \cdot v = v \cdot u. \tag{3}$$

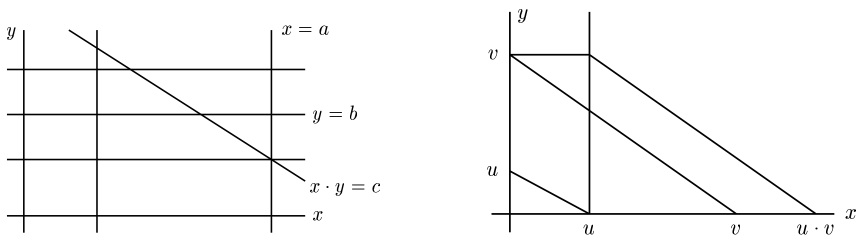

Figure 3

(T)

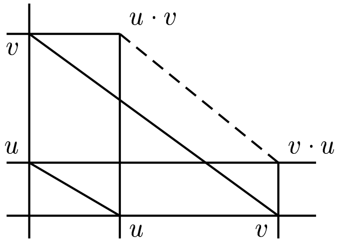

FIGURE 4

(R)

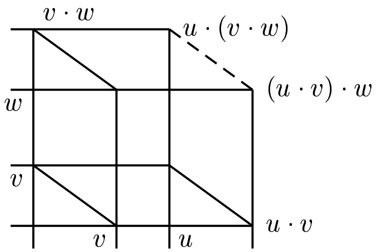

FIGURE 5

Next consider the so-called Reidemeister's figure (R) (see Figure 5). It is easy to see from Figure 5 that if all Reidemeister's figures (R) are closed, then

$$(R) \Longleftrightarrow u \cdot (v \cdot w) = (u \cdot v) \cdot w \tag{4}$$

in G. Thus the associativity is always the case for a group web.

A projective meaning of the figures (T) and (R) and their relation are considered in the books [Bl 47], [Re 68], and [P 75].

3. Quasigroups and webs

The notion of a quasigroup was introduced by Moufang [Mo 35] (see also [Su 37] where generalizations of the notion of the group are discussed) and was applied to web geometry by Bol [B 37].

A *quasigroup* q is a groupoid in which an equation $x \cdot y = z$ is uniquely solvable with respect to x and y for any $x, y, z \in q$. A quasigroup with an *identity element* is called a *loop*.

A web on $M = q \times q$ is defined exactly in the same way as it was defined for a group G in Section 2.

Suppose that M is a set whose elements we will call *points*, and $\lambda_\alpha, \alpha = 1, 2, 3$, are three sets (families) whose elements we will call *lines*. The sets M and λ_α form a *complete* 3-*web* (W, λ_α) if its elements (points and lines) satisfy the following three axioms:

(i) Any point $p = (x, y) \in M$ is incident to just one line from each family λ_α.

(ii) Any two lines of different families are incident to exactly one point of M.
(iii) Two lines of the same family λ_α are disjoint.

Note that here we used the term “complete web” (see [AG 00, Section 2.1]) instead of the term “abstract web” (see [G 90, p. 274] and [AS 92, Section 2.2]). The term “complete” is of opposite sense to “the local web” and is more precise. Note also that (iii) can be derived from (i) and (ii). One can easily prove that *the webs generated by quasigroups are complete.*

In web theory the notion of a 3-base quasigroup is very useful. Suppose that X, Y, and Z are sets of the same cardinality. The mapping

$$q : X \times Y \to Z,\ z = q(x, y),\ x \in X,\ y \in Y,\ z \in Z, \tag{5}$$

is called a 3-*base quasigroup* if this mapping is invertible with respect to both variables x and y. Let $a \in X$, $b \in Y$, $e \in Z$, and $e = q(a, b)$. We map the sets X, Y, Z into Z as follows:

$$u = q(x, b), \quad v = q(a, y). \tag{6}$$

Then on the set Z the operation

$$u \circ v = q({}^{-1}q(u, b), q^{-1}(a, v)), \tag{7}$$

where ${}^{-1}q$ and q^{-1} are the left and right inverse quasigroups for q, respectively, arises, and this operation defines a quasigroup $Z(\circ)$ on Z which is isotopic to the quasigroup q (see [Be 67], [AS 92]). Since we have

$$u \circ e = u, \quad e \circ v = v, \tag{8}$$

the quasigroup $Z(\circ)$ is a loop with the identity element e. This loop is called a *coordinate loop* connected with the point $p = (a, b)$ and denoted by L_p. There is a one-to-one correspondence between 3-webs defined on the set $M = X \times Y$ by the level sets of variables x, y, z and 3-base quasigroups.

Sabinin [Sab 88] considered a 3-web as an antiproduct $Q \times Q$ of a loop Q by itself. This antiproduct is also a loop with the unit (e, e) and with multiplication $\circ$ defined by the formula $(x, \overline{x}) \circ (y, \overline{y}) = (x \cdot y, \overline{y} \cdot \overline{x})$.

4. Bol’s figures

Let q be a binary quasigroup defined on a set Q. This quasigroup defines a complete 3-web $W(3, q)$ on the set $M = Q \times Q$. We shall say that on $W(3, q)$ the *closure condition* (T) (*respectively,* (R), (B_l), (B_r), (B_m), *and* (H)) *holds* if on $W(3, q)$ all figures (T) (respectively, (R), (B_l), (B_r), (B_m), and (H)) are closed.

(B_l)

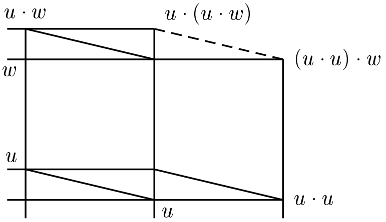

FIGURE 6

(B_r)

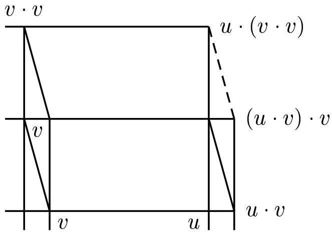

FIGURE 7

(B_m)

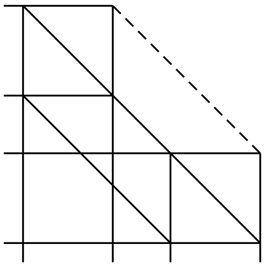

FIGURE 8

(H)

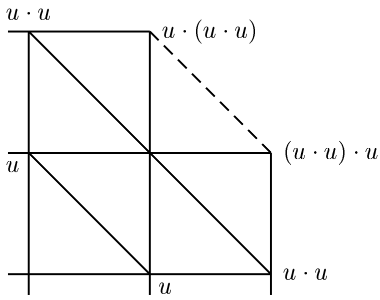

FIGURE 9

The closure conditions (T), (R), (B_l), (B_r), (B_m), and (H) are represented on Figures 4, 5, 6–9, respectively.

Note that in the notations $(T), (R), (B)$, and (H) the first letters of the words "Thomsen", "Reidemeister", "Bol", and "hexagonal" are used, and that in the notations (B_l), (B_r), and (B_m) the subindices l, r, and m are the first letters of the words "left", "right", and "middle".

A web $W(3,q)$ is a *group* (respectively, *Bol* or *hexagonal*) web if on it the closure condition (R) (respectively, any of (B_l), (B_r), (B_m) or (H)) holds. If on $W(3,q)$ all the Bol figures (B_l), (B_r), and (B_m) are closed, then it is called a *Moufang* web and denoted by (M).

The closure conditions (B_l), (B_r), and (B_m) are not independent. It is possible to prove that

$$(B_l) \ \text{ and } \ (B_r) \Longrightarrow (B_m).$$

This means that $(M) = (B_l) \cap (B_r)$.

Each of the closure figures (B_l), (B_r), and (B_m) is a particular case of the figure (R). In particular, the figure (B_r) is obtained from (R) if the leaves v and w in (R) coincide.

There exists the following dependence between the Bol closure conditions for inverse quasigroups ${}^{-1}q$ and q^{-1} (see [Be 67, p. 202] or [AS 92, Section 2.2]):

$$ {}^{-1}(B_m) \Longleftrightarrow (B_l), \quad (B_m)^{-1} \Longleftrightarrow (B_r). \tag{9}$$

It follows from Figures 6, 7 and 9 that for the Bol webs (B_l), (B_r) and the hexagonal webs (H), in any coordinate loop the following algebraic identities hold:

$$(10) \qquad (B_l) \sim (u \cdot u) \cdot w = u \cdot (u \cdot w) \qquad (\textit{left alternativity})$$

$$(11) \qquad (B_r) \sim (u \cdot v) \cdot v = u \cdot (v \cdot w) \qquad (\textit{right alternativity})$$

$$(12) \qquad (H) \sim (u \cdot u) \cdot u = u \cdot (u \cdot u) \qquad (\textit{monoassociativity})$$

(see [Ac 65] or [AS 92]). As to the Bol webs (B_m), for them an algebraic condition in the inverse quasigroups ${}^{-1}q$ and q^{-1} holds (see (9)).

Identities (10), (11), and (12) show that all coordinates loops L_p of the webs (B_l), (B_r), and (M) are left alternative, right alternative, and alternative loops, respectively. But these loops can be also characterized by universal conditions (a condition is *universal* if its fulfillment in one coordinate loop implies its fulfillment in all other coordinate loops) and also identities (10), (11), and (12). For the Moufang loops M, the left Bol loops (B_l), and the right Bol loops (B_r), the universal identities have respectively the form:

$$(13) \qquad (v \cdot u) \cdot (w \cdot v) = v \cdot ((u \cdot w) \cdot v),$$

$$(14) \qquad (u \cdot (v \cdot u)) \cdot w = u \cdot (v \cdot (u \cdot w)),$$

$$(15) \qquad u \cdot ((v \cdot w) \cdot v) = ((u \cdot v) \cdot w) \cdot v.$$

They are called the *Moufang, left Bol and right Bol identities*, respectively (see [Be 67, Chapter 6]).

Note that conditions (13)–(15) are *universal*, i.e., if they are valid in one coordinate quasigroup of a 3-web W, then they will be valid in all coordinate quasigroups of W. The same is true for the associativity condition (4). As to the commutativity condition (3) as well as conditions (10)–(12), they are not universal. This is a reason that in what follows we request that they must be satisfied in all coordinate loops of W.

It appears that algebraic properties of coordinate loops L_p of a 3-web W are connected with closure conditions introduced above (see [Ac 65], [Be 67], [Be 71], and [A 73b]). This connection can be presented in the form of Table 1.

