Loops and quasigroups: Aspects of current work and prospects for the future

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Abstract. This paper gives a brief survey of certain recently developing aspects of the study of loops and quasigroups, focussing on some of the areas that appear to exhibit the best prospects for subsequent research and for applications both inside and outside mathematics.

Keywords: quasigroup, loop, net, web, semisymmetric, Lindenbaum-Tarski duality, multiplication group, loop transversal, loop transversal code, octonions, Cayley numbers, sedenions, vector field, knot, link, quandle, surgery, Conway algebra, split extension, centraliser ring, multiplicity free, fusion rule, statistical dimension, Ising model

Classification: 20N05

Introduction

Compared to the theory of groups, the theory of quasigroups is considerably older, dating back at least to Euler's work on orthogonal Latin squares. In the first half of the twentieth century, both theories experienced comparable moderate progress. But from the nineteen-fifties to the nineteen-eighties, the theory of quasigroups was eclipsed by the phenomenal development of the theory of groups to such an extent that the former sometimes came to be considered as a minor offshoot of the latter (e.g. as witnessed by the American Mathematical Society's 1991 Subject Classification 20N05 for loops and quasigroups under the heading "Other generalizations of groups"). With the initial completion of the classification of the finite simple groups, however, attention is once again becoming more evenly divided between the two theories. The current paper aims to give a brief survey of some of the recently developing areas of research within the theory of loops and quasigroups, especially those areas with connections to other parts of mathematics and to applications outside mathematics. The topics presented are:

1. Nets and homotopy;
2. Transversals;
3. Octonions, topology, and knots;
4. Representation theory.

For background details, see [Al63], [Br58], [CPS90], [Pf90], [Sm86], [SR99].

1. Nets and homotopy

Historically, homotopy between loops and quasigroups has been treated geometrically, isotopic quasigroups corresponding to isomorphic nets ([SR99, Theorem I.4.5]). Gvaramiya and Plotkin ([GP92], [Vo99]) reduced the isotopy of quasigroups to the isomorphism of heterogeneous algebras or "automata". It is now possible to give a purely (homogeneous) algebraic treatment of homotopy, using the variety of semisymmetric quasigroups ([Sm97]). Recall that a quasigroup is *semisymmetric* if it satisfies the identity

$$(yx)y = x. \tag{1.1}$$

Each quasigroup $(Q, \cdot, /, \backslash)$ then has a *semisymmetrisation* $Q\Delta$, a semisymmetric quasigroup structure on its direct cube Q^3 in which the product of elements (x_1, x_2, x_3) and (y_1, y_2, y_3) is given by

$$(y_3/x_2, y_1 \backslash x_3, x_1 \cdot y_2). \tag{1.2}$$

A homotopy $(f_1, f_2, f_3) : (Q, \cdot, /, \backslash) \to (P, \cdot, /, \backslash)$ between quasigroups corresponds to a homomorphism

$$(f_1, f_2, f_3)\Delta : (x_1, x_2, x_3) \mapsto (x_1 f_1, x_2 f_2, x_3 f_3) \tag{1.3}$$

between their semisymmetrisations, so that two quasigroups are isotopic if and only if their semisymmetrisations are isomorphic. This observation places new importance on the variety of semisymmetric quasigroups.

Nets reappear within a duality theory for quasigroups. This duality theory is based on the so-called *Lindenbaum-Tarski duality* between sets and complete atomic Boolean algebras, the duality relating a function $f : X \to Y$ between sets to the inverse image function $f^{-1} : \wp(Y) \to \wp(X); B \mapsto f^{-1}(B)$ between their power sets ([Jt82, VI.4.6(a)], [RS96, § 7]). On a quasigroup Q, consider the binary operations of multiplication $p_3 : Q^2 \to Q; (x, y) \mapsto x \cdot y$, *left projection* $p_2 : Q^2 \to Q; (x, y) \mapsto x$, and *right projection* $p_1 : Q^2 \to Q; (x, y) \mapsto y$. Dualising, one obtains the inverse image functions $p_3^{-1} : \wp(Q) \to \wp(Q^2)$ given by multiplication, $p_2^{-1} : \wp(Q) \to \wp(Q^2)$ given by left projection, and $p_1^{-1} : \wp(Q) \to \wp(Q^2)$ given by right projection. These homomorphisms of complete atomic Boolean algebras are specified by their effects on the atoms of their domains, namely on the singleton subsets of Q:

$$(1.4) \quad \begin{cases} (1) & p_1^{-1} : \wp(Q) \to \wp(Q^2); \{b\} \mapsto \{(x, b) \mid x \in Q\}; \\ (2) & p_2^{-1} : \wp(Q) \to \wp(Q^2); \{a\} \mapsto \{(a, y) \mid y \in Q\}; \\ (3) & p_3^{-1} : \wp(Q) \to \wp(Q^2); \{c\} \mapsto \{(x, y) \mid x \cdot y = c\}. \end{cases}$$

