Loop characters

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Abstract. A survey of the basic results of loop characters is given on the lines of the treatment of the author and J.D.H. Smith for characters of quasigroups, including some recent deveplomets. One of the successes of the theory has been its suggestive influence on the theory of association schemes, group representations and the theory of the group determinant, and selected results arising are described. A section is devoted to an explanation of how the tool of loop characters has not yet been as startlingly successful as that of the early theory of group characters. This may be because in the loop case more is needed than characters and some suggestions are put forward in this direction.

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1. Introduction

In a series of papers [18]–[22] J.D.H. Smith and the present author have set out a theory of (combinatorial) characters of a finite quasigroup Q which in the case where Q is a group coincides with the usual character theory (see also [24]). The theory contains many of the features of group characters, for example row and column orthogonality, but there are also considerable differences. The character values need not be algebraic integers and whereas in the case of finite groups in characteristic zero the theory of representation by matrices, the module theory and the character theory essentially coincide there are three separate theories for quasigroups (see for example [28]). This leads to the necessity for different techniques, since a major tool to construct group characters is to take traces of matrix representations.

The object of this survey is to take a new look at the character theory when Q is a loop. The results in [18]–[22] will be discussed with some hindsight and an account of recent developments will be given. Whereas the introduction of group characters spurred a rapid development of their applications to group theory culminating in Frobenius' proof of the existence of the kernel named after him and Burnside's $p^{\alpha}q^{\beta}$ theorem, it has not yet been possible to see such a fruitful application of character theory to the theory of loops. I will try to point out reasons for this. On the other hand, the work has been useful in other directions. It has stimulated and suggested results on association schemes and combinatorics. It has given insight into some of the questions on group characters posed by Brauer in [5], and it has led to the investigation in a modern context of the group determinant which provided the motivation for Frobenius to define characters of non-commutative groups.

The history of group representation theory has been remarkable for the unexpected directions the research has taken (see for example [11] and [25]). The extension of the theory to quasigroups and loops may be seen to conform to this pattern.

Summaries of quasigroup character theory appear in [28] and [11], Section 4. Here I will refer to [11] to avoid repetition, but I will sometimes give an updated version where appropriate. I will point out specific areas in which the CFQ work has produced new results (and areas of research). I will also point out some instances where methods which have been successful for groups run into difficulties for loops.

In Section 2 loop characters are described and some illustrative examples are given. A summary of their properties is given in Section 3, together with an outline of the methods which can be used to construct loop character tables. Some of the applications of the theory as well as work in related areas which has arisen out of the theory is given in Section 4 and in Section 5 an attempt is made to point out why it has been difficult to provide applications of a similar nature to the many applications of group characters. Suggestions as to where the work can lead in the future are given in Section 6.

2. Loop characters

I refer to [1] or [7] for the background on association schemes. Let Q be a loop. Let the *mapping group* $M(Q)$ of Q be the permutation group on the set Q generated by the maps $\{L(q), R(q)\}$, $q \in Q$ where $L(q)(x) = qx$ and $R(q)x = xq$. The *inner mapping group* $I(Q)$ is the stabiliser of e in $M(Q)$. The orbits of $I(Q)$ are the *conjugacy classes* $C_1 = \{e\}, C_2, \ldots, C_k$. Let

$$\bar{C}_i = \sum_{q \in C_i} q.$$

The set $B = \{\bar{C}_i\}$ generates a commutative, associative subalgebra A of the loop algebra $\mathbf{C}Q$ and the "change of basis" matrix between B and the basis of primitive idempotents is the "$\mathcal{P}$-matrix" of an association scheme arising naturally from the class algebra. A renormalisation of this matrix gives the character table of Q. The important point is that $(M(Q), I(Q))$ forms a *Gelfand pair*, which means that a (commutative) association scheme is defined on the set Q by means of the orbits Γ_i of $M(Q)$ acting on $Q \times Q$ by

$$\sigma(q_1, q_2) = (\sigma(q_1), \sigma(q_2))$$

for $\sigma \in M(Q)$. The entries in the $\mathcal{P}$-matrix of the scheme are the eigenvalues of the incidence matrices A_i defined by $A_i(q_1, q_2) = 1$ if $(q_1, q_2) \in \Gamma_i$ and $A_i(q_1, q_2) = 0$ otherwise. In the quasigroup case the Γ_i are the classes, and for loops there is a $1:1$ correspondence between the Γ_i and the C_i given by $C_i = \{q : (e, q) \in \Gamma_i\}$.

