A-loops close to code loops are groups

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Abstract. Let Q be a diassociative A-loop which is centrally nilpotent of class 2 and which is not a group. Then the factor over the centre cannot be an elementary abelian 2-group.

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This modest note concerns diassociative A-loops that are centrally nilpotent of class 2. Its basic result can be expressed in the following way:

Proposition 1. *Suppose that Q is a diassociative A-loop with a central subloop N. If Q/N is a group of exponent 2, then Q is a group.*

The result is an offshoot of my interest in code loops and I hope that it will help to stimulate further research in the indicated direction.

It can be also understood in the context of the *Osborn problem*: Decide if every (finite) diassociative A-loop is a Moufang loop.

Osborn [6] solved this problem affirmatively for commutative, diassociative A-loops, and some progress in the general case has been recently reported by J.D. Phillips [7].

A subloop N of a loop Q is *central*, if all its elements associate and commute with all elements of Q. Suppose that $Q/N \simeq V$ and that N and V are fixed. The group N is abelian by definition, and we shall assume that V is an abelian group as well (though some of our initial observations can be easily generalized to the case of non-abelian groups). The loop Q is obviously isomorphic to one of the loops $Q(\vartheta)$, where $\vartheta : V \times V \to N$ is a mapping with $\vartheta(u, 0) = \vartheta(0, u) = 0$ for all $u \in V$, and where the binary operation of $Q(\vartheta)$ is defined by

$$(a, u) \cdot (b, v) = (a + b + \vartheta(u, v), u + v),$$

for all $a, b \in N$ and $u, v \in V$.

A loop Q is said to be a *code loop* ([5], [4], [1]), if $|N| = 2$ and V is an elementary abelian 2-group. Statements about code loops are often proved by computations in $Q(\vartheta)$, and this approach will be used also here. (It is clear that one could

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construct a proof which would not resort to ϑ-techniques. However, it is not clear if such proofs would be more transparent or more readable. Description of extensions by means of factor systems ϑ is for loops and quasigroups very natural and has been used many times, starting from [2].)

For a loop Q and $\alpha \in Q$ denote by L_α the left translation $\beta \mapsto \alpha\beta$, $\beta \in Q$, and by R_α the right translation $\beta \mapsto \beta\alpha$, $\beta \in Q$. Call Q a *left* (or *right*, or *middle*) *A-loop*, if $L_{\alpha\beta}^{-1}L_\alpha L_\beta$ (or $R_{\beta\alpha}^{-1}R_\alpha R_\beta$, or $R_\alpha^{-1}L_\alpha$) is an automorphism for all $\alpha, \beta \in Q$, respectively. A loop Q is an *A-loop* if it is simultaneously a left A-loop, a right A-loop and a middle A-loop. The primary source for A-loops is [3].

Fix now $\vartheta : V \times V \to N$, $\vartheta(u,0) = \vartheta(0,u) = 0$ for all $u \in V$, and for all $u, v, w \in V$ put

$$C(u,v) = \vartheta(u,v) - \vartheta(v,u), \quad \text{and}$$

$$A(u,v,w) = -\vartheta(u,v) + \vartheta(u,v+w) - \vartheta(u+v,w) + \vartheta(v,w).$$

Suppose that $Q = Q(\vartheta)$, $\alpha = (a,u)$ and $\beta = (b,v)$. Then $L_\alpha L_\beta$ sends (c,x) to $(a+b+c+\vartheta(u,v+x)+\vartheta(v,x), u+v+x)$, and $L_{\alpha\beta}$ sends (c,x) to $(a+b+c+\vartheta(u,v)+\vartheta(u+v,x), u+v+x)$. Hence $L_\alpha L_\beta$ equals $L_{\alpha\beta}$ if and only if $A(u,v,x) = 0$ for all $x \in V$, and we see that $Q(\vartheta)$ is a group if and only if $A(u,v,w) = 0$ for all $u, v, w \in V$.

It is now also clear that $L_{\alpha\beta}^{-1}L_\alpha L_\beta(c,x)$ equals $(A(u,v,x)+c, x)$ for all $(c,x) \in Q$. Put $\varphi = L_{\alpha\beta}^{-1}L_\alpha L_\beta$ and consider $\gamma = (c,x) \in Q$ and $\delta = (d,y) \in Q$. Then $\varphi(\gamma)\varphi(\delta)$ equals $(A(u,v,x) + A(u,v,y) + c + d + \vartheta(x,y), x+y)$, while $\varphi(\gamma\delta)$ is equal to $(A(u,v,x+y) + c + d + \vartheta(x,y), x+y)$. Comparison of the both results yields claim (i) of the following lemma. Claims (ii) and (iii) can be obtained in a similar way.