Loop manifold	Identity in a loop	Closure condition on W
Abelian groups	$u{\cdot}v{=}v{\cdot}u$ $(u{\cdot}v){\cdot}w{=}u{\cdot}(v{\cdot}w)$	(T) (Thomsen)
Groups	$(u{\cdot}v){\cdot}w{=}u{\cdot}(v{\cdot}w)$	(R) (Reidemeister)
Moufang loops	$(v{\cdot}u){\cdot}(w{\cdot}v){=}v{\cdot}((u{\cdot}w){\cdot}v)$	$(M){=}(B_l){\cap}(B_r)$
Left Bol loops	$(u{\cdot}(v{\cdot}u)){\cdot}w{=}u{\cdot}(v{\cdot}(u{\cdot}w))$	(B_l) (left Bol)
Right Bol loops	$u{\cdot}((v{\cdot}w){\cdot}v){=}((u{\cdot}v){\cdot}w){\cdot}v$	(B_l) (right Bol)
Monoassociative loops	$u^2{\cdot}u{=}u{\cdot}u^2$	(H) (hexagonal)

TABLE 1

The equivalence of the identities in the second column of Table 1 and the corresponding closure conditions for all cases, except the first one, immediately follows from the definition of multiplication in the loop L_p (see [Ac 65], [Be 67], [Be 71], and [AS 92]). For condition (T) the situation is more complicated.

The closure conditions indicated in Table 1 are related by the following implication:

$$
\begin{tikzcd}
 & & & (B_l) \arrow[dr, Rightarrow] & \\
(T) \arrow[r, Rightarrow] & (R) \arrow[r, Rightarrow] & (M) \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (B_r) \arrow[r, Rightarrow] & (H) \\
 & & & (B_m) \arrow[ur, Rightarrow] &
\end{tikzcd}
\tag{16}
$$

When in the next section we will turn from complete 3-webs to geometric webs $W(3,2,r)$ defined on a differentiable manifold M, we will come to the following situation. The quasigroups $q(\cdot)$ associated with a web $W(3,2,r)$ are defined locally (not globally) but they are differentiable quasigroups (see [A 73b]). The coordinate loops L_p for a web $W(3,2,r)$ become local differentiable loops. As we did for complete webs, on a web $W(3,2,r)$ we can also consider closure conditions but we must restrict ourselves to the construction of *sufficiently small closure figures that entirely belong to the manifold M.*

It appeared that *for two-dimensional geometric* 3*-webs* $W(3,2,1)$ *all closure conditions indicated in Table* 1 *are equivalent* since for such webs we have the implication

$$(P) \Longleftrightarrow (H), \quad (P) \Longleftrightarrow (T)$$

(see [BB 38, §2], or [Ac 65] or [AS 92, Section 2.2]).

For webs $W(3,2,r)$ in the cases $r = 2,3$, the conditions (R) and (M) are equivalent. Only if $r \geq 4$, do webs $W(3,2,r)$ give geometric realizations for each of the closure conditions of Table 1. This was noted in [C 36] and studied in detail in works of Akivis and his followers (see [AS 92]).

Note also that *algebraically all the conditions of Table* 1 *are not equivalent*. This gives rise to the following problem: *Construct geometric* 3*-webs for which the closure conditions indicated above are not equivalent.* Chern [C 36] gave a particular solution of this problem, and Akivis [A 69b] (see also [G 88] and [AS 92]) gave its complete solution.

Finally note that the Bol paper [B 35] and the Chern paper [C 36] were the first works in the theory of multidimensional webs where the theory of invariants of a 3-web of dimension 4 and $2r$, respectively, was constructed. At the end of 1960s, the study of multidimensional 3-webs continued. In 1969 Akivis published the papers [A 69a], [A 69b], and these papers were followed by an extensive series of his papers as well as of his students' papers (see [AS 92] or [AG 00] for further references). In 1973 Goldberg began the study of $(n+1)$-webs $W(n+1,n,r)$ of codimension r on an (nr)-dimensional manifold (see [G 73] or [G 88] or [AG 00]).

5. Multidimensional 3-webs and local quasigroups

Let M be a C^s-manifold of dimension $2r$, $r \geq 1$, $s \geq 3$. We say that a 3-*web* $W = (M, \lambda_\alpha), \alpha = 1,2,3$, is given in M if

(a) three foliations λ_α of codimension r are given in M; and

(b) three leaves (of λ_α) passing through a point $p \in M$ are in general position, i.e., any two of the three tangent spaces to the leaves at the point p intersect each other only at the point p.

In a neighborhood U_p of a point $p \in M$ the foliations λ_α (the sets of leaves) become fibrations. Denote by X_α, $\dim X_\alpha = r$, their *local bases.*

Two webs $W(3,2,r)$ and $\widetilde{W}(3,2,r)$ with domains $D \subset X^{2r}$ and $\widetilde{D} \subset \widetilde{X}^{2r}$ are *locally equivalent* if there exists a local diffeomorphism $\varphi : D \to \widetilde{D}$ of their domains such that $\varphi(\lambda_\alpha) = \widetilde{\lambda}_\alpha$, $\alpha = 1,2,3$, where λ_α and $\widetilde{\lambda}_\alpha$ are foliations of $W(3,2,r)$ and $\widetilde{W}(3,2,r)$, respectively.

Suppose that two fibers $F_\alpha \subset X_\alpha$ and $F_\beta \subset X_\beta$ intersect each other at a point $p \in M$. Then there is a unique fiber $F_\gamma \subset X_\gamma$ passing through p. So, in a neighborhood U_p of a point p, there are six correspondences

$$q_{\alpha\beta} : X_\alpha \times X_\beta \to X_\gamma.$$

They define a *local* 3-*base quasigroup.* There are the following relations between these quasigroups:

$$(17)\quad q_{12} = q, \quad q_{13} = q^{-1}, \quad q_{32} = {}^{-1}q, \quad q_{21} = q^*, \quad q_{23} = ({}^{-1}q)^*, \quad q_{31} = (q^{-1})^*,$$

where $q^*(x,y) = q(y,x)$ and ${}^{-1}q$ and q^{-1} are the left and right inverse quasigroups for q, respectively. The latter means that if $z = q(x,y)$, then $x = {}^{-1}q(z,y)$ and $y = q^{-1}(x,z)$. Quasigroups (17) are the *coordinate quasigroups* of a web $W = (M, \lambda_\alpha)$.

The local diffeomorphisms

$$J_\alpha : X_\alpha \to Q, \quad \dim Q = r,$$

define on Q a local quasigroup with operation $\circ$:

$$J_1(x_1) \circ J_2(x_2) = J_3(q(x_1,x_2)), \; x_1 \in X_1, \; x_2 \in X_2.$$

The triple $J = (J_\alpha)$ is called an *isotopy* of the quasigroup $q(\cdot)$ upon a quasigroup $Q(\circ)$, and the quasigroups q and Q are called *isotopic* to each other.

From the definitions of equivalent webs and isotopic quasigroups it follows that *two webs are equivalent if and only if their corresponding coordinate quasigroups are isotopic.*

Therefore, there exists a family of isotopic quasigroups corresponding to a given 3-web. Web geometry studies properties of webs corresponding to those properties of quasigroups which are invariant under isotopies.

From results of Section 3 it follows in particular that with any point $p \in M$ we can associate the coordinate loop $L_p(\circ)$ which is isotopic to the quasigroup q, and $\dim L_p = r$. Conversely, suppose that a local differentiable loop L_p is given on an r-dimensional manifold X, and $e \in X$ is its unit. The loop L_p defines a smooth 3-web W whose foliations are defined by the equations $u = u_0$, $v = v_0$ and $u \circ v = w_0$, where $u_0, v_0, w_0 \in X$. A local loop at the point $p = (e, e)$ for the web W we have constructed is the original loop L_p, and all other local loops of W are isotopic to L_p.

Example. Grassmann 3-*web* on the Grassmannian $G(1, r{+}1)$, $\dim G(1, r{+}1) = 2r$.

Consider three hypersurfaces in a projective space P^{r+1} of dimension $r + 1$: $X_\alpha \subset P^{r+1}$, $\dim X_\alpha = r$. Suppose that M, $\dim M = 2r$, is a domain on the Grassmannian $G(1, r + 1)$ formed by the straight lines intersecting all X_α (see Figure 10). Consider bundles F_α of straight lines with their centers at $x_\alpha \in X_\alpha$, $\dim F_\alpha = r$. They will be leaves of 3 foliations λ_α defining a 3-web (M, λ_α). Such a 3-web is called a *Grassmann* 3-*web* (see [A 73a] and [AS 92]).

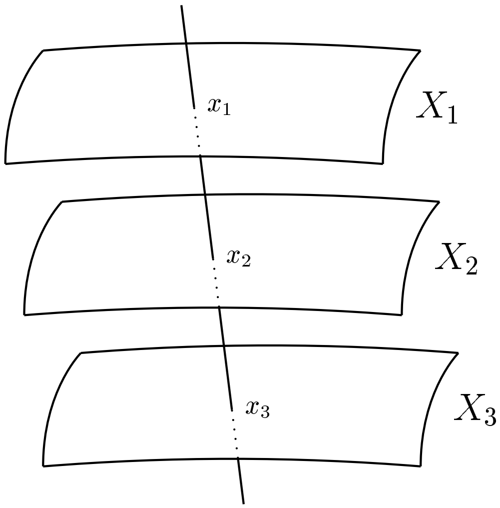

FIGURE 10

6. Local loops

1. Consider a local differentiable loop $Q(\cdot)$ of class $C^s, s \geq 3$, and in a neighborhood U_e of its identity e define a coordinate system in such a way that the coordinates of the point e are equal to zero. Denote the coordinates of points

u and v from U_e by u^i and v^i, respectively. Suppose that the product $u \cdot v$ also belongs to the neighborhood U_e. Then the coordinates of this product are differentiable functions of the coordinates of its factors u and v:

$$(u \cdot v)^i = f^i(u^j, v^k). \tag{18}$$

We will also write equations (18) in the form $u \cdot v = f(u, v)$ using the same symbol u for the set of coordinates (u^i) of the point $u \in Q$.

Using the Taylor formula in a neighborhood of the point e, we can expand the function $f(u, v)$. Since all the coordinates of the identity element e are zero, the function f satisfies the conditions $f(u, 0) = u$ and $f(0, v) = v$. If we restrict ourselves to third-order terms, we can write the Taylor formula in the form

$$u \circ v = u + v + \Lambda(u, v) + \frac{1}{2} M(u, u, v) + \frac{1}{2} N(u, v, v) + o(\rho^3), \tag{19}$$

where $\rho = \max\limits_i(|u^i|, |v^i|)$, $\Lambda(u, v)$ is a bilinear form, $M(u, v, w)$ and $N(u, v, w)$ are trilinear forms, and

$$M(u, v, w) = M(v, u, w), \quad N(u, v, w) = N(u, w, v) \tag{20}$$

(see [As 92, Section 2.4], or [AG 00, Section 2.6]).

Local coordinates u^i in a local loop L_p are defined up to transformations of the form

$$'u^i = {'u^i}(u^j), \tag{21}$$

where

$$'u^i(0) = 0, \quad \left.\frac{\partial('u^i)}{\partial u^j}\right|_0 = \alpha^i_j, \quad \det(\alpha^i_j) \neq 0. \tag{22}$$

These transformations are called *admissible*. They preserve expansion (19).