In net terminology, $p_1^{-1}\{b\}$ in (1.4)(1) is just the 1-*line* labelled by b. Similarly, $p_2^{-1}\{a\}$ in (1.4)(2) is the 2-*line* labelled by a, while $p_3^{-1}\{c\}$ in (1.4)(3) is the 3-*line* labelled by c. Thus the dual of a quasigroup is a net. The dual object is a set that decomposes as the direct product of any two of a set of three (isomorphic) images [SR99, p. 88]. This definition of a net is purely categorical, and may thus be interpreted in the category of topological spaces or other categories of geometric interest. It offers new approaches to the problem of coordinatising configurations in web geometry that has been studied by Akivis, Goldberg, Shelekov et al. ([Ak92], [Go88]). In geometric contexts, one should replace the "combinatorial" Lindenbaum-Tarski duality by a duality appropriate to the context, e.g. Gelfand duality between compact Hausdorff spaces and C*-algebras ([Jt82, IV.4]), or the duality between (locally defined) smooth maps and their pullbacks acting on differential forms ([Ol95, p. 26]).

2. Transversals

Loops, or more generally right loops, appear naturally as algebraic structures on transversals or sections of a subgroup of a group. This observation, going back to R. Baer [Ba39], lies at the heart of much current research on loops, e.g. work of Karzel, Kreuzer, Strambach, Wefelscheid et al. in geometry ([Ka93], [Kr98], [KW94], [Na94]) or work of Kikkawa, Sabinin, Ungar et al. in differential geometry and analysis ([CPS, Chapter XII], [Fr94], [Ki98], [Sa72-98], [Un91-94]). It is also closely related to the question of when or how a given group (action) arises as the multiplication group (action) of a quasigroup or loop, cf. work of Drápal, Kepka, Niemenmaa, Phillips et al. ([CPS90, § III.6], [Dr93-94], [Ke93-97], [Ni95d-97], [NK90-94], [NR92], [NV94], [PS99]).

Let H be a subgroup of a group $(G, \cdot, /, \backslash, 1)$, and let T be a right transversal to H in G such that 1 represents H. Thus the group is partitioned as $G = \dot{\bigcup}_{t \in T} Ht$. Define a map $\varepsilon : G \to T; g \mapsto g^\varepsilon$ by

$$g \in Hg^\varepsilon, \tag{2.1}$$

so that g^ε or $g\varepsilon$ is the unique representative in T for the right coset of H that contains g. It is also convenient to define a map $\delta : G \to H; g \mapsto g^\delta$ by

$$g = g^\delta g^\varepsilon. \tag{2.2}$$

Note that $1^\delta = 1^\varepsilon = 1$. Moreover $h^\delta = h$ and $h^\varepsilon = 1$ for h in H, while $t^\delta = 1$ and $t^\varepsilon = t$ for t in T. Now define a binary multiplication $*$ and a binary right division $\|$ on T by

$$t * u = (tu)\varepsilon, \;\; t\|u = (t/u)\varepsilon \tag{2.3}$$

for t, u in T, i.e. by $tu \in H(t * u)$ and $t/u = tu^{-1} \in H(t\|u)$. Then $(T, *, \|, 1)$ is a right loop. If this right loop happens to be two-sided, then the transversal T is called a *loop transversal*. To within right loop isomorphism, every right loop Q may be obtained via (2.3) from the transversal $T = R(Q)$ to the stabiliser H of 1 in the right multiplication group G of Q.

The loop theoretical concept of a loop transversal even has fruitful applications within abelian groups. In coding theory, the loop transversal $(T,*)$ to a linear code $(C,+)$ within an abelian group channel $(G,+)$ is the set of errors corrected by the code ([Sm92l], [SR99, I § 4.4]). Using the notation of (2.2): If a word g is received, it is decoded to the codeword $g\delta$ under the assumption that the transmitted codeword was subjected to the error $g\varepsilon$. The most efficient way to define a code in a good channel is to specify the loop structure (an abelian group) on the set of errors. For elements $t_1, t_2, \ldots$ of T, define $\sum_{i=1}^{m} t_i$ inductively by $\sum_{i=1}^{0} t_i = 0$ and $\sum_{i=1}^{m} t_i = t_m + \sum_{i=1}^{m-1} t_i$. Define $\prod_{i=1}^{m} t_i$ inductively by $\prod_{i=1}^{0} t_i = 0$ and $\prod_{i=1}^{m} t_i = t_m * \prod_{i=1}^{m-1} t_i$.

Principle of Local Duality. *Let T be a loop transversal to a linear code C in a channel G. Suppose that T is a set of generators for G. Then*

$$C = \{\sum_{i=1}^{m} t_i - \prod_{i=1}^{m} t_i \,|\, t_1, \ldots, t_m \in T\}.$$

Note that the Principle of Local Duality may be used to obtain the code C as soon as the loop structure $(T,*)$ is specified.

Example. Consider the length 3 binary repetition code $C = \{000, 111\}$. Interpret G as $\mathbb{Z}_2^3$. The right transversal $T = \{000, 001, 010, 100\}$ is the set of errors corrected by C. The abelian group multiplication $*$ on T given by (2.3) has the table

*	000	001	010	100
000	000	001	010	100
001	001	000	100	010
010	010	100	000	001
100	100	010	001	000

.

