Groups, transversals, and loops

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Abstract. A family of loops is studied, which arises with its binary operation in a natural way from some transversals possessing a "normality condition".

Keywords: loops, groups, transversals

Classification: Primary 20N05; Secondary 20D99

§1. Introduction

The study of loops leads in a natural way to the study of transversals of subgroups, for example the works of: Karzel, Kepka, Kiechle, Kinyon, Niemenmaa, Phillips, Sabinin, Ungar, and et al. [6], [10], [11], [13], [15], [16]. Loops have also played an important role in the study of groups such as in Conway's celebrated construction of the Fischer-Griess monster group using Parker's Moufang loop of order 2^{13}. This led to Griess's construction of binary code loops by a double cover elementary abelian 2-group ([3], [8]). From the above one can see that there is a strong connection between the study of group and loops.

Sabinin has shown that every left loop arises with its binary operation in a natural way from some special transversal of a subgroup in certain groups ([16]). In this paper I will look at a family of loops which arises with its binary operation in a natural way from some transversals which possesses a "normality condition". Subgroups and subsets of groups with normality conditions such as subnormality, seminormality, and etc. have interested me for a long time ([4], [5]).

§2. AP-loop as transversals of groups

Definition 2.1. A *groupoid* ([1]) is a nonempty set with a binary operation. An automorphism of the groupoid $(S, \odot)$ is a bijection of S that respects the binary operation $\odot$ in S. The set of all automorphisms of $(S, \odot)$ forms a group denoted by $Aut(S, \odot)$.

Definition 2.2. A *left loop* is a groupoid $(S, \odot)$ with an identity element in which the equation $a \odot x = b$ possesses a unique solution for the unknown x. $(S, \odot)$ is a *loop* if $y \odot a = b$ also possesses a unique solution.

Definition 2.3. Given a left loop P and an ordered pair $(a,b) \in P \times P$ we get a bijection $\delta_{a,b} : P \to P$ defined by $a \odot (b \odot x) = (a \odot b) \odot \delta_{a,b}x$ for any $x \in P$. Here $\delta_{a,b}$ is a correction to associativity called a *left inner mapping* ([1]). Let $AS_\delta(P)$ be the group generated by all the bijections $\delta_{a,b}$ (Note that $(a \odot b) \odot x = a \odot (b \odot \delta_{a,b}^{-1}x)$).

Note. It is shown in [16] that if L_a for $a \in P$ is the left translation by a, then $\delta_{a,b} = L_{(a\odot b)}^{-1} L_a L_b$ where the product is in the permutation group L_P generated by all left translations.

Definition 2.4. We will call a left loop (or a loop) a *left A_l-loop* (or *A_l-loop* respectively) if for all $(a,b) \in P \times P$, $\delta_{a,b} \in Aut(P, \odot)$.

Definition 2.5. A groupoid $(P, \odot)$ has the *left inverse property* if for each $a \in P$, there is a unique $a^{-1} \in P$ such that $a^{-1} \odot (a \odot b) = b$ for all b in P.

Definition 2.6. A left A_l-loop P (or an A_l-loop) is a *left AP-loop* (or *AP-loop* respectively) if it possesses the left inverse property.

Definitions of two well known AP-loops are presented below:

Definition 2.7 (*A_l-Bol-loop = Gyrogroup* [15], [17])**.** A groupoid $(G, \odot)$ is an *A_l-Bol-loop* if its binary operation satisfies the following axioms. In G there is at least one element, 1, called a left identity, satisfying

(G1) $1 \odot a = a$ Left Identity

for all $a \in G$. There is an element $1 \in G$ satisfying axiom $(G1)$ such that for each a in G there is an x in G, called a left inverse of a, satisfying

(G2) $x \odot a = 1.$ Left Inverse

Moreover, for any $a, b, z \in G$ there exists a unique element $\delta_{a,b}z \in G$ such that

(G3) $a \odot (b \odot z) = (a \odot b) \odot \delta_{a,b}z.$

If $\delta_{a,b}$ denotes the map $\delta_{a,b}\colon G \to G$ given by $z \mapsto \delta_{a,b}z$ then

(G4) $\delta_{a,b} \in Aut(G, \odot),$

(G5) $\delta_{a,b} = \delta_{a\odot b,b}.$ Left Loop Property

Definition 2.8 (*Gyrocommutative Gyrogroup = K-loop = Bruck-loop* [9], [15], [17])**.** The A_l-Bol-loop $(G, \oplus)$ is a *Bruck loop* if for all $a, b \in G$,

(G6) $a \odot b = \delta_{a,b}(b \odot a).$

In case P is a left A_l-loop, since $\delta_{a,b} \in Aut(P, \odot)$, Sabinin's [16] "semidirect product" becomes:

Definition 2.9. Let $P = (P, \odot)$ be a left A_l-loop, and let $AS_\delta(P) \le H \le Aut(P, \odot)$. The *semidirect product group*

$$P \rtimes_l H$$

is the set of ordered pairs (x, X), where $x \in P$ and $X \in H$, with the binary operation given by

$$(x, X)(y, Y) = (x \odot Xy, \delta_{x,Xy} XY).$$

Below is a corollary to Sabinin's Theorem 2 [16] about "semidirect products":

Corollary 2.10. *Let $(P, \odot)$ be a left A_l-loop, and let $AS_\delta(P) \leq H \leq Aut(P, \odot)$. Then $P \rtimes H$ is a group.*

Definition 2.11. A set B is a transversal in a group G (all transversals in this article are left transversals) of a subgroup H of G if every $g \in G$ can be written uniquely as $g = bh$ where $b \in B$ and $h \in H$. Let $b_1, b_2 \in B$ be any two elements of B, and let

$$b_1 b_2 = (b_1 \odot b_2) h(b_1, b_2)$$

be the unique decomposition of the element $b_1 b_2 \in G$, where $b_1 \odot b_2 \in B$ and $h(b_1, b_2) \in H$, determining (i) a binary operation, $\odot$, in B, called the *loop or transversal operation* of B induced by G, and (ii) a map $h\colon B \times B \to H$, called the *transversal map*. The element $h(b_1, b_2) \in H$ is called the element of H determined by the two elements b_1 and b_2 of its transversal B in G. A *transversal groupoid* $(B, \odot)$ of H in G is a groupoid formed by a transversal B of H in G with its transversal operation $\odot$.

Definition 2.12. A transversal groupoid $(B, \odot)$ of a subgroup H in a group G is an *A_l-transversal* of H in G if

(i) $1_G \in B$, 1_G being the identity element of G;
(ii) B is normalized by H, $H \subseteq N_G(B)$, that is, $hBh^{-1} \subseteq B$ for all $h \in H$.

Note. If an A_l-transversal is also a subgroup, then it is a normal subgroup. So we see that an A_l-transversal possesses a "normality condition".

Theorem 2.13. *Let $(B, \odot)$ be an A_l-transversal groupoid of a subgroup H in a group G. Then, for any $a, b, x \in B$, $(a \odot b) \odot \delta_{a,b} x = a \odot (b \odot x)$ and $\delta_{a,b} \in Aut(B, \odot)$.*

Proof: Let $a, b \in B$ be any two elements of B, and let $ab = (a \odot b)h(a, b)$ be the unique decomposition of the element $ab \in G$, where $a \odot b \in B$ and $h(a, b) \in H$. Let $\delta_{a,b} x = x^{h(a,b)} = h(a, b)x(h(a, b))^{-1}$ for all $x \in B$.

For all $a, b, c \in B$ we have in G,

$$(ab)c = a(bc). \tag{2.1}$$

Employing the uniqueness of the decomposition for both sides of (2.1) we have

$$\begin{aligned} (ab)c &= (a \odot b)h(a, b)c \\ &= (a \odot b)\delta_{a,b} c h(a, b) \\ &= ((a \odot b) \odot \delta_{a,b} c) h(a \odot b, \delta_{a,b} c) h(a, b) \end{aligned} \tag{2.2}$$

on one hand, and

$$a(bc) = a(b \odot c)h(b, c) = (a \odot (b \odot c))h(a, b \odot c)h(b, c) \tag{2.3}$$

on the other hand. It follows from (2.1)–(2.3) and from the uniqueness of the decomposition that

$$(a \odot b) \odot \delta_{a,b}c = a \odot (b \odot c).$$

We now have to show that

$$(x \odot y)^{h(a,b)} = x^{h(a,b)} \odot y^{h(a,b)}$$

for all $a, b, x, y \in B$.

More generally, however, we will verify the desired identity for any $k \in H$ regardless of whether or not k possesses the form $k = h(a, b)$. We will thus show that

$$(x \odot y)^k = x^k \odot y^k$$

for any $k \in H$. Clearly, we have in G

$$(xy)^k = x^k y^k.$$

Employing the unique decomposition $G = BH$, we have

$$(xy)^k = ((x \odot y)h(x, y))^k = (x \odot y)^k h(x, y)^k$$

on one hand, and

$$x^k y^k = (x^k \odot y^k)h(x^k, y^k)$$

on the other hand. It follows from the above and from the uniqueness of the decomposition $G = BH$ that

$$(x \odot y)^k = x^k \odot y^k,$$

which completes the proof. □

Theorem 2.14. *An A_l-transversal B of a subgroup H in G that possesses the left inverse property is a left AP-loop under the loop operation.*

Proof: We have to show that $(B, \odot)$ satisfies axioms (G1)–(G4) of Definition 2.7 and is a left loop. Axioms (G3) and (G4) are verified in Theorem 2.13, and we get (G2) from the left inverse property.