The form $\Lambda(u, v)$ in (19) is not invariant under admissible coordinate transformations (21) in Q. But this form allows us to construct an invariant bilinear skew-symmetric form

$$A(u, v) = \Lambda(u, v) - \Lambda(v, u), \tag{23}$$

which is called the *torsion form* of Q. It is defined in the space $T_e \times T_e$ with its values in T_e, where T_e is the tangent space to the loop Q at the identity element e.

Similarly, the forms M and N occurring in (19) are not invariant under admissible coordinate transformations (21) in Q, but they allow us to construct an invariant trilinear form

$$B(u, v, w) = M(u, v, w) - N(u, v, w) + \Lambda(\Lambda(u, v), w) - \Lambda(u, \Lambda(v, w)), \tag{24}$$

which is called the *curvature form* of Q.

Both forms, $A(u,v)$ and $B(u,v,w)$, are tensor forms. They are connected by the following identity:

$$\operatorname{alt} B(u,v,w) = \frac{1}{2} \operatorname{alt} A(A(u,v),w) = \frac{1}{6} J(u,v,w), \tag{25}$$

called the *generalized Jacobi identity* (see [AS 92, Section 2.4], or [AG 00, Section 2.4]). The term "generalized" is used here since in the case when a loop Q is a group (i.e., if the associativity holds) the identity (25) becomes the Jacobi identity.

2. Let us find an algebraic meaning of the forms $A(u,v)$ and $B(u,v,w)$. In a loop Q we consider the *left and right commutators*

$$\alpha_l(u,v) = {}^{-1}(v \circ u) \circ (u \circ v), \quad \alpha_r(u,v) = (u \circ v) \circ (v \circ u)^{-1}, \tag{26}$$

as well as the *left and right associators*

$$\begin{aligned} \beta_l(u,v,w) &= {}^{-1}(u \circ (v \circ w)) \circ ((u \circ v) \cdot w), \\ \beta_r(u,v,w) &= ((u \circ v) \circ w) \circ (u \circ (v \circ w))^{-1}. \end{aligned} \tag{27}$$

The following result shows a relation between the commutators (26) and the associators (27) and the torsion form $A(u,v)$ and the curvature form $B(u,v,w)$: *Up to infinitesimals of second and third order respectively, the commutators and associators of the loop $Q(\circ)$ coincide with the forms $A(u,v)$ and $B(u,v,w)$ defined above, i.e., the following relations hold*:

$$\alpha_l(u,v) = A(u,v) + o(\rho^2), \quad \alpha_r(u,v) = A(u,v) + o(\rho^2), \tag{28}$$

and

$$\beta_l(u,v,w) = B(u,v,w) + o(\rho^3), \quad \beta_r(u,v,w) = B(u,v,w) + o(\rho^3). \tag{29}$$

For the proof of this result see [AS 92, Section 2.4].

It follows from relations (28) and (29) that $A(u,v)$ is the principal part of the left and right commutators, and $B(u,v,w)$ is the principal part of the left and right associators of the loop $Q(\circ)$.

7. Local algebras of a 3-web (M, λ_α)

1. The commutators and associators of the loop $Q(\circ)$ considered above allow us to define, in the tangent space $T_e(Q)$, binary and ternary operations which are called the *commutation* and the *association*, respectively. Consider two smooth lines $u(t)$ and $v(t)$ on the loop $Q(\circ)$ that pass through its unit e, parametrize these lines in such a way that $u(0) = v(0) = e$, and denote the tangent vectors to these curves at the point e by ξ and η. In order not to make our notation complicated, we identify a vector with a set of its coordinates:

$$\xi = \lim_{t \to 0} \frac{u(t)}{t}, \quad \eta = \lim_{t \to 0} \frac{v(t)}{t}, \quad \xi, \eta \in T_e(Q).$$

Construct two more curves on the loop $Q(\circ)$:

$$\alpha_l(t) = A(u(t), v(t)) + o(t^2) \text{ and } \alpha_r(t) = A(u(t), v(t)) + o(t^2),$$

where α_l and α_r are the commutators (26) of the loop $Q(\circ)$. These lines also pass through the point e for $t = 0$. The curves $\alpha_l(t)$ and $\alpha_r(t)$ have the common tangent vector $\zeta = A(\xi, \eta)$ at the point e:

$$\zeta = \lim_{t\to 0} \frac{\alpha_l(t)}{t^2} = \lim_{t\to 0} \frac{\alpha_r(t)}{t^2} = A(\xi, \eta).$$

The vector ζ is called the *commutator* of the vectors $\xi \in T_e(Q)$ and $\eta \in T_e(Q)$ and denoted by $[\xi, \eta]$. Therefore, we have

$$[\xi, \eta] = A(\xi, \eta). \tag{30}$$

It follows that the commutation is bilinear and skew-symmetric:

$$[\xi, \eta] = -[\eta, \xi]. \tag{31}$$

Next, consider on Q three smooth curves $u(t), v(t)$, and $w(t)$ parametrized in such a way that $u(0) = v(0) = w(0) = e$ and denote the tangent vectors to these curves at the point e by $\xi, \eta, \zeta \in T_e(Q)$:

$$\xi = \lim_{t\to 0} \frac{u(t)}{t}, \quad \eta = \lim_{t\to 0} \frac{v(t)}{t}, \quad \zeta = \lim_{t\to 0} \frac{w(t)}{t}.$$

Construct two more curves $\beta_l(t)$ and $\beta_r(t)$ on the loop $Q(\circ)$ both also passing through the point e:

$$\beta_l(t) = B(u(t), v(t), w(t)) + o(t^3) \text{ and } \beta_r(t) = B(u(t), v(t), w(t)) + o(t^3), \tag{32}$$

where β_l and β_r are the associators (27) of the loop $Q(\circ)$. The curves $\beta_l(t)$ and $\beta_r(t)$ have the common tangent vector at the point e:

$$\theta = \lim_{t\to 0} \frac{\beta_l(t)}{t^3} = \lim_{t\to 0} \frac{\beta_r(t)}{t^3} = B(\xi, \eta, \zeta).$$

The vector θ is called the *associator* of the vectors $\xi, \eta, \zeta \in T_e(Q)$. It follows from the previous formula that

$$(\xi, \eta, \zeta) = B(\xi, \eta, \zeta), \tag{33}$$

and relation (24) shows that the commutators and associators are connected by the relation

$$\operatorname{alt}(\xi, \eta, \zeta) = \frac{1}{2} \operatorname{alt}[[\xi, \eta], \zeta] = \frac{1}{6} J(\xi, \eta, \zeta), \tag{34}$$

where "alt" denote the alternating mean, and

$$J(\xi,\eta,\zeta) = [[\xi,\eta],\zeta] + [[\eta,\zeta],\xi] + [[\zeta,\xi],\eta]$$

is the Jacobian of the elements ξ, η, and ζ. Relation (34) is called the *generalized Jacobi identity* for the pair of forms $[\xi,\eta]$ and (ξ,η,ζ).

2. The operations of commutation $[\ ,\ ]$ and association $(\ ,\ ,\ )$ connected by the generalized Jacobi identity (34) that were introduced above define in the tangent space $T_e(Q)$ of the loop $Q(\circ)$ a binary-ternary algebra $\mathcal{A}$ which is a local algebra of the loop $Q(\circ)$. This algebra was introduced by Akivis in [A 78] during his study of multidimensional 3-webs and their local coordinate quasigroups. Later on in [HS 86a], [HS 86b] this algebra was named the *Akivis algebra*. Hofmann and Strambach in [HS 86a], [HS 86b], [HS 90] considered the correspondence between real analytic loops and their local Akivis algebras (see [HS 86a], [HS 86b], [HS 90]). It turns out that in the general case an Akivis algebra does not define an analytic loop uniquely (see [HS 86a], [HS 86b], [HS 90]).

However, for some special classes of quasigroups this correspondence is 1-to-1. In particular, if a differentiable loop $Q(\circ)$ is a local Lie group G, then its associators β_l and β_r defined by formulas (32) are identically equal to e. In this case the operation of association in the space T_e is trivial, i.e., $(\xi,\eta,\zeta) = 0$ for any vectors ξ,η, and ζ from T_e. Moreover, identity (34) becomes the Jacobi identity

$$J(\xi,\eta,\zeta) = 0,$$

and the tangent Akivis algebra of the loop $Q(\circ)$ becomes a Lie algebra $L(G)$. Namely this was the reason for calling relation (34) the generalized Jacobi identity.

It is well-known that for an arbitrary Lie algebra L, there exists an associative algebra B such that L can be isomorphically embedded into the commutator algebra of B.

Akivis [A 76] posed the following problem: *Generalize this construction to the binary-ternary algebras* $\mathcal{A}$. Recently Shestakov (1998) solved this problem (see [Sh 99a], [Sh 99b], [Sh ta]). He proved that *an arbitrary Akivis algebra can be isomorphically embedded into the algebra of commutators and associators of a certain nonassociative algebra* B.

8. Special classes of 3-webs and their local algebras

1. In Section 2 we considered different special classes of 3-webs. The following table shows the structure of local loops and local algebras, indicates the principal operations for these classes, and provides references.

Closure condition	Local loop	Local algebra	Principal operations	References
(T)	Abelian group	Trivial	—	[A 69b]
(R)	Lie group	Lie algebra	$[\xi, \eta]$	[A 69b]
(M)	Moufang loop	Mal'cev algebra	$[\xi, \eta]$	[AS 71], [Sag 61]
(B_l)	Left Bol loop	Bol algebra	$[\xi, \eta]$, (ξ, η, ζ)	[F 78], [SM 82]
(B_r)	Right Bol loop	Bol algebra	$[\xi, \eta]$, (ξ, η, ζ)	[F 78], [SM 82]
(H)	Monoassociative loop	Local algebra of hexagonal 3-web	$[\xi, \eta]$, (ξ, η, ζ), $(\xi, \eta, \zeta, \theta)_1$, $(\xi, \eta, \zeta, \theta)_2$	[Sh 89b]

Table 2

2. We will comment Table 2 now and find special features of the local Akivis algebras $\mathcal{A}$ associated with 3-webs on which the closure conditions indicated in Table 2 are satisfied.

(i) If condition (T) holds on a web W, then by (3), the forms $A(u,v)$ and $B(u,v,w)$ defined by (23) and (24) vanish. This implies that the commutators and the associators of $\mathcal{A}$ vanish too, that is, for arbitrary vectors $\xi,\ \eta,\ \zeta \in T_e(Q)$, we have

$$[\xi,\ \eta] = 0,\ (\xi,\ \eta,\ \zeta) = 0. \tag{35}$$

As a result, the Akivis algebras $\mathcal{A}$ become *trivial*.

(ii) If condition (R) holds on a web W, then by (4), the form $B(u,v,w)$ defined by (24) vanishes. This implies that the associators of all local Akivis algebras vanish too,

$$(\xi,\ \eta, \zeta) = 0. \tag{36}$$

As we showed earlier, the generalized Jacobi identity (34) becomes the regular Jacobi identity. As a result, the Akivis algebras $\mathcal{A}$ become the *Lie algebras* that are isomorphic copies of a given Lie algebra.