Note that the table may be summarised by the specification that the map

(2.4) $$s : (T,*) \to (\mathbb{Z}_2^2, +); 001 \mapsto 01, 010 \mapsto 10, 100 \mapsto 11$$

is an abelian group homomorphism. As an illustration of local duality, the non-trivial codeword is obtained as $111 = (001 + 010) - (001 * 010)$. □

In general, the key step in constructing linear codes by the loop transversal method is to specify the *syndrome*, a monomorphism such as (2.4) from $(T,*)$ to an abelian group. The simplest approach is to use a greedy algorithm. Record-breaking codes have been obtained in this way, and the method offers completely automatic constructions of binary and ternary Golay codes ([HH96], [HS96]).

3. Octonions, topology, and knots

Probably the most important non-associative loop structures are those given by multiplication of non-zero octonions (possibly of norm one, possibly split, possibly over finite fields). The real octonions appear as part of the series: real numbers, complex numbers, quaternions, octonions. With each doubling of the dimension, there is a progressive loss of structure: order, commutativity, associativity. Adams' work on vector fields on spheres ([Ad60], [Ad62]) showed that while the seven-sphere supports a vector field of dimension seven (spanned by multiplication by basic non-identity octonions of norm one), the fifteen-sphere only supports a vector field of dimension at most eight. This result, showing that there is no sixteen-dimensional real division algebra, has long been accepted as the final word on the subject. However, recent work ([Sm95]) has shown that one may obtain a sixteen-dimensional structure, the *sedenions*, by relaxing the requirement of right distributivity. (This structure, which does preserve the Euclidean norm, is not to be confused with that obtained by applying the Cayley-Dickson process to the octonions. The latter structure does not preserve the norm.) The underlying Euclidean space of the sedenions may be realised as the direct sum $\mathbb{K} \oplus \mathbb{K}f$ of two copies of the octonions, with respective identities 1 and f. The octonions are embedded in the sedenions by

$$\mathbb{K} \to \mathbb{K} \oplus \mathbb{K}f; x \mapsto x + 0f. \tag{3.1}$$

For elements $z = x + yf$ and $w = u + vf$ of $\mathbb{K} \oplus \mathbb{K}f$, the product $z \cdot w$ is defined as

$$\textbf{if} \quad z \in \mathbb{K} \quad \textbf{then} \quad zu + vzf \quad \textbf{else} \quad (xy \cdot uy^{-1} - y\bar{v}) + (y\bar{u} + vy^{-1} \cdot xy)f. \tag{3.2}$$

The sedenions of norm one form a left loop on the fifteen-sphere (actually a loop almost everywhere). In this left loop, the multiplications by f and the seven basic non-identity octonions of norm one yield a vector field of the maximal dimension eight allowed by Adams' theorem. The problem of classifying the finite two-sided subloops of this left loop remains open, as does the search for the laws satisfied by the left loop.

Another major interface between topology and the theory of (one-sided) quasigroups is provided by knot theory: work of Conway, Fuad, Joyce, Matveev, et al. ([Cw69], [Fu98], [Jy82], [Mt82], [Sm92q]). Recall that a *knot* (in the narrowest mathematical sense) may be visualised as a knotted piece of string whose ends have been joined seamlessly to make an endless knotted loop. A *link* may be visualised as a collection of knots, possibly linked by encircling each other. More precisely, the knot is the image of the circle S^1 under a continuous, piecewise linear embedding into the three-sphere S^3, Euclidean space $\mathbb{R}^3$ together with an additional point at infinity. The link is the image of a disjoint union of circles under such an embedding. Two links are said to be *equivalent* if there is a homeomorphism from S^3 to itself taking one link to the other. The links are said to have the same *type* if there is such a homeomorphism that additionally preserves the orientation of the three-sphere. The basic problem of knot theory is to determine when two knots are equivalent, or when two oriented links have the same type. One-sided quasigroups enter knot theory at two levels: at the "local" level a one-sided quasigroup may be an equivalence invariant associated with a single knot, while at the "global" level an invariant of a particular kind may be associated with each link, oriented links of the same type having the same invariant, and then the set of these invariants (for all possible oriented links) may carry a right quasigroup structure.

At the local level, an idempotent, right distributive right quasigroup (a *quandle* in Joyce's terminology) forms an invariant determining knots to within equivalence ([Jy82], [Mt82], [Mt91, § 8.2]). Note that quandles constitute a variety in the sense of universal algebra. To associate a quandle with a knot, consider the knotted loop of string laid flat on a table in such a way that no two crossings coincide. Suppose that this knotted loop has been used as the design for the construction of a racetrack, with small bridges corresponding to the crossings. Now imagine that an earthquake has destroyed all the bridges, leaving only disjoint stretches of track on the ground. Assign a variable to each such stretch, and then form the free quandle over the set of variables obtained. The quandle invariant associated with the knot is a quotient of this free quandle by relations specified at each former bridge location. If the stretch q at such a location is now blocked by the stretch r, then the variable assigned to the stretch that used to continue q over the bridge is identified with the right quasigroup product qr.