Example. Let Q be the simple Moufang loop of order 120.

The $\mathcal{P}$-matrix of the corresponding association scheme is

$$\begin{array}{rrr} 1 & 63 & 56 \\ 1 & -9 & 8 \\ 1 & 3 & -4 \end{array}$$

and the character table is

$$\begin{array}{rrr} 1 & 1 & 1 \\ \sqrt{35} & -\sqrt{5/7} & \sqrt{5/7} \\ \sqrt{81} & 2/\sqrt{(21)} & -\sqrt{3/7} \end{array}$$

The renormalisation between the two matrices is as follows. The entry in the (i,j)th position of the character table is $\sqrt{f_i/n_j}$ times the corresponding entry of the $\mathcal{P}$-matrix, where n_j is the size of the jth conjugacy class of Q and f_i is the degree of the corresponding irreducible subconstituent of the permutation representation of $M(Q)$ acting on Q.

Further examples.

There is a commutative loop of order 6 with the weak inverse property, whose unbordered multiplication table is the following latin square.

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	3	6	5	2	1
5	6	1	2	4	3
6	5	2	1	3	4

It has a normal subloop of order 2. Its character table is

$$\begin{array}{rrcc} 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 \\ 1 & 1 & \omega^2 & \omega \\ \sqrt{3} & -\sqrt{3} & 0 & 0 \end{array}$$

where $\omega = (-1 + \sqrt{3}i)/2$.

The character table of the smallest Moufang loop (of order 12) is

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 2\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 \end{array}$$

More examples are given in [18]–[22] and in [13].

3. Properties of loop characters and methods of calculation

Let the (i,j)th entry in the character table of the loop Q be $\chi_i(j)$. We refer to χ_i as the ith *basic character* and $\chi_i(j)$ as its value on the jth class. A *class function* on Q is a function from Q to $\mathbf{C}$ which is constant on each conjugacy class. The basic characters form a basis for the set of class functions.

The character table determines the following information.

(1) The set of normal subloops. The *kernel* of a basic character χ is the union of the classes C_i for which $\chi(i) = \chi(e)$. A subloop P of a loop Q is normal if and only if P is the intersection of the kernels of a finite set of characters.

(2) The centre (the union of singleton classes).

(3) The upper and lower central series.

Row and column orthogonality. As in the group case, we have the following.

Theorem 3.1.

$$\sum_{q\in Q} \chi_i(q)\chi_j(q) = \delta_{i,j}|Q|$$

$$\sum_{i=1}^{r} \chi_i(k)\chi_i(m) = \delta_{k,m}|Q|/|C_k|.$$

If Q and P are groups and if $f : Q \to P$ is a homomorphism from Q onto P and ϕ is an irreducible (i.e. basic) character of P it follows from the fact that ϕ is associated to a matrix representation that $\chi = f\phi$ (composition) is an irreducible character of Q. The analogous statement remains true for loop characters, but a different proof is needed. A *linear character* of a loop Q is defined to be a basic character ϕ such that $\phi(e) = 1$. Again, as in the group case a character is linear if and only if it arises from a character of an abelian group by composition with a homomorphism, and again a new proof is required.

A major divergence of the loop character theory is that the entries in the character table are no longer algebraic integers, as in the group case, although the $\mathcal{P}$-matrix entries are algebraic integers. The order of the classes and the degree of the characters are integers dividing the order of a group G. In the loop case, it is well known that the order of a class need not divide the order of the loop and as may be seen from the examples in the previous section the degree of a character χ $(= \chi(1))$ need not be an integer.

Induction and Frobenius reciprocity.