(iii) If condition (M) holds on a web W, then the Moufang identity (13) holds. Since a Moufang web is always a Bol web, we have conditions (10) and (11) which imply that the associators of the Akivis algebras of a Moufang web satisfy the conditions

$$(\xi,\xi,\eta) = 0, \quad (\xi,\eta,\eta) = 0. \tag{37}$$

The generalized Jacobi identity (34) is reduced to the form

$$(\xi,\ \eta,\ \zeta) = \frac{1}{6} J(\xi,\ \eta,\ \zeta), \tag{38}$$

i.e., the ternary operation in a local Akivis algebra $\mathcal{A}$ is expressed in terms of its binary operation. It can be proved (see [Sag 61] or [AS 71]) that in this case the commutator of an Akivis algebra will satisfy the so-called *Sagle identity*:

$$[[\xi,\ \eta],\ [\xi,\ \zeta]] = [[[\xi,\ \eta],\ \zeta],\ \xi] + [[[\eta,\ \zeta],\ \xi],\ \xi] + [[[\zeta,\ \xi],\ \xi],\ \eta]. \tag{39}$$

Thus in this case an Akivis algebra becomes a *Mal'cev algebra*. It was proved in [Ku 70], [Ku 71] (see also [AS 71]) that a Moufang loop (and a Moufang web) is completely defined by its Mal'cev algebra given in the tangent space $T_e(L_p)$. Earlier Mal'cev [Ma 55] established a similar relationship between the binary Lie loops and Mal'cev algebras.

(iv) If the left Bol condition (B_l), which is expressed algebraically by equation (14), holds on a web W, then for all its local loops condition (10) of left alternativity holds. Identity (10) implies that the associators of the Akivis algebras of a left Bol web satisfy the condition

$$(\xi, \xi, \eta) = 0. \tag{40}$$

Fedorova [F 78] proved that condition (40) implies that for a left Bol web all multilinear forms of order higher than 3, that can be constructed by prolongation of equation (19), are expressed in terms of the forms $A(u, v)$ and $B(u, v, w)$.

Sabinin and Mikheev [SM 82, 85] found all closed form conditions which the operations $[\xi, \eta]$ and (ξ, η, ζ) of a local algebra of a left Bol web satisfy. They called such binary-ternary algebras the *Bol algebras*.

We will give here a new definition of Bol algebras which is equivalent to the definition given above. Consider the Akivis algebra $\mathcal{A}$ of a Bol 3-web. As for the general 3-webs, a binary operation in the algebra $\mathcal{A}$ is defined by (30). As to a ternary operation, instead of defining it by (33), we will introduce a new ternary operation $\{\ ,\ ,\ \}$ in $T_e(L_p)$ by setting

$$\{\xi,\ \eta,\ \zeta\} = \frac{1}{4}(\xi,\ \eta,\ \zeta) - \frac{1}{2}[[\eta,\ \zeta],\ \xi]. \tag{41}$$

This operation satisfies the following properties:

$$\begin{aligned} &a)\quad \{\xi,\ \eta,\ \eta\} = 0,\\ &b)\quad \{\xi,\ \eta,\ \zeta\} + \{\eta,\ \zeta,\ \xi\} + \{\zeta,\ \xi,\ \eta\} = 0,\\ &c)\quad \{\{\zeta,\ \theta,\ \psi\},\ \xi,\ \eta\} = \{\{\zeta,\ \xi,\ \eta\},\ \theta,\ \psi\}\\ &\qquad + \{\zeta,\ \{\theta,\ \xi,\ \eta\},\ \psi\} + \{\zeta,\ \theta,\ \{\psi, \xi,\ \eta\}\},\\ &d)\quad \{[\zeta,\ \theta],\ \xi,\ \eta\} - \{[\xi,\ \eta],\ \zeta,\ \theta\} - [\{\eta,\ \zeta, \theta\},\ \xi]\\ &\qquad + [\{\xi,\ \zeta,\ \theta\},\ \eta] + [[\xi\ \eta],\ [\zeta,\ \theta]] = 0 \end{aligned} \tag{42}$$

(see [AS 92, Section 4.1]). A linear space T with given binary and ternary operations, $[\ ,\ ]$ and $\{\ ,\ ,\ \}$, satisfying identities (42), is called a *Bol algebra*.

It is easy to see that these two definitions of the Bol algebras are equivalent. Thus, the Akivis algebras of the Bol 3-web (B_l) are Bol algebras.

Sabinin and Mikheev [SM 85] also proved that a left Bol loop is completely determined by a left Bol algebra. This implies that a 3-web W on which the closure condition (B_l) holds is also completely determined by a left Bol algebra. This last result was proved also in [F 78] by means of another method.

Of course, similar results are valid for the right and middle Bol 3-webs.

(v) If the hexagonality condition (H) holds on a web W (i.e., a web W is hexagonal), then all its local loops are monoassociative (see Table 1). This implies that the associators of local algebras of W satisfy the condition

$$(\xi, \xi, \xi) = 0. \tag{43}$$

Shelekhov (see [S 89a], [S 89b], [S 89c]) proved that this implies that the expansion (19) generates not only the commutator and the associator in a local algebra but also two quaternary operations, and all other terms of this expansion can be expressed in terms of these 4 operations. These 4 operations determine a local binary-ternary-quaternary algebra in the space $T_e(L_p)$. However, for a local algebra of a hexagonal 3-web (H), a complete system of identities, which these 4 operations satisfy, is unknown. Mikheev [Mi 96] proved that one of two quaternary operations found by Shelekhov in [S 89a], [S 89b], [S 89c] can be expressed in terms of the other one and the binary and ternary operations.

3. In Section 5 we defined a Grassmann 3-web on the Grassmannian $G(1, r+1)$. We will find now local algebras of such webs. Such a web was defined in a projective space P^{r+1} by a triple of hypersurfaces $X_\alpha, \alpha = 1, 2, 3$ that are in general position. Its elements are the straight lines intersecting each of X_α at a point, and its fibers are the bundles of straight lines with their centers located at X_α (see Figure 10). A coordinate quasigroup $q : X_1 \times X_2 \to X_3$ is a mapping of X_1 and X_2 into X_3 by means of straight lines of P^{r+1}.

Akivis [A 73a] proved that the basic operations (30) and (33) of the local algebra of a Grassmann 3-web can be written as follows

$$\begin{cases} [\xi, \eta] = a(\xi)\eta - a(\eta)\xi, \\ (\xi, \eta, \zeta) = h_1(\eta, \zeta)\xi + h_2(\zeta, \xi)\eta + h_3(\xi, \eta)\zeta, \end{cases} \tag{44}$$

where $a(\xi)$ is a linear form, and h_α are symmetric bilinear forms, which are *the second fundamental forms* of the hypersurfaces $X_\alpha \subset P^{r+1}$.

The converse is also valid: *If the operations in all local algebras of a* 3*-web* (M, λ_α) *have the forms indicated above, then such a web is Grassmannizable*, i.e., it is equivalent to a Grassmann 3-web.

Suppose now that on a Grassmann 3-web, one of the closure conditions indicated in Table 1 holds. Then we have the following classification (see [A 73a] or [AS 92, Section 3.3]).

a) If condition (H) holds (i.e., if W is a hexagonal Grassmann 3-web), then

$$h_1 + h_2 + h_3 = 0, \tag{45}$$

and the hypersurfaces X_α belong to an algebraic surface of third order.

b) If condition (B_l) holds (i.e., if W is a left Bol Grassmann 3-web), then

$$h_1 + h_2 = 0, \quad h_3 = 0, \tag{46}$$

and the hypersurfaces X_1 and X_2 belong to a hyperquadric, and the hypersurface X_3 is a hyperplane.

c) If condition (R) holds (i.e., if W is a group Grassmann 3-web), then

$$h_\alpha = 0, \tag{47}$$

and all hypersurfaces X_α are hyperplanes in general position.

d) If condition (T) holds (i.e., if W is a parallelizable 3-web), then

$$h_\alpha = 0, \quad a = 0, \tag{48}$$

and all hypersurfaces X_α are hyperplanes of a pencil with an $(r-1)$-dimensional vertex.

Note that Grassmann 4-webs $W(4,3,r)$ were studied in [AG 74], Grassmann $(n+1)$-webs $W(n+1,n,r)$, $n > 2$, in [G 75c], and Grassmann 4-webs $W(4,2,r)$ in [G 82b] (see also [G 88, Sections 5.1 and 7.6]).

4. In the theory of Lie groups the following Third Lie's converse theorem is well-known: *A Lie algebra completely determines a local Lie group.* Our previous considerations prove that similar theorems are valid for Moufang loops, Bol loops, and monoassociative loops: *the Moufang loops are determined completely by their local Mal'cev algebras, the Bol loops by their local Bol algebras, and the monoassociative loops by local algebras of hexagonal webs.*

Note that for Moufang's loops, the converse Third Lie Theorem was proved by different methods by Kuz'min [Ku 70], [Ku 71] and by Akivis and Shelekhov [AS 71]. For Bol's loops, the converse Third Lie Theorem was proved independently by Fedorova [F 78] and Sabinin and Mikheev [SM 82].

Note also that differential geometric structures defined on the manifold M, $\dim M = 2r$, by the webs indicated in Table 2 are closed in the sense of the paper [A 75]. As a result, the theorems analogous to the converse part of Lie's Third Fundamental Theorem are valid for them.

9. $(n+1)$-webs and local n-quasigroups

1. In [G 75a], [G 75b], [G 76], [G 88] Goldberg found a close relationship between $(n+1)$-webs and their n-quasigroups. For simplicity, in this section we will consider 4-webs and ternary quasigroups.

Let $M = X^{3r}$ be a differentiable manifold of dimension $N = 3r$. We shall say that a 4-*web* $W(4,3,r)$ *of codimension* r is given in an open domain $D \subset X^{3r}$ by a set of 4 foliations of codimension r which are in general position.

Two webs $W(4,3,r)$ and $\widetilde{W}(4,3,r)$ with domains $D \subset X^{3r}$ and $\widetilde{D} \subset \widetilde{X}^{3r}$ are *locally equivalent* if there exists a local diffeomorphism $\varphi : D \to \widetilde{D}$ of their domains such that $\varphi(\lambda_\xi) = \widetilde{\lambda}_\xi$, $\xi = 1,2,3,4$, where λ_ξ and $\widetilde{\lambda}_\xi$ are foliations of $W(4,3,r)$ and $\widetilde{W}(4,3,r)$, respectively.