At the global level, entropic right quasigroup structures are associated with the set of all oriented links ([Cw69], [Mt91, § 8.3]). (Recall that an algebra is *entropic* if each operation, as a map to the algebra from a direct power of the algebra, is a homomorphism [SR99, pp. 63, 318].) The topological basis for the association is a general method for constructing a given oriented link from an (oriented) *unlink* U_c with c components, a collection of c unknotted, unlinked, oriented loops. The method is known as *surgery*. Mixing metaphors, it will be described using the racetrack idea introduced above. (In this context, the term "earthmoving" would be more appropriate than "surgery".) Suppose that an oriented link, the image of d copies of S^1, is used in place of a knot as the design for a racecourse. This means that there may be d different races held at once on the course, one on each of d independent closed tracks that may sweep over and under one another. Moreover, the orientation of the link determines a fixed direction for each track. A *right-hand crossing* is a bridge from which a driver racing on the top track would see the cars beneath passing from his right to his left. A *left-hand crossing* is a bridge from which a driver on the top track would see the cars beneath racing from her left to her right. An ordered triple (K_R, K_L, K_0) of links constitutes a *surgery triple* if the three corresponding racecourses could be obtained from one another by a construction project limited to the environment of a single bridge.

In this project, K_R would have a right-hand crossing at the bridge. Then K_L would be obtained on replacing the bridge by an underpass, thereby making a left-hand crossing at the location. The course K_0 would be obtained from K_R by removing the bridge altogether. Traffic that formerly approached on the upper track would be diverted to leave the location on the lower track, while traffic that formerly approached on the lower track would be diverted to leave the location on the upper track. Each oriented link K may be obtained from an unlink K^r by a series

$$K^r \mapsto K^{r-1} \mapsto \cdots \mapsto K^1 \mapsto K^0 = K \tag{3.3}$$

of such projects, in which each step represents conversion at a certain location from one to another of the three components of a surgery triple.

A *Conway algebra* $(Q, \cdot, /, \{u_n \mid n \in \mathbb{Z}^+\})$ ([Mt91, Definition 8.5]) is an entropic right quasigroup $(Q, \cdot, /)$ with a countable set $\{u_n \mid n \in \mathbb{Z}^+\}$ of constants, satisfying the identities $u_n \cdot u_{n+1} = u_n/u_{n+1} = u_n$ for each positive integer n. Suppose that an element $w(K)$ of Q is associated with each oriented link K in such a way that

$$w(U_c) = u_c \tag{3.4}$$

for each positive integer c, and that

$$w(K_R) = w(K_L) \cdot w(K_0) \tag{3.5}$$

for each surgery triple (K_R, K_L, K_0). Then $w(K)$ is a well-defined invariant of oriented links [Mt91,Th.8.4], readily computed by means of (3.3). The most famous invariants of this kind are the skein polynomials ([FH85]). For them, the corresponding Conway algebra appears as a reduct of a pointed quasigroup. Recall that a *pique* is a "pointed idempotent quasigroup", a quasigroup with an idempotent element selected by a nullary operation ([CPS90, p. 105]). Let $(Q, \cdot, /, \backslash, e)$ be the free entropic pique on the singleton $\{u_1\}$. Define the operation $x\%y = y/x$, the opposite of right division. Define the constants u_n inductively from the generator u_1 by $u_{n+1} = u_n \cdot u_n$. Then $(Q, \%, \backslash, \{u_n \mid n \in \mathbb{Z}^+\})$ is a Conway algebra, and the skein polynomial is defined there by (3.3)–(3.5) ([Sm91]).

4. Representation theory

The representation theory of loops and quasigroups currently comprises three separate topics: permutation representations, modules, and characters.

The study of quasigroup permutation representations is still in its infancy ([Sm99]). At present, the theory is limited to the construction of transitive permutation representations, analogous to the permutation representation of a group Q (with subgroup P) on the homogeneous space

$$P \backslash Q = \{Px \mid x \in Q\} \tag{4.1}$$

by the actions

$$R_{P\backslash Q}(q) : P \setminus Q \to P \setminus Q\,;\; Px \mapsto Pxq \tag{4.2}$$

for elements q of Q. Let P be a subquasigroup of a quasigroup Q. Recall that the *relative left multiplication group* $\mathrm{LMlt}_Q P$ of P in Q is the subgroup of the multiplication group of Q generated by all the left multiplications by elements of P. Let $P \setminus Q$ be the set of orbits of the permutation group $\mathrm{LMlt}_Q P$ on the set Q. If Q is a group, and P is nonempty, then this notation is consistent with (4.1). Let A be the incidence matrix of the membership relation between the set Q and the set $P \setminus Q$ of subsets of Q. (In particular, the rows of A are indexed by elements of Q, while the columns of A are indexed by the orbits of $\mathrm{LMlt}_Q P$.) Let A^+ be the pseudoinverse of the matrix A ([Pe55]). For each element q of Q, right multiplication in Q by q yields a permutation of Q. Let $R_Q(q)$ be the corresponding permutation matrix. Define a new matrix

$$R_{P\backslash Q}(q) = A^+ R_Q(q) A. \tag{4.3}$$

[In the group case, the matrix (4.3) is just the permutation matrix given by the permutation (4.2).] Then in the transitive permutation representation of the quasigroup Q, each quasigroup element q yields a Markov chain on the state space $P \setminus Q$ with transition matrix $R_{P\backslash Q}(q)$ given by (4.3). The set of convex combinations of the states from $P \setminus Q$ forms a complete metric space, and then the actions (4.3) of the quasigroup elements form an iterated function system or IFS in the sense of fractal geometry ([Bn88, p. 82]).