The formula for induction is simpler in the quasigroup setting. A class of a quasigroup is an orbit Γ_i of $M(Q)$ acting on $Q \times Q$ (see above). Let Q be a loop with subloop P. Let the classes of P (regarded as subsets of $P \times P$) be $\{\Delta_1, \ldots, \Delta_r\}$. Suppose that ϕ is a class function on P. Then the *induced class function* ϕ^Q is defined by

$$\phi_P^Q(C_j)/|Q \times Q| = \phi(C_j \cap H \times H)/|H \times H|.$$

In the above $\phi(C) = \sum_{q \in C} \phi(q)$ for any subset C of Q, and $\phi(q) = \phi(j)$ where $q \in C_j$.

The following hold:

(a) Transitivity of induction. Let H be a subloop of P.

$$(\phi_H^P)_P^Q = \phi_H^Q.$$

(b) Frobenius reciprocity. As for group characters define the inner product (ϕ, ψ) of class functions ϕ and ψ by

$$(\phi, \psi)_Q = \sum_{q \in Q} \phi(q)\psi(q)^*$$

where * denotes the complex conjugate. Let ϕ be a class function on P and ψ be a class function on Q. The *restriction* ψ_P of ψ to P is defined in the obvious manner. Then

$$(\phi, \psi_P)_P = (\phi_P^Q, \psi)_Q.$$

At this point there is a divergence from group characters. If χ is a basic character of Q, χ_P is a linear combination of basic characters of P but the coefficients need not be integers. Similarly, if ψ is a basic character of P ψ_P^Q need not be a linear combination of positive integral multiples of basic characters of Q. This means that one must be much more careful in using induction to deduce the existence of basic characters. Another major technique in group character theory also fails to generalise. If ϕ and ψ are class functions the product $\phi\psi$ is defined by

$$\phi\psi(q) = \phi(q)\psi(q).$$

In the group case $\phi\psi$ is always a sum of irreducible characters with positive integral coefficients. For loop characters this is no longer true. For a loop Q the it coefficient ring is defined to be the ring over $\mathbf{Z}$ generated by $(\chi_i\chi_j, \chi_k)$. This ring seems to be of interest and several conjectures about it are given in [13].

There is the following analogue of the theorem of Artin for group characters. A set of subloops $\{P_i, i = 1, \dots N\}$ is defined to be *protrusive* if for each conjugacy class C_i there exists a j such that $P_j \cap C_i \neq \emptyset$. For groups an obvious protrusive set is the set of all cyclic subgroups, and this generalises to the set of singly generated subloops of a loop (although this is no longer the case for quasigroups, see [19]).

Theorem 3.2. *Let $\{P_i, i = 1, \dots N\}$ be a protrusive set of subgroups of Q. A basic character χ of Q satisfies*

$$\chi_i = \sum_j b_j \psi_{P_j}^Q$$

where ψ is a basic character of P_j and the b_j are algebraic integers.

It is readily seen that Artin's theorem is the special case where the protrusive set is as in the above example.

Methods of calculation.

As with group characters, a variety of different methods has been used to calculate loop character tables. Below are some of the methods which have been applied.

(1) A direct calculation of the eigenvalues of the adjacency matrices. This is effective only in small cases.

(2) The calculation of the structure constants of the class algebra. The structure constants $\{a_{ij}^k\}$, which are non-zero integers, are defined by

$$\bar{C}_i\bar{C}_j = \sum a_{ij}^k \bar{C}_k.$$

A matrix representation of the class algebra can be obtained by representing $\bar{C}_i$ by the matrix whose (j,k)th entry is a_{ij}^k and the $\mathcal{P}$-matrix entries are again the eigenvalues. This method is similar to that by which Frobenius calculated the character table of $PSL(2,p)$ ([9]).

(3) If Q has a homomorphic image whose character table is known, a portion of the character table of Q may be calculated by using the composition with the homomorphism as described above.

(4) Under certain circumstances induction may be used to obtain basic characters, although care must be taken in the case where the coefficient ring is not integral.

(5) Fusion. It often happens that the character table of a quasigroup Q_1 is a "fusion" of the table of a quasigroup Q_2 of the same order. In [20] a condition that a table has a fusion is obtained, the *magic rectangle* condition. This method may be applied most easily when Q_1 is a loop with conjugacy classes which are unions of conjugacy classes of a group Q_2. The character table of Q_1 may then be obtained from that of Q_2, which can be calculated by the much easier methods of group theory. Fusion is also used to obtain the character tables of various families of loops arising from extensions of groups discussed in [13].