Next we will define local differentiable ternary quasigroups and loops. Let X_ξ, $\xi = 1,2,3,4$, be differentiable manifolds of the same dimension r. Let

$$f : X_1 \times X_2 \times X_3 \to X_4 \tag{49}$$

be a mapping satisfying the following conditions: if $a_4 = f(a_1, a_2, a_3)$, then

(i) for any neighborhood U_4 of a_4, there exist neighborhoods U_α of a_α, $\alpha = 1,2,3$, such that for any $x_\alpha \in U_\alpha$, the value of the function $f(x_1,x_2,x_3)$ is defined and $f(x_1,x_2,x_3) = x_4 \in U_4$;

(ii) for any neighborhood U_α of a_α, where α is fixed, there exist neighborhoods U_β of a_β, $\beta \neq \alpha$ and U_4 of a_4 such that for any $x_\beta \in U_\beta$ and $x_4 \in U_4$, the equation $f(x_1,x_2,x_3) = x_4$ is solvable for x_α, and $x_\alpha \in U_\alpha$.

If the manifolds X_ξ and the function f are of class C^k, then we say that there is given a 4-*base local differentiable ternary quasigroup* $Q(f)$ (abbreviation: l.d.t. quasigroup).

Suppose that $X_1 = X_2 = X_3 = X_4 = X$, and there exists at least one element $e \in X$ such that

$$f(x,e,e) = f(e,x,e) = f(e,e,x) = x. \tag{50}$$

Then the l.d.t. quasigroup $Q(f)$ is called a *l.d.t. loop*. If in addition for any x_1, x_2, x_3, x_4, x_5, we have

$$f(f(x_1,x_2,x_3),x_4,x_5) = f(x_1,f(x_2,x_3,x_4),x_5) = f(x_1,x_2,f(x_3,x_4,x_5)), \tag{51}$$

then the l.d.t. quasigroup $Q(f)$ is said to be an *l.d.t. group* (see [G 75a], [G75 b], [G 76] or [G 88, Section 3.1]).

Example. The equations

$$x_4^i = f^i(x_1^j, x_2^k, x_3^l), \qquad i,j,k,l = 1,2,3, \tag{52}$$

in some domain $D \subset \mathbf{R}^{3r}$ define an l.d.t. quasigroup if in D

$$\det\left(\frac{\partial f^i}{\partial x_\alpha^j}\right) \neq 0, \qquad \alpha = 1,2,3. \tag{53}$$

In this case X_α is the projection of D onto the r-dimensional subspace defined by the axes Ox_α^i, $i = 1,\dots,r$, and X_4 is the range of the functions f^i.

2. Let $Q(f)$ be a ternary quasigroup given on a set Q. On the set $S = Q \times Q \times Q$ it defines a web $W(4, Q)$ formed by 4 foliations λ_ξ whose leaves are determined by the equations

$$
(54) \qquad \begin{aligned} F_\alpha &= \{(x_1, x_2, x_3) | x_\alpha = a_\alpha, \quad \alpha = 1, 2, 3\}, \\ F_4 &= \{(x_1, x_2, x_3) | f(x_1, x_2, x_3) = a_4\}, \end{aligned}
$$

where a_α and a_4 are constants. Such a web is called *complete*.

Conversely, let a complete 4-web W formed on a set S by 4 foliations λ_ξ with bases X_ξ, $\xi = 1, 2, 3, 4$, such that $S = X_{\xi_1} \times X_{\xi_2} \times X_{\xi_3}$, $\xi_1, \xi_2, \xi_3 = 1, 2, 3, 4$. Then we have a mapping: $q_{\xi_1 \xi_2 \xi_3} : X_{\xi_1} \times X_{\xi_2} \times X_{\xi_3} \to X_{\xi_4}$ for which the corresponding leaves pass through the same point $p \in S$. This mapping gives a 4*-base quasigroup* (see [Be 71, Section 1.1], [G 75a], [G 75b], [G 76], and [G 88, Section 3.1]).

If each foliation is an r-dimensional differentiable manifold, then $S = X^{3r}$, and we obtain an l.d.t. quasigroup $Q(f)$ defined by a web $W(4, 3, r)$. It is easy to see that there are 4! of such ternary quasigroups. They are called *coordinate quasigroups* of a 4-web W.

Now suppose that there are given 4 differentiable and locally invertible mappings (diffeomorphisms) $g_\xi : X_\xi \to \overline{X}_\xi$. Then we obtain an l.d.t. quasigroup $\overline{Q}(\overline{f})$

$$
(55) \qquad \overline{f} : \overline{X}_1 \times \overline{X}_2 \times \overline{X}_3 \to \overline{X}_4
$$

if we put

$$
(56) \qquad \overline{x}_4 = g_4(x_4) = g_4(f(x_1, x_2, x_3)) = g_4(f(g_1^{-1}(\overline{x}_1), g_2^{-1}(\overline{x}_2), g_3^{-1}(\overline{x}_3))).
$$

The quasigroup $\overline{Q}(\overline{f})$ is called *isotopic* to the quasigroup $Q(f)$.

From the definitions of equivalent 4-webs and isotopic ternary quasigroups it follows that *two* 4*-webs are equivalent if and only if their corresponding coordinate ternary quasigroups are isotopic*.

Therefore, there exists a family of isotopic ternary quasigroups corresponding to a given 4-web. The theory of 4-webs studies properties of webs corresponding to those properties of ternary quasigroups which are invariant under isotopies.

3. *Example.* A parallelizable web $PW(4, 3, r)$ is a web which is equivalent to a web formed by 4 foliations λ_ξ of parallel $(2r)$-planes of an affine space A^{3r}.

A coordinate ternary quasigroup of a web $PW(4, 3, r)$ is isotopic to an abelian ternary group. Conversely, if a coordinate ternary quasigroup of a web $W(4, 3, r)$ is isotopic to an abelian ternary group, then the web $W(4, 3, r)$ is parallelizable.

It is easy to prove (see [AG 00, Section 2]) that web equations of a parallelizable web $PW(4, 3, r)$ can be reduced to the form $x_4^i = \sum_\alpha x_\alpha^i$, and conversely.

Suppose that we have a ternary quasigroup (49) and $f(a_1, a_2, a_3) = a_4$. We map the sets X_ξ into X_4 as follows:

$$
(57) \qquad u_1 = f(x_1, a_2, a_3), \quad u_2 = f(a_1, x_2, a_3), \quad u_3 = f(a_1, a_2, x_3), \quad u_4 = x_4.
$$

It is easy to prove (see [Be 72, Section 1.2], [G 88, Section 3.1], and [G 90]) that as a result of this isotopy, we obtain a ternary loop on the set X_4 with the unit $e = a_4$. In fact, denote the three inverse operations for the quasigroup f by f_α. Then it follows from (57) that

$$x_1 = f_1(u_1, a_2, a_3),\ x_2 = f_2(a_1, u_2, a_3),\ x_3 = f_3(a_1, a_2, u_3),\ u_\alpha \in X_4, \tag{58}$$

where f_α is the inverse function of f with respect to the argument x_α. On the set X_4 in a neighborhood of e we can define now the operation $u_1u_2u_3 = F(u_1, u_2, u_3)$ in the following way:

$$\begin{aligned} u_4 &= f(x_1, x_2, x_3) = f(f_1(u_1, a_2, a_3), f_2(a_1, u_2, a_3), f_3(a_1, a_2, u_3)) \\ &= F(u_1, u_2, u_3). \end{aligned} \tag{59}$$

For the operation F we have

$$F(u_1, e, e) = u_1, \quad F(e, u_2, e) = u_2, \quad F(e, e, u_3) = u_3. \tag{60}$$

The above isotopy maps the quasigroup $Q(f)$ into a ternary loop L_p, $p = (a_1, a_2, a_3)$ which is called a *a coordinate ternary loop* of the web $W(4, 3, r)$. It can be defined for any point $p = (a_1, a_2, a_3)$.

4. We shall assume that the points a_α correspond to zero coordinates. Then by (60) we have

$$F(u_1, 0, 0) = u_1,\ F(0, u_2, 0) = u_2, \quad F(0, 0, u_3) = u_3. \tag{61}$$

We now write the Taylor expansions of the function F. Taking into account (61) and restricting ourselves to second-order terms, we have the following expansion for the function F:

$$F(u_1, u_2, u_3) = u_1 + u_2 + u_3 + \frac{1}{2}(\underset{12}{\lambda} u_1u_2 + \underset{23}{\lambda} u_2u_3 + \underset{31}{\lambda} u_3u_1) + o(\rho^2), \tag{62}$$

where $\rho = \max\limits_i(|u_1^i|,\ |u_2^i|,\ |u_3^i|)$.

As in the binary case, local coordinates u^i in a local loop L_p are defined up to transformations of the form

$$'u^i = {}'u^i(u^j), \tag{63}$$

where

$$'u^i(0) = 0, \quad \left.\frac{\partial('u^i)}{\partial u^j}\right|_0 = \alpha^i_j, \quad \det(\alpha^i_j) \neq 0.$$

These transformations are called *admissible*. They preserve expansion (62).

The coefficients of (62) are partial derivatives of the functions F at $u_\alpha = 0$:

$$\underset{\alpha\beta}{\lambda}{}_{jk} = \left.\frac{\partial^2 F}{\partial u^j_\alpha \partial u^k_\beta}\right|_{u_\gamma = 0}. \tag{64}$$

These coefficients are not tensors under transformations (63), but they generate some tensors.

5. Goldberg has proposed the problem of finding an algebraic construction in the tangent bundle of the coordinate n-loop of a web $W(n+1,n,r)$, $n>2$, similar to the construction of the Akivis algebra for webs $W(3,2,r)$ for binary loops. Smith [Sm 88] (see also [G 88, Section 3.7]) found such a construction and established a correspondence between real analytic n-loops and appropriate algebraic objects. As in the binary case, neither this correspondence is bijective.

10. Special classes of multidimensional 4-webs and local differentiable ternary quasigroups

1. There are many special classes of multidimensional $(n+1)$-webs $W(n+1,n,r)$ and l.d. n-quasigroups. For simplicity, in this section we consider 4-webs $W(4,3,r)$ and local differentiable ternary quasigroups and some of their special classes. For others, the reader is referred to [G 75a], [G 76] (see also [G 88, Chapter 7]).

It is important to note that as we saw in the binary case, *all special classes of webs $W(n+1,n,r)$ are not distinct for $r=1$, but are distinct for $r>1$.*

In this section, we will use the notations x,y,z instead of x_1, x_2, x_3.

We will call a ternary quasigroup *reducible* if its ternary operation can be reduced to two binary operations. For example, consider one of the three possible reducibilities:

$$u=f(x,y,z)=g(x,\varphi(y,z))=x\circ(y\cdot z). \tag{65}$$

Reducibility of this kind of n-quasigroups was studied from the algebraic point of view in [San 65], [BSa 66], [Be 72, Section 5.3], and [Ra 60].

We will call a 4-web $W(4,3,r)$ *reducible* if at least one of its coordinate ternary quasigroups is reducible.