Modules over a quasigroup Q are defined as abelian groups in the slice category $\underline{\underline{V}}/Q$. Here $\underline{\underline{V}}$ is a variety containing the quasigroup Q, and $\underline{\underline{V}}$ is construed as a category whose morphisms are the homomorphisms between the quasigroups in $\underline{\underline{V}}$ ([Sm86], [SR99, III § 2]). If Q is a group and $\underline{\underline{V}}$ is the category $\underline{\underline{\mathrm{Gp}}}$ of groups, then such an object $\pi : E \to Q$ is just the projection $\pi : Q[M \to Q;\ (q, m) \mapsto q$ from the split extension $Q[M$ of a module M over the group Q in the traditional sense.

Modules play a key role in the extension theory and cohomology of loops and quasigroups (work of Eilenberg, Mac Lane, Johnson, Leedham-Green et al., e.g. [Dh95], [EM47], [Jh74], [JL90], [LM76], [Sm76, Chapter 6].) An elementary version of this extension theory is used in the construction of loops from codes, e.g. as a step in the construction of the Monster from the binary Golay code (work of Conway, Griess, Parker et al. [As94]). For an application to relativity theory, see [SU96].

The character theory for quasigroups is to some extent (but not completely) located within the theory of association schemes ([BI84]). The action of the multiplication group $\mathrm{Mlt}Q$ on a finite quasigroup Q of order n is "multiplicity free". In other words, extending the action by linearity, the $\mathbb{C}\mathrm{Mlt}Q$-module $\mathbb{C}Q$ decomposes as a direct sum of mutually inequivalent irreducible submodules. Thus the *centraliser ring* $V(\mathrm{Mlt}Q,Q)$, the ring $\mathrm{End}_{\mathbb{C}\mathrm{Mlt}Q}\mathbb{C}Q$ of $\mathbb{C}\mathrm{Mlt}Q$-endomorphisms of the module $\mathbb{C}Q$, is a commutative $\mathbb{C}$-algebra. As such, it decomposes as a direct sum of copies of $\mathbb{C}$. One usually identifies the endomorphisms of $\mathbb{C}Q$ that form the elements of $V(\mathrm{Mlt}Q,Q)$ with their matrices in terms of the basis Q of $\mathbb{C}Q$. There are then two natural bases for $V(\mathrm{Mlt}Q,Q)$: the set $\{e_1,\ldots,e_s\}$ of orthogonal idempotents yielding the decomposition $V(\mathrm{Mlt}Q,Q)=\mathbb{C}e_1\oplus\cdots\oplus\mathbb{C}e_s$, and the set $\{a_1,\ldots,a_s\}$ of incidence matrices for the corresponding orbits $C_1,\ldots,C_s$ of $\mathrm{Mlt}Q$ in its diagonal action on Q^2. Conventionally, one takes e_1 to be the $n\times n$ matrix in which each entry is n^{-1}, and one takes a_1 to be the $n\times n$ identity matrix, the incidence matrix of the diagonal orbit C_1 of $\mathrm{Mlt}Q$ on Q. Set $n_i=\mid C_i\mid/n$ and $f_i=\mathrm{tr}(e_i)$ for $1\le i\le s$. Define the $s\times s$ matrix $[\xi_{ij}]$ by

$$a_i=\sum_{j=1}^{s}\xi_{ij}e_j \tag{4.4}$$

for $1\le i\le s$. Then define the $s\times s$ matrix $\Psi=[\psi_{ij}]$ by

$$\psi_{ij}=\sqrt{f_i}\xi_{ji}/n_j. \tag{4.5}$$

If Q is a group with identity element 1, then Ψ is the ordinary character table of Q, in the sense that ψ_{ij} is the value taken by the i-th irreducible character of Q on each element q of Q such that $(1,q)\in C_j$. In the general case, the matrix Ψ is thus called the *character table* of the quasigroup Q. These character tables provide the foundation for extending ordinary character theory from groups to quasigroups. Further details are available in [Jh92], [JS84-89x], [JSS90], [Sm86-90q].