Given $b \in B$ we get

$$b = 1b = (1 \odot b)h(1, b)$$

Hence (G1) is verified. Given $a \odot x = b$ we have a unique solution $a^{-1} \odot b$. □

Note. If B is an A_l-transversal with $B = B^{-1}$, then B is an left AP-loop.

Definition 2.15. An A_l-transversal B with $B = B^{-1}$ is called an *AP-transversal* = *gyrotransversal* ([6]).

Example 2.16. Let $P \subset S_n$ where P consists of the identity permutation and all 2-cycles of the form $(1, i)$ for $i = 2, \ldots, n$. Let H be the stabilizer of 1 in S_n. Then P is a transversal of H, and is, in fact, an AP-transversal, but P is not a loop.

Note. $P = P^{-1}$ and if $a \in P$ and $h \in H$, then $hah^{-1} \in P$.

§3. A family of AP-loops

In the literature there are many examples of A_l-Bol-loops and K-loops which are all necessarily AP-loops. In this section we will look at a family of AP-loops that are not A_l-Bol-loops (and thus not K-loops).

Definition 3.1 (Diagonal transversals)**.** Let K be a group and let $G = K \rtimes Inn(K)$ be the semidirect product group of K and $Inn(K)$, where $Inn(K)$ is the inner automorphism group of K whose generic element α_k denotes conjugation by $k \in K$ (i.e. $\alpha_k x = kxk^{-1}$). Then, the *diagonal transversal* D generated by K (in G) is the subset of G given by

$$D = \{(k, \alpha_k) | k \in K\} \subset G$$

which is a transversal of $Inn(K)$ in G. Any element $(k, \alpha_k) \in D$ is determined by a corresponding element $k \in K$. We therefore use the notation

$$D(k) = (k, \alpha_k)$$

to denote the elements of D.

Theorem 3.2. *A diagonal transversal with its transversal operation is an AP-transversal.*

PROOF: ([6, Theorem 3.2]). $\square$

Definition 3.3 (Associated Left Gyrogroups) [6]**.** The associated left gyrogroup of a group $(K, \cdot)$ from Theorem 3.2 is the left A_l-loop $(K, \odot)$. The operation $\odot$ is given in terms of the group operation $\cdot$ by $a \odot b = ab^a = a^2ba^{-1}$ for all $a, b \in K$. This corresponds to the transversal operation of the diagonal transversal.

Theorem 3.4. *If K is a nilpotent group, then $(K, \odot)$ is an AP-loop.*

PROOF: It will suffice to show that the equation $x \odot a = b$ has a unique solution for x. It is true when $(K, \cdot)$ is abelian, and an induction on the class (divide out the center) does the rest. $\square$

Theorem 3.5. *The associated left gyrogroup $(K, \odot)$ of a group $(K, \cdot)$ is a group if and only if $(K, \cdot)$ is nilpotent of class 2.*

PROOF: ([6, Theorem 3.6]). $\square$

Definition 3.6. Given $a, b \in E$ let $[a,_1 b] = [a, b]$ and inductively $[a,_{n+1} b] = [[a,_n b], b]$. A group E is n-Engel group ([14]) $[a,_n b] = 1$ for all $a, b \in E$.

Theorem 3.7. *Let $(K, \odot)$ be the associated left gyrogroup of a group $(K, \cdot)$. Then $(K, \odot)$ is a A_l-Bol-loop = gyrogroup if and only if $(K, \cdot)$ is central by a 2-Engel group.*

Proof: ([6, Theorem 3.7]). □

Hence, for any nilpotent group $(K, \cdot)$ of class ≥ 5, the associated groupoid $(K, \odot)$ is an AP-loop but not A_l-Bol-loop, since it is nilpotent but not central by a 2-Engel group.

§4. A multiplication table and a look at normal subgroups of loops

Example 4.1 (A A_l-Bol-loop multiplication table)**.** The lowest order of a nilpotent group of class 3 which is not of class 2 is 16. Using the software package MAGMA ([2]) we found three non-isomorphic nilpotent groups of order 16 which are of class 3 but are not of class 2. Their associated left gyrogroup generate three non-K-loops (i.e., non-Bruck-loops) A_l-Bol-loops of order 16, denoted by K_{16}, L_{16}, and M_{16}. The multiplication table of K_{16}, is presented in Table I, where the elements $k_i \in K_{16}$, $i = 1, 2, \ldots, 16$, are denoted by their subscripts.