It was proved (see [Ra 60]) that reducibility (65) is equivalent to the closure condition

$$f(x_1,y_2,z_1)=f(x_1,y_1,z_2)\Longrightarrow f(x_2,y_2,z_1)=f(x_2,y_1,z_2), \tag{66}$$

and to the following identity in the loop $L(u_1,u_2,u_3)$:

$$F(x,e,y)=F(x,y,e). \tag{67}$$

The geometric meaning of condition (66) can be seen from Figure 11, where the leaves of the first three foliations of a 4-web are presented as coordinate planes of some Cartesian coordinate system $Oxyz$, the point $O(x_1,y_1,z_1)$ is the origin, and the points M_1, M_2, M_3, and M_4 have the coordinates: $M_1(x_1,y_2,z_1)$, $M_2(x_1,y_1,z_2)$, $M_3(x_2,y_2,z_1)$, $M_4(x_2,y_1,z_2)$. Condition (66) means that if the points M_1 and M_2 belong to a leaf of the fourth foliation λ_4, then the points M_3 and M_4 also belong to a leaf of the fourth foliation λ_4.

If we apply (64), we obtain condition (67) in the form of partial differential equations for the functions $u^i=F^i(x^j,y^k,z^l)$. They represent a necessary and

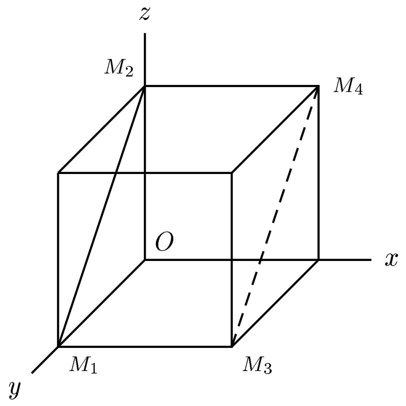

FIGURE 11

sufficient condition for a smooth vector function of 3 vector variables to be a superposition of two vector functions of 2 variables each. These conditions were obtained by Goldberg in [G 75a], [G 76] (see also [G 88, Section 4.1]) for $n \geq 3$. In the case $n = 3$ and $r = 1$ for reducibility (66) they have the form of the equation

$$\frac{F_{xy}}{F_y} = \frac{F_{xz}}{F_z} \tag{68}$$

(see [G 75a] or [G 88, Section 4.1]), which is a necessary and sufficient condition for a smooth function $F(x, y, z)$ of three variables to be a superposition of two functions of two variables each, that is, to have the form

$$F(x, y, z) = g(x, \varphi(y, z)). \tag{69}$$

Equations (68) and (69) were first obtained by Goursat [Go 99] and are usually discussed in monographs.

Goldberg in [G 76] (see also [G 88, Sections 4.1 and 4.2]) developed an algorithm for finding a set of invariant tensorial conditions for webs $W(n + 1, n, r)$ and corresponding n-quasigroups that are reducible of different kinds.

Belousov ([Be 72, p. 217]) posed the following problem: *Construct examples of irreducible n-quasigroups for $n > 3$. Do there exist irreducible n-quasigroups for any $n > 3$?*

Invariant tensorial conditions for reducible webs $W(n+1, n, r)$ and the corresponding n-quasigroups obtained in [G 75a], [G 76], [G 88] show that reducible $(n + 1)$-webs (and therefore, reducible n-quasigroups) form a special class of $(n + 1)$-webs (of n-quasigroups) characterized by the invariant conditions mentioned above. However, if these invariant conditions do not hold, then the $(n+1)$-web and its coordinate n-quasigroups are irreducible. It is not difficult to construct concrete examples of irreducible local differentiable n-quasigroups.

2. We shall call a 4-web $W(4,2,r)$ a *group* 4-web if at least one of its coordinate ternary quasigroups is a ternary group. Goldberg in [G 75a], [G 76] (see also [G 88, Section 4.3]) found invariant characterizations for the group $(n+1)$-webs, $n \geq 2$.

For $n=3$ we find from the results mentioned above that *for a complete* 4-*web to be a group web, it is necessary and sufficient that the web be doubly reducible* (i.e., it is reducible in two different forms). For example, for one of three possible double reducibilities, we have

$$f(x,y,z) = y \circ x \circ z, \tag{70}$$

where the operation $\circ$ is determined by the formula $x \circ y = F(x,y,e)$. The closure condition in the corresponding ternary quasigroup and the identity in the corresponding ternary loop that are equivalent to (70) are

$$\begin{aligned} f(x_2,y_1,z_1) = f(x_1,y_2,z_1), \quad & f(x_1,y_1,z_3) = f(x_3,y_1,z_1) \\ & \Longrightarrow f(x_2,y_1,z_3) = f(x_3,y_2,z_1) = f(x_1,y_2,z_3) \end{aligned} \tag{71}$$

and

$$F(e,x,y) = F(x,e,y) = F(y,x,e), \tag{72}$$

(see [Ra 60]). The geometric meaning of condition (71) can be seen in Figure 12, where the points O, M_1, M_2, M_3, M_4, M_5, M_6, M_7 have the following coordinates: $O(x_1,y_1,z_1)$, $M_1(x_2,y_1,z_1)$, $M_2(x_1,y_2,z_1)$, $M_3(x_1,y_1,z_3)$, $M_4(x_3,y_1,z_1)$, $M_5(x_2,y_1,z_3)$, $M_6(x_3,y_2,z_1)$, $M_7(x_1,y_2,z_3)$.

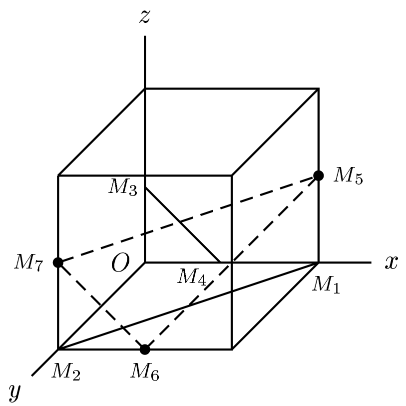

FIGURE 12

3. In [G 75a], [G 76] (see also [G 88, Section 4.4]) Goldberg considered different kinds of $(2n+2)$-hedrality of webs $W(n+1,n,r)$, found closure conditions in a coordinate loop $L(a_1,\dots,a_n)$, and an invariant characterization for them, and relations among them as well as their relations with group webs $W(n+1,n,r)$.

In the case $n=3$, $(2n+2)$-hedral webs become *octahedral*. They are characterized by any of the following conditions:

$$f(x_2,y_1,z_1)=f(x_1,y_2,z_1)=f(x_1,y_1,z_2) \Longrightarrow f(x_1,y_2,z_2)=f(x_2,y_1,z_2)=f(x_2,y_2,z_1), \tag{73}$$

$$F(x,x,e)=F(x,e,x)=F(e,x,x) \tag{74}$$

(see [Ra 60]). The invariant characterization for octahedral webs $W(4,3,r)$ was found in [G 75a] (see also [G 88, Section 4.4]).

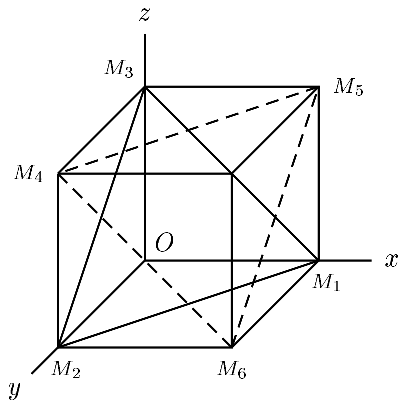

FIGURE 13

Geometrically, condition (73) expresses the fact that if the points O, M_1, M_2 lie on a leaf of the foliation λ_4, then the points M_3, M_4, M_5 also lie on a leaf of the foliation λ_4 (see Figure 13). The points O, M_1, M_2, M_3, M_4, and M_5 are the vertices of an octahedron formed by leaves of the 4-web. For this reason a 4-web satisfying condition (73) is called octahedral. Thus condition (73) means that if seven triples of six points O, M_1, M_2, M_3, M_4, and M_5 lie on leaves of the 4-web, then the eighth triple of points also lie on a leaf of the 4-web.

5. For parallelizable 4-webs we have the following characterizations:

$$f(x_2,y_1,z_1)=f(x_1,y_2,z_1),\quad f(x_3,y_1,z_1)=f(x_1,y_3,z_1)=f(x_1,y_1,z_3) \Longrightarrow f(x_2,y_3,z_1)=f(x_3,y_2,z_1)=f(x_2,y_1,z_3)=f(x_1,y_2,z_3), \tag{75}$$

$$F(x,y,e)=F(y,e,x)=F(e,x,y), \tag{76}$$

$$f(x,y,z)=x\circ y\circ z \tag{77}$$

(see [Ra 60]), where the operation $\circ$ is the operation of a commutative group. For parallelizable webs $W(4,3,r)$, the invariant characterization were also obtained in [G 75a] (see also [G 88, Section 4.3]).

The geometric meaning of conditions (75) is evident from Figure 14: if the points M_1, M_2 and M_3, M_4, M_5 lie on leaves of the foliation λ_4, then the points M_6, M_7, M_8, and M_9 also lie on a leaf of the foliation λ_4.

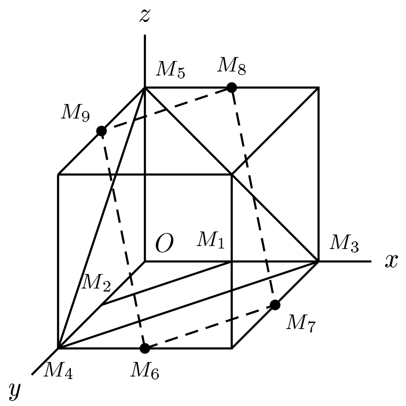

FIGURE 14

11. Webs $W(4,2,r)$ and a pair of orthogonal quasigroups

1. Consider also a 4-web $W(4,2,r)$ on a manifold X^{2r} (see [G 77], [G 80], [G 82a] or [G 88, Chapter 7]). Such a web contains 4 3-subwebs formed by any 3 of its 4 foliations. Denote by $[\xi_1, \xi_2, \xi_3]$ the subweb formed by the foliations λ_{ξ_1}, λ_{ξ_2}, and λ_{ξ_3}. Two binary quasigroups are connected with such a 4-web. In fact, the surfaces $x^j = a^j$ of the foliation λ_1 and the surfaces $y^k = b^k$ of the foliation λ_2 (a^j and b^k are constants) define the point (a^j, b^k) through which one leaf of λ_3 and one leaf of λ_4 pass. Hence two binary quasigroups A and B are defined:

$$z^i = A^i(x^j, y^k), \tag{78}$$

and

$$u^i = B^i(x^j, y^k). \tag{79}$$

These quasigroups are *orthogonal* (see [Be 71, Section 6.1]). This means that the system of equations $A^i(x^j, y^k) = \alpha^i$ and $B^i(x^j, y^k) = \beta^i$ has a unique solution. The constant parameters of the leaves of the foliations λ_1 and λ_2 through the point of intersection of the leaves $z^i = \alpha^i$ and $u^j = \beta^j$ of the foliations λ_3 and λ_4 produce this solution.

Goldberg [G 82a] (see also [G 88, Section 7.3]) proved that applying certain admissible transformations, one can make the quasigroup A to be a loop, and the unit of this loop will be a left unit of the quasigroup B.