Quasigroup characters implement the so-called "fusion rules" of quantum field theory ([CP94, Definition 5.2.8]). Let Λ be a set equipped with an involution $\lambda\mapsto\lambda^*$ and a distinguished element ω such that $\omega^*=\omega$. Then a set of *fusion rules* indexed by Λ is a collection $\{N_{\lambda\mu,\nu}\}_{\lambda,\mu,\nu\in\Lambda}$ of non-negative integers satisfying the following conditions:

$$\left\{\begin{array}{lll}(1) & \forall\lambda,\mu\in\Lambda, N_{\lambda\mu,\nu}=0 & \text{except for finitely many } \nu;\\ (2) & N_{\lambda\mu,\nu}=N_{\mu\lambda,\nu}; & \\ (3) & \sum_{\alpha\in\Lambda}N_{\lambda\alpha,\beta}N_{\mu\nu,\alpha}=\sum_{\alpha\in\Lambda}N_{\lambda\mu,\alpha}N_{\alpha\nu,\beta}; & \\ (4) & N_{\lambda\omega,\mu}=N_{\omega\lambda,\mu}=\delta_{\lambda,\mu}; & \\ (5) & N_{\lambda\mu,\nu}=N_{\mu^*\lambda^*,\nu^*}; & \\ (6) & N_{\lambda\mu^*,\omega}=\delta_{\lambda,\mu}. & \end{array}\right. \tag{4.6}$$

In the quasigroup implementation, the set of orbits of $\mathrm{Mlt}Q$ on Q^2 is the index set Λ for the fusion rules, equipped with the involution * given by permutation of components in Q^2, and with the diagonal orbit as the distinguished element ω of Λ. The set of fusion rules is then the set $\{c_{ij,k}\}_{1\le i,j,k\le s}$ of structure constants for the centraliser ring $V(\mathrm{Mlt}Q, Q)$ with respect to the basis $\{a_1, \ldots, a_s\}$ consisting of the incidence matrices of the orbits.

For a group Q, the dimensions of the (irreducible) characters are integral. For a more general quasigroup Q, this need no longer be the case. For instance, suppose that Q is the (additively written) cyclic group of order 4, considered as a quasigroup $(\mathbb{Z}_4, -)$ under the operation of subtraction. Note that $\mathrm{Mlt}Q$ is the dihedral group D_4 ([SR99, I Example 2.1.2]). Then the dimensions of the characters of Q are $1, 1, \sqrt{2}$ ([Sm86, Example 537]). (As in group theory, the order of the quasigroup is the sum of the squares of the irreducible character dimensions.) It is significant that this non-integrality is quite typical of statistical dimensions in quantum field theory. For example, consider the conformal field theory describing the scaling limit of the Ising model at the critical point ([CP94, Example 5.2.12], [MS90]). This theory has three physical representations ρ_0, ρ_1, $\rho_{1/2}$, with respective statistical dimensions $1, 1, \sqrt{2}$ ([CP, Example 11.3.22], [MS90, (1.57)]). Using the notation of [Sm86, 537], let a_1 be the 4×4 identity matrix, the incidence matrix of the diagonal orbit on Q^2. Let a_2, a_3 be the respective incidence matrices of the orbits of $(0, 2)$ and $(0, 1)$. Comparing [Sm86, 537] with [CP, Example 5.2.12], [MS90, Theorem 4], one then sees that the quasigroup $(\mathbb{Z}_4, -)$ yields the fusion rules of the conformal field theory under the assignments $\rho_0 \mapsto a_0$, $\rho_1 \mapsto a_2$, $\rho_{1/2} \mapsto a_3/\sqrt{2}$.

References

[Ad60] Adams J.F., *On the non-existence of elements of Hopf invariant one*, Ann. of Math. **72** (1960), 20–104.

[Ad62] Adams J.F., *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.

[Ak92] Akivis M.A., Shelekov A.M., *Geometry and Algebra of Multidimensional Three-Webs*, Kluwer, Dordrecht, 1992.

[Al63] Albert A.A. (ed.), *Studies in Modern Algebra*, Prentice-Hall, Englewood Cliffs, NJ, 1963.

[As94] Aschbacher M., *Sporadic Groups*, Cambridge University Press, Cambridge, 1994.

[Ba39] Baer R., *Nets and groups I*, Trans. Amer. Math. Soc. **46** (1939).

[BI84] Bannai E., Ito T., *Algebraic Combinatorics I*, Benjamin/Cummings, Menlo Park, CA, 1984.

[Bn88] Barnsley M.F., *Fractals Everywhere*, Academic Press, San Diego, CA, 1988.

[Br58] Bruck R.H., *A Survey of Binary Systems*, Springer, Berlin, 1958.

[CP94] Chari V., Pressley A.N., *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994.

[CPS90] Chein O., Pflugfelder H.O., Smith J.D.H. (Eds.), *Quasigroups and Loops: Theory and Applications*, Heldermann, Berlin, 1990.

[Cw69] Conway J.H., *An enumeration of knots and links*, in "Computational Problems in Abstract Algebra", (ed. J. Leech), Pergamon Press, 1969, pp. 329–358.

[Dh95] Dharwadker A., Smith J.D.H., *Split extensions and representations of Moufang loops*, Comm. Algebra **23** (1995), 4245–4255.