Lemma 1. (i) *$Q(\vartheta)$ is a left A-loop if $A(u,v,x+y) = A(u,v,x) + A(u,v,y)$ for all $x, y, u, v \in V$;*

(ii) *$Q(\vartheta)$ is a right A-loop if $A(x+y,v,u) = A(x,v,u) + A(y,v,u)$ for all $x, y, u, v \in V$;*

(iii) *$Q(\vartheta)$ is a middle A-loop if $C(u,x+y) = C(u,x)+C(u,y)$ for all $x, y, u \in V$.*

The loop $Q(\vartheta)$ is a group, if $A(u,v,w) = 0$ for all $u, v, w \in V$. If $Q(\vartheta)$ is a left A-loop, then each pair $u, v \in V$ determines, by Lemma 1 (i), a group homomorphism $V \to N$. All such homomorphisms are trivial if and only if the loop $Q(\vartheta)$ is a group. Since finite groups of coprime order admit only trivial homomorphisms, Lemma 1 yields the following corollary:

Corollary 1. *Let Q be a finite centrally nilpotent loop of class 2 which is not a group, and let Z be its centre. If $|Z|$ and $|Q/Z|$ are coprime, then Q is neither a left A-loop, nor a right A-loop. Furthermore, if $|Z|$ and $|Q/Z|$ are coprime, then Q is a middle A-loop if and only if it is commutative.*

It will be useful to express the equation of Lemma 1 (i) just in terms of ϑ. We obtain the equality:

$$\begin{aligned}(1)\qquad & \vartheta(u,v)+\vartheta(u,v+x+y)-\vartheta(u+v,x+y)+\vartheta(v,x+y)\\ &\quad -\vartheta(u,v+x)+\vartheta(u+v,x)-\vartheta(v,x)\\ &\quad -\vartheta(u,v+y)+\vartheta(u+v,y)-\vartheta(v,y)=0.\end{aligned}$$

Lemma 2. *Suppose that V is of exponent 2. Then $Q(\vartheta)$ is diassociative if and only if*

$$\vartheta(u,u+v)=\vartheta(u,u)-\vartheta(u,v),\ \vartheta(u+v,u)=\vartheta(u,u)-\vartheta(v,u),\ \textit{and}$$
$$2\vartheta(v,u)=2\vartheta(u,v)$$

are true for all $u,v\in U$.

Proof: Consider $u,v\in V$. By our assumption, $2u=2v=0$, and hence the equations $A(u,u,v)=0$ and $A(v,u,u)=0$ are equivalent to equations $\vartheta(u,u)-\vartheta(u,u+v)-\vartheta(u,v)=0$ and $-\vartheta(v,u)-\vartheta(v+u,u)+\vartheta(u,u)=0$, respectively. If these equalities are true, then $A(u,v,u)=0$ is equivalent to $2\vartheta(v,u)=2\vartheta(u,v)$, as $A(u,v,u)=-\vartheta(u,v)+\vartheta(u,v+u)-\vartheta(u+v,u)+\vartheta(v,u)=(\vartheta(v,u)-\vartheta(u,v))+(\vartheta(u,u)-\vartheta(u,v)-\vartheta(u,u)+\vartheta(v,u))=2\vartheta(v,u)-2\vartheta(u,v))$.

It is now clear that the direct implication of the lemma holds. To prove the converse implication means to show that the ϑ-equalities imply $A(u,v,w)=0$ whenever u, v and w belong to a subgroup of V with ≤ 4 elements. One clearly has $A(u,v,w)=0$ if $0\in\{u,v,w\}$. From the first part of the proof one gets the cases $u=v$, $v=w$ and $u=w$, and so the only remaining case is the case when 0, u, v and w are pairwise different. However, the assumed equalities yield $A(u,u+v,v)=-\vartheta(u,u+v)+\vartheta(u,u)-\vartheta(v,v)+\vartheta(u+v,v)=\vartheta(u,v)-\vartheta(u,v)=0$. $\square$

We are now ready to prove Proposition 1. Assume that V is of exponent 2 and that $Q(\vartheta)$ is a diassociative A-loop.

Consider (1) for the case $v+x+y=0$, and note that the first row of (1) then yields $\vartheta(u,v)-\vartheta(u+v,v)+\vartheta(v,v)$, which is, by Lemma 2, equal to $2\vartheta(u,v)=2\vartheta(u,x+y)$. Furthermore, $\vartheta(u,v+x)+\vartheta(u,v+y)$ equals $\vartheta(u,x)+\vartheta(u,y)$, and $\vartheta(v,x)+\vartheta(v,y)-\vartheta(u+v,x)-\vartheta(u+v,y)$ equals $\vartheta(x+y,x)+\vartheta(x+y,y)-\vartheta(x+y+u,x)-\vartheta(y+x+u,y)=\vartheta(x,x)-\vartheta(y,x)+\vartheta(y,y)-\vartheta(x,y)-\vartheta(x,x)+\vartheta(y+u,x)-\vartheta(y,y)+\vartheta(x+u,y)=-\vartheta(y,x)-\vartheta(x,y)+\vartheta(y+u,x)+\vartheta(x+u,y)$. Hence

$$\begin{aligned}(2)\qquad 2\vartheta(u,x+y) &=\\ &\vartheta(u,x)-\vartheta(y,x)+\vartheta(u,y)-\vartheta(x,y)+\vartheta(y+u,x)+\vartheta(x+u,y)\end{aligned}$$

holds for all $x,y,u\in V$.