Table I (*The A_l-Bol-loop K_{16}*)

$\circ$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	2	1	7	8	6	5	12	11	9	10	16	15	13	14
4	4	3	1	2	8	7	5	6	11	12	10	9	15	16	14	13
5	5	6	7	8	4	3	1	2	16	15	13	14	10	9	12	11
6	6	5	8	7	3	4	2	1	15	16	14	13	9	10	11	12
7	7	8	6	5	1	2	3	4	14	13	16	15	11	12	10	9
8	8	7	5	6	2	1	4	3	13	14	15	16	12	11	9	10
9	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
10	10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7
11	11	12	10	9	15	16	14	13	4	3	1	2	8	7	5	6
12	12	11	9	10	16	15	13	14	3	4	2	1	7	8	6	5
13	13	14	15	16	12	11	9	10	7	8	6	5	1	2	3	4
14	14	13	16	15	11	12	10	9	8	7	5	6	2	1	4	3
15	15	16	14	13	9	10	11	12	5	6	7	8	4	3	1	2
16	16	15	13	14	10	9	12	11	6	5	8	7	3	4	2	1

K_{16} has only one non-identity left inner mapping, A, whose transformation table is given in Table II.

Table II (*The automorphism A of K_{16}*)

$1 \to 1$	$5 \to 5$	$9 \to 10$	$13 \to 14$
$2 \to 2$	$6 \to 6$	$10 \to 9$	$14 \to 13$
$3 \to 3$	$7 \to 7$	$11 \to 12$	$15 \to 16$
$4 \to 4$	$8 \to 8$	$12 \to 11$	$16 \to 15$

The left inner mapping $\delta_{a,b}$ generated by any $a, b \in K_{16}$ is either A or the identity automorphism denoted by 1. The left inner mapping table for $\delta_{a,b}$ is presented in Table III.

Table III (*The left inner mapping $\delta_{a,b}$ of K_{16}*)

δ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	A	A	A	A	A	A	A	A
6	1	1	1	1	1	1	1	1	A	A	A	A	A	A	A	A
7	1	1	1	1	1	1	1	1	A	A	A	A	A	A	A	A
8	1	1	1	1	1	1	1	1	A	A	A	A	A	A	A	A
9	1	1	1	1	A	A	A	A	1	1	1	1	A	A	A	A
10	1	1	1	1	A	A	A	A	1	1	1	1	A	A	A	A
11	1	1	1	1	A	A	A	A	1	1	1	1	A	A	A	A
12	1	1	1	1	A	A	A	A	1	1	1	1	A	A	A	A
13	1	1	1	1	A	A	A	A	A	A	A	A	1	1	1	1
14	1	1	1	1	A	A	A	A	A	A	A	A	1	1	1	1
15	1	1	1	1	A	A	A	A	A	A	A	A	1	1	1	1
16	1	1	1	1	A	A	A	A	A	A	A	A	1	1	1	1

Definition 4.2. A subloop X of a loop P is a *normal subgroup* ([7]) of P if it is a normal subloop which is in the middle nucleus i.e.

(i) $\delta_{a,x} = 1$ for all $x \in X$ and $a \in P$;
(ii) $\delta_{a,b}(X) \subseteq X$ for all $a, b \in P$;
(iii) $a \bigodot X = X \bigodot a$ for all $a \in P$.

Note. If X is a normal subgroup of a loop P, then X is a group with group operation given by the restriction of $\odot$ to X.

Note. Since a normal subgroup X of P is a normal subloop, P/X forms a factor loop.

Theorem 4.5. *If $(P, \odot)$ is a A_l-Bol-loop, then P has a normal subgroup Ξ such that P/Ξ is a Bruck-loop.*

Proof: ([7, Theorem 4.11]). □

Example 4.6. The 4×4 upper left corner of Table I forms a multiplication table of a group, H. The group H is a normal subgroup of K_{16}. The quotient K_{16}/H turns out to be an abelian group. Hence, we have in hand an example of an extension of a group by another group that gives a non-associative structure (that is, the A_l-Bol-loop K_{16}). It is an extension which is far from being trivial since H and K_{16}/H are groups while K_{16} is a non-Bruck-loop A_l-Bol-loop.

§5. Open questions

Question 5.1. Are there any nontrivial AP-transversals that are loops in finite simple groups (The answer is positive in the infinite case [12], but I suspect that it is negative in the finite case)?

Question 5.2. For which groups is the associated left gyrogroup an AP-loop?

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(*Received* September 7, 1999, *revised* January 10, 2000)