[Dr93] Drápal A., Kepka T., *Multiplication groups of quasigroups and loops I*, Acta Univ. Carolin. Math. Phys. **34** (1993), 85–99.

[Dr94] Drápal A., Kepka T., Maršálek P., *Multiplication groups of quasigroups and loops II*, Acta Univ. Carolin. Math. Phys. **35** (1994), 9–29.

[EM47] Eilenberg S., Mac Lane S., *Algebraic cohomology groups and loops*, Duke Math. J. **14** (1947), 435–463.

[FH85] Freyd P., Hoste J., Lickorish W.B.R., Millett K., Ocneanu A., Yetter D., *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239–246.

[Fr94] Friedman Y., Ungar A.A., *Gyrosemidirect product structure of bounded symmetric domains*, Res. Math. **26** (1994), 28–31.

[Fu98] Fuad T.S.R., *Representations of right quasigroups*, in "Proceedings of the Second Asian Mathematical Conference 1995 (Nakhon Ratchasima)", World Sci. Publishing, River Edge, NJ, 1998, pp. 128–131.

[Go88] Goldberg V.V., *Theory of Multicodimensional* $(n+1)$*-Webs*, Kluwer, Dordrecht, 1988.

[GP92] Gvaramiya A.A., Plotkin B.I., *The homotopies of quasigroups and universal algebras*, in "Universal Algebra and Quasigroup Theory", (eds. A. Romanowska and J.D.H. Smith), Heldermann, Berlin, 1992, pp. 89–99.

[HH96] Hsu F.-L., Hummer F.A., Smith J.D.H., *Logarithms, syndrome functions, and the information rate of greedy loop transversal codes*, J. of Comb. Math. and Comb. Comp. **22** (1996), 33–49.

[HS96] Hummer F.A., Smith J.D.H., *Greedy loop transversal codes, metrics, and lexicodes*, J. of Comb. Math. and Comb. Comp. **22** (1996), 143–155.

[Jh74] Johnson K.W., *Varietal generalizations of Schur multipliers, stem extensions and stem covers*, J. Reine angew. Math. **270** (1974), 169–183.

[Jh92] Johnson K.W., *Some historical aspects of the representation theory of groups and its extension to quasigroups*, in "Universal Algebra and Quasigroup Theory", (eds. A. Romanowska and J.D.H. Smith), Heldermann, Berlin, 1992, pp. 101–117.

[JL90] Johnson K.W., Leedham-Green C.R., *Loop cohomology*, Czech. Math. J. **40** (1990), 182–194.

[JS84] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups*, European J. Combin. **5** (1984), 43–50.

[JS86] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups II: induced characters*, European J. Combin. **7** (1986), 131–137.

[JS89f] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups III: quotients and fusion*, European J. Combin. **10** (1989), 47–56.

[JS89s] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups IV: products and superschemes*, European J. Combin. **10** (1989), 257–263.

[JS89x] Johnson K.W., Smith J.D.H., *Characters of finite quasigroups V: linear characters*, European J. Combin. **10** (1989), 449–456.

[JSS90] Johnson K.W., Smith J.D.H., Song S.Y., *Characters of finite quasigroups VI: critical examples and doubletons*, European J. Combin. **11** (1990), 267–275.

[Jt82] Johnstone P.T., *Stone Spaces*, Cambridge University Press, Cambridge, 1982.

[Jy82] Joyce D., *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), 37–65.

[Ka93] Karzel H., Wefelscheid H., *Groups with an involutory antiautomorphism and K-loops*, Res. Math. **23** (1993), 338–354.

[Ke93] Kepka T., Niemenmaa M., *On loops with cyclic inner mapping groups*, Arch. Math. **60** (1993), 233–236.

[Ke97] Kepka T., Phillips J.D., *Connected transversals to subnormal subgroups*, Comment. Math. Univ. Carolinae **38** (1997), 223–230.

[Ki98] Kikkawa M., *Collected Works of Michihiko Kikkawa*, Shimane University, Matsue, Shimane, 1998.

[Kr98] Kreuzer A., *Inner mappings of Bruck loops*, Math. Proc. Camb. Phil. Soc. **123** (1998), 53–57.

[KW94] Kreuzer A., Wefelscheid H., *On K-loops of finite order*, Res. Math. **25** (1994), 79–102.

[LM76] Leedham-Green C.R., McKay S., *Baer-invariants, isologism, varietal laws and homology*, Acta Math. **137** (1976).

[MS90] Mack G., Schomerus V., *Conformal field algebras with quantum symmetry from the theory of superselection sectors*, Comm. Math. Phys. **134** (1990), 139–196.

[Mt82] Matveev S.V., *Distributive groupoids in the theory of knots* (Russian), Mat. Sb. **119** (1982), 78–88.

[Mt91] Matveev S.V., Fomenko A.T., *Algorithmic and Computer Methods in Three-Dimensional Topology* (Russian), Moscow University Press, Moscow, 1991; English translation by M. Hazewinkel and M. Tsaplina: *Algorithmic and Computer Methods for Three-Manifolds*, Kluwer, Dordrecht, 1997.