2. Next we consider special classes of webs $W(4,2,r)$. There are 4 classes of webs $W(4,2,r)$ satisfying one of Desargues closure conditions D_ξ, 6 classes satisfying one of Desargues closure conditions $D_{\xi\eta}$, and 12 classes satisfying one of triangle conditions (Δ). Webs satisfying each of these conditions were studied by Goldberg [G 82a] (see also [G 88, Section 7.4]). We consider here a class from each group and a class of group 4-webs which is connected with the first group of 4-webs.

(D_1)

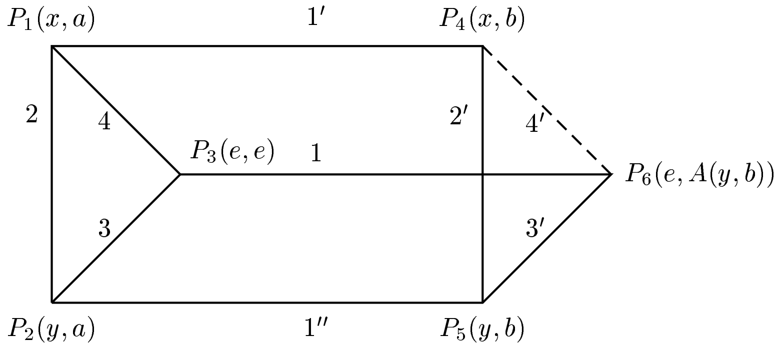

FIGURE 15

We shall say that on a web $W(4,2,r)$ the *Desargues closure condition* D_1 is realized if at any point P_3 of a web domain, for any leaves $2, 2' \in \lambda_2$, the figure represented in Figure 15 is closed.

If A is chosen as a loop with the unit e and B is a quasigroup having e as its left unit, then the following conditional identity corresponds to Figure 15:

$$A(y,a)=0,\ \ B(x,a)=0 \Longrightarrow B(x,b)=A(y,b). \tag{80}$$

(D_{12})

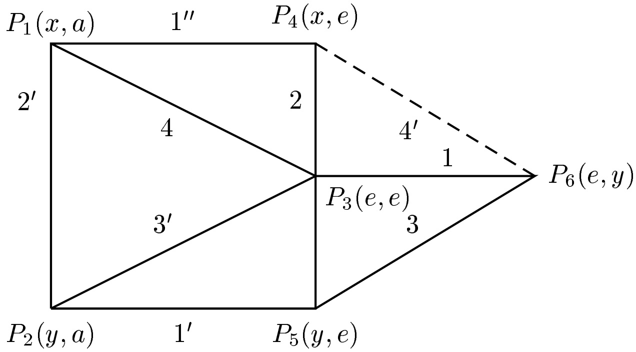

FIGURE 16

We shall say that on a web $W(4,2,r)$ the *Desargues closure condition* D_{12} is realized if at any point P_3 of a web domain, for any leaf $2' \in \lambda_2$, the figure represented in Figure 16 is closed.

The following conditional identity corresponds to Figure 16:

$$A(y,a)=0,\ \ B(x,a)=0 \Longrightarrow B(x,0)=y. \tag{81}$$

Note that if on a web $W(4,2,r)$ the figures D_{12} are closed, then the 3-subwebs $[1,2,3]$ and $[1,2,4]$ are Bol webs (B_r).

We shall say that on a web $W(4,2,r)$ the *triangle closure condition* (Δ) : $\begin{pmatrix} 1\ 2 \mid 3 \\ 1\ 2 \mid 4 \end{pmatrix}$ is realized if at any point P_3 of a web domain the figure represented in Figure 17 is closed. The following conditional identity corresponds to Figure 17:

$$B(x,x)=0. \tag{82}$$

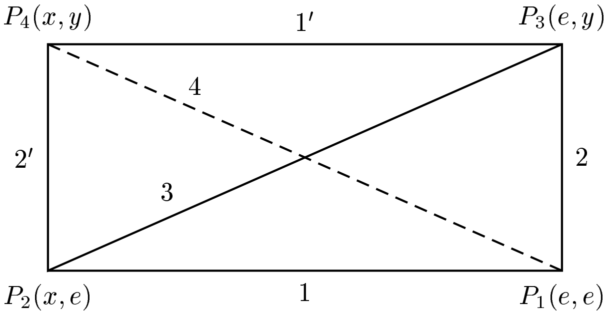

FIGURE 17

In Figures 15–17 segments represent the leaves of the corresponding foliations of the web, and coordinates are introduced in such a way that the vertices have the indicated coordinates. In particular, in Figures 15 and 16, P_3 has coordinates (e,e).

In [G 77], [G 80] (see also [G 88, Section 7.2]) Goldberg introduced and studied *group* 4-*webs* $W(4,2,r)$. They can be defined as 4-*webs for which all* 3-*subwebs are group webs and the basis affinor is covariantly constant on the whole web.* The closure conditions D_1 and D_2 allow him (see [G 82a] or [G 88, Section 7.4]) to give the following characterization of group 4-webs: *A web* $W(4,2,r)$ *is a group* 4-*web if and only if both the figures* D_1 *and* D_2 *are closed.*

In conclusion we will give the identities in the pairs of orthogonal quasigroups A and B which are equivalent to the closure conditions D_ξ, $D_{\xi\eta}$, and (Δ).

To this end, we denote by $\cdot$ and $+$ the operations in A and B and by ${}^{-1}a$, a^{-1}, and $-a$, $a-$ the left and right inverse elements in A and B, respectively. Then using (80)–(82), and similar conditional identities, we obtain the identities mentioned above.

Recall that e is the unit of the loop A and the left unit of B, that is, $a\cdot e = e\cdot a = a$, $e+a=a$.

We will list now some of these identities (see [G 82a] or [G 88, Section 7.4], where a complete list of these identities is given):

$$
\begin{array}{lrl}
(D_1) & -a+b & = {}^{-1}a \cdot b, \\
(D_2) & b+(a-) & = b \cdot a^{-1} + e, \\
(D_3) & b + {}^{-1}b \cdot (a \cdot b^{-1} + e) & = a + b^{-1}, \\
(D_{12}) & {}^{-1}a & = -a + e, \\
(D_{21}) & (a-) & = a^{-1} + e, \\
(D_{24}) & -(a^{-1}) + (a-) & = [-(a^{-1})] \cdot a^{-1} + e, \\
\begin{pmatrix} 1 & 2 & | & 3 \\ 1 & 2 & | & 4 \end{pmatrix} & a+a & = e, \\
\begin{pmatrix} 1 & 2 & | & 4 \\ 1 & 2 & | & 3 \end{pmatrix} & {}^{-1}a & = a + e, \\
\begin{pmatrix} 3 & 4 & | & 1 \\ 3 & 4 & | & 2 \end{pmatrix} & (-a) \cdot a^{-1} & = {}^{-1}a + a, \\
\begin{pmatrix} 2 & 3 & | & 1 \\ 2 & 3 & | & 4 \end{pmatrix} & a^{-1} & = a \cdot (a-), \\
\begin{pmatrix} 1 & 4 & | & 2 \\ 1 & 4 & | & 3 \end{pmatrix} & {}^{-1}a + a & = a + e.
\end{array}
\tag{83}
$$

References

[Ac 65] Aczel J., *Quasigroups, nets and nomograms*, Adv. Math. **1** (1965), no. 3, 383–450.

[A 69a] Akivis M.A., *The canonical expansions of the equations of a local analytic quasigroup*, Dokl. Akad. Nauk SSSR **188** (1969), no. 5, 967–970 (Russian); English transl.: Soviet Math. Dokl. **10** (1969), no. 5, 1200–1203.

[A 69b] Akivis M.A., *Three-webs of multidimensional surfaces*, Trudy Geometr. Sem. **2** (1969), 7–31 (Russian).

[A 73a] Akivis M.A., *The local differentiable quasigroups and three-webs that are determined by a triple of hypersurfaces*, Sibirsk. Mat. Zh. **14** (1973), no. 3, 467–474 (Russian); English transl.: Siberian Math. J. **14** (1973), no. 3, 319–324.

[A 73b] Akivis M.A., *Local differentiable quasigroups and three-webs of multidimensional surfaces*, in "Studies in the Theory of Quasigroups and Loops", pp. 3–12 (Russian), Izdat. "Shtiintsa", Kishinev, 1973.

[A 75] Akivis M.A., *Closed G-structures on a differentiable manifold*, Problems in Geometry, Vol. **7**, 69–79 (Russian), Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1975.

[A 76] Akivis M.A., *The local algebras of a multidimensional three-web*, Sibirsk. Mat. Zh. **17** (1976), no. 1, 5–11 (Russian); English transl.: Siberian Math. J. **17** (1976), no. 1, 3–8.

[A 78] Akivis M.A., *Geodesic loops and local triple systems in a space with an affine connection*, Sibirsk. Mat. Zh. **19** (1978), no. 2, 243–253 (Russian); English transl.: Siberian Math. J. **19** (1978), no. 2, 171–178.

[AG 74] Akivis M.A., Goldberg V.V., *The 4-web and the local differentiable ternary quasigroup that are determined by a quadruple of surfaces of codimension two*, Izv. Vyssh. Uchebn. Zaved. Mat. **1974**, no. 5(144), 12–24 (Russian); English transl.: Soviet Math. (Iz. VUZ) **18** (1974), no. 5, 9–19.

[AG 00] Akivis M.A., Goldberg V.V., *Differential geometry of webs*, Chapter 1 in "Handbook of Differential Geometry", Vol. 1, Elsevier, 2000, pp. 1–152.

[AS 71] Akivis M.A., Shelekhov A.M., *Local differentiable quasigroups and connections that are associated with a three-web of multidimensional surfaces*, Sibirsk. Mat. Zh. **12** (1971), no. 6, 1181–1191 (Russian); English transl.: Siberian Math. J. **12** (1971), no. 6, 845–852.

[AS 92] Akivis M.A., Shelekhov A.M., *Geometry and Algebra of Multidimensional Three-Webs*, translated from the Russian by V.V. Goldberg, Kluwer Academic Publishers, Dordrecht, 1992, xvii+358 pp.

[Ba 90] Barlotti A., *Geometry of quasigroups*, Chapter VIII in "Quasigroups and Loops: Theory and Applications", O. Chein, H.O. Pflugfelder, and J.D.H. Smith, Eds., Heldermann-Verlag, Berlin, 1990, pp. 197–203.

[BS 83] Barlotti A., Strambach K., *The geometry of binary systems*, Adv. Math. **49** (1983), no. 1, 1–105.

[Be 67] Belousov V.D., *Foundations of the Theory of Quasigroups and Loops*, Izdat. "Nauka", Moscow, 1967, 223 pp. (Russian).

[Be 71] Belousov V.D., *Algebraic Nets and Quasigroups*, Izdat. "Shtiintsa", Kishinev, 1971, 165 pp. (Russian).