Finally, it may be pointed out that the character table is an invariant of isotopy, so that it is in fact an invariant of the underlying 3-web corresponding to a loop. The connection between characters and 3-webs remains to be explored.

4. Applications of the ideas

(1) The characters of loops have provided a set of examples which have motivated the discovery of new techniques and definitions in the theory of association schemes. As mentioned above, the crucial result is that for any loop one obtains a Gelfand pair.

(a) Several authors had voiced the need for a definition of an induced character for an association scheme. The definition given in [19] carries over to arbitrary association scheme characters and the details are set out in the note [23].

(b) Fusion and the magic rectangle condition. The results in [20] also carry over to arbitrary association schemes and have motivated further work (see for example [29]).

(c) The calculation of the character tables of the family of simple Moufang loops (the Paige loops) carried out in the thesis of Song began a series of papers where the character tables of the association schemes of various families of classical groups acting on subgroups. See [2], [3] and [4].

(2) Group character theory. In [5] R. Brauer suggested a series of problems on group characters, pointing out that although the theory is successful there are many aspects of it which are not well understood. Loop character tables provide examples of tables which satisfy all the obvious criteria that group tables satisfy, but which do not arise from groups. This gives an explanation as to why it has been difficult to find a set of conditions on a character table which ensure that it comes from a group. Other light on these questions has been shed by the work on group determinants described below.

(3) Loop determinants and latin square determinants. Since many loops have the same character table, it is natural to search for a finer invariant. In the group case the initial object of the study which produced group characters was the group determinant, which is precisely such an object, since a group is determined by its determinant ([8]). A loop determinant may be defined as follows. If Q is a loop of order n assign variables $x_q, q \in Q$ to the elements. Define the *loop matrix* X_Q as a matrix whose rows and columns are indexed by the elements of Q, whose $(p,q)^{th}$ entry is $\{x_{p\rho(q)}\}$ (where $\rho(q)$ is defined by $q\rho(q) = e$). One may think of X_Q as being obtained from the latin square representing the unbordered multiplication table of the loop by rearranging the columns so that e appears in the diagonal and replacing q by x_q for all q in Q. The *loop determinant* Θ_Q is $\det(X_Q)$.

A study of latin square determinants (which may be thought of as loop determinants) was begun in [10] and [12]. As in the group case there is a $1:1$ correspondence between factors of the determinant and basic characters, but the factors may not be irreducible. Define the equivalence relation $\mathcal{R}$ on the set of loops of a given order to be the smallest relation including isotopy and transposition (Q_1 is the transpose of Q_2 if the loop multiplication table of Q_1 is the transpose of that of Q_2). $\mathcal{R}$-equivalent loops necessarily have equivalent determinants, and loops of order at most 7 with equivalent determinants are $\mathcal{R}$ equivalent, but a calculation for the latin squares of order 8 in [15] has shown that of the 842,227 $\mathcal{R}$ classes all except 37 have inequivalent determinants, the exceptional 37 classes having only 12 distinct determinants. One might conjecture that the loop determinant "almost" determines the loop.

One of the important consequences of this work is that it has stimulated a new look at the group determinant. It would take this discussion too far from its roots to describe this work in any detail and the reader is referred to the recent survey [14]. As mentioned above, the work has led to further understanding of the difficulty of solving some of the Brauer problems for group characters and interesting applications of the theory of norm-type forms on algebras have been made. Mention may also be made of the *weak Cayley table* of a group G which is defined in [16] to be the table with rows and columns indexed by the elements of G whose $(g,h)^{th}$ entry is the conjugacy class containing gh. This is a finer invariant than the character table, and the information in it is equivalent to that in the irreducible 2-characters of G.