[Na94] Nagy P.T., Strambach K., *Loops as invariant sections in groups, and their geometry*, Canad. J. Math. **46** (1994), 1027–1056.

[Ni95d] Niemenmaa M., *On loops which have dihedral* 2*-groups as inner mapping groups*, Bull. Austr. Math. Soc. **52** (1995), 153–160.

[Ni95t] Niemenmaa M., *Transversals, commutators and solvability in finite groups*, Boll. Un. Mat. Ital. A(7) **9** (1995), 203–208.

[Ni96] Niemenmaa M., *On the structure of the inner mapping groups of loops*, Comm. Algebra **24** (1996), 135–142.

[Ni97] Niemenmaa M., *On connected transversals to subgroups whose order is a product of two primes*, European J. Combin. **18** (1997), 915–919.

[NK90] Niemenmaa M., Kepka T., *On multiplication groups of loops*, J. Algebra **135** (1990), 112–122.

[NK92] Niemenmaa M., Kepka T., *On connected transversals to abelian subgroups in finite groups*, Bull. London Math. Soc. **24** (1992), 343–346.

[NK94] Niemenmaa M., Kepka T., *On connected transversals to abelian subgroups*, Bull. Austral. Math. Soc. **49** (1994), 121–128.

[NR92] Niemenmaa M., Rosenberger G., *On connected transversals in infinite groups*, Math. Scand. **70** (1992), 172–176.

[NV94] Niemenmaa M., Vesanen A., *On connected transversals in the projective special linear group*, J. Algebra **166** (1994), 455–460.

[Ol95] Olver P.J., *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge, 1995.

[Pe55] Penrose R., *A generalised inverse for matrices*, Proc. Camb. Phil. Soc. **51** (1955), 406–413.

[Pf90] Pflugfelder H.O., *Quasigroups and Loops: Introduction*, Heldermann, Berlin, 1990.

[PS99] Phillips J.D., Smith J.D.H., *Quasiprimitivity and quasigroups*, Bull. Austral. Math. Soc. **59** (1999), 473–475.

[RS96] Romanowska A.B., Smith J.D.H., *Semilattice-based dualities*, Studia Logica **56** (1996), 225–261.

[Sa72] Sabinin L.V., *The geometry of loops*, Math. Notes **12** (1972), 799–805.

[Sa98] Sabinin L.V., *Quasigroups, geometry and nonlinear geometric algebra*, Acta Appl. Math. **50** (1998), 45–66.

[Sm76] Smith J.D.H., *Mal'cev Varieties*, Springer, Berlin, 1976.

[Sm86] Smith J.D.H., *Representation Theory of Infinite Groups and Finite Quasigroups*, Les Presses de l'Université de Montréal, Montreal, 1986.

[Sm88] Smith J.D.H., *Quasigroups, association schemes, and Laplace operators on almost-periodic functions*, in "Algebraic, Extremal and Metric Combinatorics 1986", (M.-M. Deza, P. Frankl and I.G. Rosenberg, Eds.), Cambridge University Press, Cambridge, 1988, pp. 205–218.

[Sm90e] Smith J.D.H., *Entropy, character theory and centrality of finite quasigroups*, Math. Proc. Camb. Phil. Soc. **108** (1990), 435–443.

[Sm90q] Smith J.D.H., *Combinatorial characters of quasigroups*, in "Coding Theory and Design Theory Part I: Coding Theory", (ed. D. Ray-Chaudhuri), Springer, New York, NY, 1990, pp. 163–187.

[Sm91] Smith J.D.H., *Skein polynomials and entropic right quasigroups*, Demonstratio Math. **24** (1991), 241–246.

[Sm92l] Smith J.D.H., *Loop transversals to linear codes*, J. of Comb., Info. and System Sci. **17** (1992), 1–8.

[Sm92q] Smith J.D.H., *Quasigroups and quandles*, Discrete Mathematics **109** (1992), 277–282.

[Sm95] Smith J.D.H., *A left loop on the* 15-*sphere*, J. Algebra **176** (1995), 128–138.

[Sm97] Smith J.D.H., *Homotopy and semisymmetry of quasigroups*, Algebra Universalis **38** (1997), 175–184.

[Sm99] Smith J.D.H., *Quasigroup actions: Markov chains, pseudoinverses, and linear representations*, Southeast Asian Bull. Math. **23** (1999), 1–11.

[SR99] Smith J.D.H., Romanowska A.B., *Post-Modern Algebra*, Wiley, New York, NY, 1999.

[SU96] Smith J.D.H., Ungar A.A., *Abstract space-times and their Lorentz groups*, J. Math. Phys. **37** (1996).

[Un91] Ungar A.A., *Thomas precession and its associated grouplike structure*, Amer. J. Phys. **50** (1991), 824–834.

[Un94] Ungar A.A., *The holomorphic automorphism group of the complex disk*, Aeq. Math. **47** (1994), 240–254.

[Vo99] Voutsadakis G., *Categorical models and quasigroup homotopies*, Elsevier Preprint, 1999.

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