[Be 72] Belousov V.D., *n-ary Quasigroups*, Izdat. "Shtiintsa", Kishinev, 1972, 227 pp. (Russian).

[BSa 66] Belousov V.D., Sandik M.D., *n-ary quasigroups and loops*, Sibirsk. Mat. Zh. **7** (1966), no. 1, 31–54 (Russian); English transl.: Siberian Math. J. **7** (1966), no. 1, 24–42.

[Bl 28] Blaschke W., *Thomsens Sechseckgewebe. Zueinander diagonale Netze*, Math. Z. **28** (1928), 150–157.

[Bl 47] Blaschke W., *Projective Geometrie*, Wolfenbüttel, Hannover, 1947, 160 pp.; 2nd ed., Wolfenbüttel, Hannover, 1948, 160 pp.; 3rd ed., Birkhäuser, Basel-Stuttgart, 1954, 197 pp.

[Bl 55] Blaschke W., *Einführung in die Geometrie der Waben*, Birkhäuser-Verlag, Basel-Stuttgart, 1955, 108 pp.; Russian transl.: GITTL, Moskva, 1959, 144 pp.

[BB 38] Blaschke W., Bol G., *Geometrie der Gewebe*, Springer-Verlag, Berlin, 1938, viii+339 pp.

[B 35] Bol G., *Über 3-Gewebe in vierdimensionalen Raum*, Math. Ann. **110** (1935), 431–463.

[B 37] Bol G., *Gewebe und Gruppen*, Math. Ann. **114** (1937), 414–431.

[C 36] Chern S.S., *Eine Invariantentheorie der Dreigewebe aus r-dimensionalen Mannigfaltigkeiten in* $\mathbf{R}_{2r}$, Abh. Math. Sem. Univ. Hamburg **11** (1936), no. 1–2, 333–358.

[F 78] Fedorova V.I., *A condition defining multidimensional Bol's three-webs*, Sibirsk. Mat. Zh. **19** (1978), no. 4, 922–928 (Russian); English transl.: Siberian Math. J. **19** (1978), no. 4, 657–661.

[G 73] Goldberg V.V., *$(n+1)$-webs of multidimensional surfaces*, Dokl. Akad. Nauk SSSR **210** (1973), no. 4, 756–759 (Russian); English transl.: Soviet Math. Dokl. **14** (1973), no. 3, 795–799.

[G 75a] Goldberg V.V., *An invariant characterization of certain closure conditions in ternary quasigroups*, Sibirsk. Mat. Zh. **16** (1975), no. 1, 29–43 (Russian); English transl.: Siberian Math. J. **16** (1975), no. 1, 23–34.

[G 75b] Goldberg V.V., *Local ternary quasigroups that are connected with a four-web of multidimensional surfaces*, Sibirsk. Mat. Zh. **16** (1975), no. 2, 247–263 (Russian); English transl.: Siberian Math. J. **16** (1975), no. 2, 190–202.

[G 75c] Goldberg V.V., *The $(n+1)$-webs defined by $n+1$ surfaces of codimension $n-1$*, Probl. Geom., Vol. 7 (Russian), Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1975, pp. 173–195.

[G 76] Goldberg V.V., *Reducible, group, and $(2n+2)$-hedral $(n+1)$-webs of multidimensional surfaces*, Sibirsk. Mat. Zh. **17** (1976), no. 1, 44–57 (Russian); English transl.: Siberian Math. J. **17** (1976), no. 1, 34–44.

[G 77] Goldberg V.V., *On the theory of four-webs of multidimensional surfaces on a differentiable manifold* X_{2r}, Izv. Vyssh. Uchebn. Zaved. Mat. **1977**, no. 11(186), 15–22 (Russian); English transl.: Soviet Math. (Iz. VUZ) **21** (1977), no. 11, 97–100.

[G 80] Goldberg V.V., *On the theory of four-webs of multidimensional surfaces on a differentiable manifold* X_{2r}, Serdica **6** (1980), no. 2, 105–119 (Russian).

[G 82a] Goldberg V.V., *Multidimensional four-webs on which the Desargues and triangle figures are closed*, Geom. Dedicata **12** (1982), no. 3, 267–285.

[G 82b] Goldberg V.V., *Grassmann and algebraic four-webs in a projective space*, Tensor (N.S.) **38** (1982), 179–197.

[G 88] Goldberg V.V., *Theory of Multicodimensional* $(n + 1)$*-Webs*, Kluwer Academic Publishers, Dordrecht, 1988, xxii+466 pp.

[G 90] Goldberg V.V., *Local differentiable quasigroups and webs*, Chapter X in "Quasigroups and Loops: Theory and Applications" O. Chein, H.O. Pflugfelder, and J.D.H. Smith, Eds., Heldermann-Verlag, Berlin, 1990, pp. 263–311.

[Go 99] Goursat E., *Sur les équations du second ordre à n variables, analogues à l'équation de Monge–Ampère*, Bull. Soc. Math. France **27** (1899), 1–34.

[HS 86a] Hofmann K.H., Strambach K., *Lie's fundamental theorems for local analytical loops*, Pacific J. Math. **123** (1986), no. 2, 301–327.

[HS 86b] Hofmann K.H., Strambach K., *The Akivis algebra of a homogeneous loop*, Mathematika **33** (1986), no. 1, 87–95.

[HS 90] Hofmann K.H., Strambach K., *Topological and analytic loops*, Chapter IX in "Quasigroups and Loops: Theory and Applications", O. Chein, H.O. Pflugfelder, and J.D.H. Smith, Eds., Heldermann-Verlag, Berlin, 1990, pp. 205–262.

[Kn 32] Kneser H., *Gewebe und Gruppen*, Abh. Math. Sem. Univ. Hamburg **9** (1932), 147–151.

[Ku 70] Kuz'min E.N., *La relation entre les algebras de Malcev et les boucles de Moufang analytiques*, C.R. Acad. Sci. Paris Sér. A **271** (1970), no. 23, 1152–1155.

[Ku 71] Kuz'min E.N., *On relation between Mal'tsev algebras and analytic Moufang loops*, Algebra i Logika **10** (1971), no. 1, 3–22 (Russian); English transl.: Algebra and Logic **10** (1971), no. 1, 1–14.

[Lie 93] Lie S., *Vorlesungen über kontinuierliche Gruppen mit geometrischen und anderen Anwendungen*, Teubner, Leipzig, 1893, xii+810 pp.; reprint Chelsea Publishing Co., Bronx, N.Y., 1971, xii+810 pp.

[Ma 55] Mal'cev A.I., *Analytical loops*, Mat. Sb. **36** (1955), 569–576 (Russian).

[Mi 96] Mikheev P.O., *On a problem of Chern-Akivis-Shelekhov on hexagonal three-webs*, Aequationes Math. **51** (1996), no. 1–2, 1–11.

[MS 90] Mikheev P.O., Sabinin L.V., *Quasigroups and differential geometry*, Chapter XII in "Quasigroups and Loops: Theory and Applications", O. Chein, H.O. Pflugfelder, and J.D.H. Smith, Eds., Heldermann-Verlag, Berlin, 1990, pp. 357–430.

[Mo 35] Moufang R., *Zur Struktur von Alternativ Körpern*, Math. Ann. **110** (1935), 416–430.

[NS 94] Nagy P., Strambach K., *Loops as invariant sections in groups, and their geometry*, Canad. J. Math. **46** (1994), no. 5, 1027–1056.

[P 75] Pickert G., *Projective Ebenen*, 2nd edition, Springer-Verlag, Berlin, 1975, ix+371 pp.

[Ra 60] Rado F., *Generalisation of space webs for certain algebraic structures*, Studia Univ. Babeş-Bolyai Ser. Math. Phys. **1960**, no. 1, pp. 41–55 (Roumanian).

[Re 28] Reidemeister K., *Gewebe und Gruppen*, Math. Z. **29** (1928), 427–435.

[Re 68] Reidemeister K., *Vorlesungen Über Grundlagen der Geometrie*, Springer-Verlag, Berlin-New York, 1968, x+148 pp.

[Sab 98] Sabinin L.V., *Quasigroups, geometry and nonlinear geometric algebra*, Acta Appl. Math. **50** (1998), no. 1–2, 45–66.

[SM 82] Sabinin L.V., Mikheev P.O., *Analytical Bol loops*, Webs and Quasigroups (Russian), Kalinin. Gos. Univ., Kalinin, 1982, pp. 102–109.

[SM 85] Sabinin L.V., Mikheev P.O., *The Theory of Smooth Bol Loops*, Univ. Druzhby Narodov, Moscow, 1985, 81 pp.

[Sag 61] Sagle A.A., *Mal'tsev algebras*, Trans. Amer. Math. Soc. **101** (1961), no. 3, 426–458.

[San 65] Sandik M.D., *Completely reducible n-quasigroups*, Izv. Akad. Nauk Moldav. SSR Ser. Fiz.-Tekhn. Mat. Nauk **7** (1965), 55–67 (Russian).

[S 89a] Shelekhov A.M., *Classification of multidimensional three-webs according to closure conditions*, Probl. Geom. **21**, 109–154, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989 (Russian); English transl.: Soviet Math. **55** (1991), no. 6, 2140–2168.

[S 89b] Shelekhov A.M., *The G-structures associated to a hexagonal 3-web is closed*, J. Geom. **35** (1989), 169–176.

[S 89c] Shelekhov A.M., *On the closed G-structure associated to a hexagonal 3-web*, in "Differential Geometry and Its Applications", Dubrovnik, 1988, Univ. Novi Sad, Novi Sad, 1989, pp. 323–326.

[Sh 99a] Shestakov I.P., *Any Akivis algebra is linear*, Geom. Dedicata **77** (1999), no. 2, 215–223.

[Sh 99b] Shestakov I.P., *Linear representability of Akivis algebras*, Dokl. Akad. Nauk **368** (1999), no. 1, 21–23.

[Sh ta] Shestakov I.P., *Speciality and deformation of algebras*, Proceedings of the Intern. Algebraic Conf. dedicated to the memory of A.G. Kurosh, Moscow, 1998, to appear.

[Su 37] Sushkevich A.K., *Theory of Generalized Groups*, GNTI, Kharkiv–Kiev, 1937, 176 pp.

[Sm 88] Smith J.D.H., *Multilinear algebras and Lie's theorem for formal n-loops*, Arch. Math. (Basel) **51** (1988), 169–177.

[T 27] Thomsen G., *Un teoreme topologico sulle schiere di curve e una caratterizzazione geometrica delle superficie isotermo-asintotiche*, Boll. Un. Mat. Ital. Bologna **6** (1927), 80–85.

Department of Mathematics, Jerusalem College of Technology—Mahon Lev, Havaad Haleumi St., P.O.Box 16031, Jerusalem 91160, Israel

E-mail: akivis@avoda.jct.ac.il

Department of Mathematical Sciences, New Jersey Institute of Technology, University Heights, Newark, N.J. 07102, USA

E-mail: vlgold@m.njit.edu

(*Received* October 15, 1999)