(4) The idea of a sharp character for a group arises from permutation groups. A permutation group is sharply transitive if and only if the permutation character is sharp. This definition was generalised by Cameron and Kiyota in [6] to arbitrary group characters, and results have been obtained by several authors on the kinds of groups which have a sharp character with a given set of values. A basic result is the following generalisation of a well-known result of Burnside: if χ is any faithful character of a group and $L = \{l_1, l_2, \ldots, l_r\}$ is the set of values taken on by χ on non-identity elements, and if

$$f_\chi(x) = \prod_{i=1}^{r}(x - l_i)$$

then

$$f_\chi(\chi) = m\rho$$

where ρ is the regular character and m is an integer. If $m = 1$ then χ is sharp. This result has a counterpart for loop characters, and in [13] examples are given of sharp characters for loops. Independent work of Strunkov pointed out that one may obtain a dual of the above result for classes, and in [17] it is shown that Strunkov's result holds for any association scheme. A consequence for loops is that if r is the number of distinct values in the column of the $\mathcal{P}$-matrix corresponding to a "faithful" class C_i of a loop Q then every element of Q may be expressed as a word in the elements of C_i of length at most r, where a class C_i is faithful if no other entry in the i^{th} column of the $\mathcal{P}$-matrix is equal to the first entry. This work has also led to interesting conjectures related to the Burnside ring of a group.

5. Limitations of the theory

The following are some of the reasons behind the difficulty of applying character theory to arbitrary loops.

(1) The *trivial* character table (for a loop of order n) is

$$\begin{array}{cc} 1 & 1 \\ \sqrt{(n-1)} & -1/\sqrt{(n-1)} \end{array}$$

"Almost all" loops have the trivial table. However many special families do not, for example Moufang loops and any loop which is not simple cannot have the trivial table.

(2) It is difficult to relate properties such as associativity, Moufang, Bol, or validity of Lagrange's theorem to the character table. For any loop there are usually many loops with the same table (for example the loops of order 8 discussed in [10] all have the same character table), and although there has not been much research in this direction one suspects that such loops can have widely differing algebraic properties. Note that associativity can be determined from the loop determinant.

(3) One application of characters in group theory has been in the construction of sporadic simple groups. A possible character table is constructed for a simple group and the group is then constructed (as in the case of the Monster). It has proved to be difficult to use the same strategy for the construction of simple loops with a given property. For example, the calculation of the character tables of the Paige loops was carried out by working with Gelfand pairs (G, H) with G a classical group. Such a pair exists where $[G : H] = 36$ and if a loop transversal to H in G exists the corresponding character table is that of a simple loop. An unsuccessful attempt was made in collaboration with Smith to construct such a loop transversal with the extra conditions which would produce a Bol loop with this table (although it remains possible that such a transversal can be constructed).

(4) The multiplicative structure of the "character ring" and induced characters are not easy to control.

(5) There is as yet no analogue of the Burnside ring (i.e. the ring of permutation representations of a group) for a loop. However in [27] a candidate for a permutation representation of a quasigroup is offered.

6. Suggestions for future work

The following suggestions may be regarded as being very tentative, given the lack of predictability of group representation theory.

(1) Loop determinants. As was pointed out in the previous section, the loop determinant appears to be a reasonably good invariant for an isotopy class of loops. Analogues of the k-characters discussed in [14] do exist for loops but the new results and perspectives in group theory have diverted effort away from understanding the loop case. The weak Cayley table of a loop would also seem to be an interesting object of study.

(2) Lagrange's theorem. The connection between cosets and permutation representations may give a method to prove that a loop (probably with severely restricted properties) satisfies Lagrange's theorem. The hope is that character induction, the methods in [27] and other methods above may be applied to show that Moufang loops of small order satisfy the conclusion of the theorem.

(3) The orbits of $M(Q)$ acting on $Q^{(r)}$ (rth Cartesian product) and superschemes. How much more loop properties can these orbits determine, and is there always an r so that the orbits on $Q^{(r)}$ and $Q^{(r+1)}$ contain the same information?

I conclude with the following problems.

Problem 1. Characterise loops which have the same character table as a group.

A non-associative loop cannot have the same character table as an abelian group, so the smallest character table to consider is that of the symmetric group S_3. There is one loop (necessarily of order 6) with the same table, which is weak inverse, generalised Moufang and generalised Bol. The loops of order 8 discussed in [12] all have the same character table as the two non-abelian groups of that order.

Problem 2. Characterise the loops which have the same character table as a Moufang loop.

Problem 3. Characterise the loops for which the coefficient ring is $\mathbf{Z}$.

The two Moufang loops whose character tables are given in Section 2 do not have integral coefficient rings. There are examples in [13] of families which have integral coefficient ring.

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