By Lemma 1 (iii), $C(x,u+y)$ equals $C(x,u)+C(x,y)$, and hence $\vartheta(y+u,x)=\vartheta(x,u+y)+\vartheta(u,x)-\vartheta(x,u)+\vartheta(y,x)-\vartheta(x,y)$. Substitute now this expression of

$\vartheta(y+u,x)$ to (2). One gets $2\vartheta(u,x+y) = 2\vartheta(u,x) + \vartheta(u,y) - 2\vartheta(x,y) + \vartheta(x,u+y) + \vartheta(x+u,y) - \vartheta(x,u)$, and so we see that

$$
\begin{aligned}
(3) \qquad 2(-\vartheta(u,x) + \vartheta(u,x+y) + \vartheta(x,y)) \; &= \\
-\vartheta(x,u) + \vartheta(x,u+y) + \vartheta(x+u,y) + \vartheta(u,y)&
\end{aligned}
$$

holds for all $u,x,y \in V$.

By adding $-2\vartheta(x+u,y)$ to the both sides of (3), we obtain

$$2A(u,x,y) = A(x,u,y) \quad \text{for all} \quad x,y,u \in V.$$

However, $2A(u,x,y) = A(u,x,y) + A(u,x,y) = A(u,x,y+y) = A(u,x,0) = 0$ by Lemma 1(i), and so we see that $A(x,u,y)$ is always zero, and that is what was needed to prove Proposition 1.

Note that our proof did not use the fact that Q is a right A-loop. In our last statement we give an explanation.

Proposition 2. *Let Q be a loop that is centrally nilpotent of class 2. If Q is simultaneously a left A-loop and a middle A-loop, then it is also a right A-loop.*

Proof: One can obviously assume that Q equals $Q(\vartheta)$. When the equality $A(x+y,v,u) - A(x,v,u) - A(y,v,u) = 0$ is expressed in terms of ϑ, one gets

$$
\begin{aligned}
(4) \qquad -\vartheta(v,u) - \vartheta(x+y+v,u) + \vartheta(x+y,u+v) - \vartheta(x+y,v)& \\
+\vartheta(x+v,u) - \vartheta(x,u+v) + \vartheta(x,v)& \\
+\vartheta(y+v,u) - \vartheta(y,u+v) + \vartheta(y,v) &= 0.
\end{aligned}
$$

Summing up (1) and (4) one obtains

$$
\begin{aligned}
(5) \qquad C(u,v) + C(u,v+x+y) - C(u+v,x+y) + C(v,x+y)& \\
-C(u,v+x) + C(u+v,x) - C(v,x)& \\
-C(u,v+y) + C(u+v,y) - C(v,y) &= 0.
\end{aligned}
$$

Since $C(x,u+v) = C(x,u) + C(x,v)$ and $C(u+v,x) = C(u,x) + C(v,x)$ are true, by Lemma 1(iii), for all $x,u,v \in V$, we see that (5) is satisfied. Since (1) holds by Lemma 1 (i), the equality (4) must be also true (for all $x,y,u,v \in V$), and hence Q is a right A-loop, by Lemma 1 (ii). □

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References

[1] Aschbacher M., *Sporadic Groups*, Cambridge Tracts in Mathematics **104**, Cambridge University Press, 1994.
[2] Bruck R.H., *Contributions to the theory of loops*, Trans. Amer. Math. Soc. **60** (1946), 245–354.
[3] Bruck R.H., Paige L.J., *Loops whose inner mappings are automorphisms*, Ann. of Math. **63** (1954), 308–323.
[4] Chein O., Goodaire E.G., *Moufang loops with a unique non-identity commutator* (*associator, square*), J. Algebra **130** (1990), 369–384.
[5] Griess R.L., Jr., *Code loops*, J. Algebra **100** (1986), 224–234.
[6] Osborn M.J., *A theorem on A-loops*, Proc. Amer. Math. Soc. **9** (1958), 347–349.
[7] Phillips J.D., *On Moufang A-loops*, Comment. Math. Univ. Carolinae **41** (2000), 371-375.

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