Connected transversals — the Zassenhaus case

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Abstract. In this short note, it is shown that if A, B are H-connected transversals for a finite subgroup H of an infinite group G such that the index of H in G is at least 3 and $H \cap H^u \cap H^v = 1$ whenever $u, v \in G \setminus H$ and $uv^{-1} \in G \setminus H$ then $A = B$ is a normal abelian subgroup of G.

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Throughout this extremely short note, let H be a subgroup of a group G such that the index of H in G is at least 3 and $H \cap H^u \cap H^v = 1$ whenever $u, v \in G \setminus H$ and $uv^{-1} \in G \setminus H$. Furthermore, let A, B be subsets of G such that $AH = G = BH$ and $a^{-1}b^{-1}ab \in H$ for all $a \in A$ and $b \in B$. In his remarkable paper [3], A. Drápal showed that then $A = B$ is an abelian subgroup of G, provided that G is finite. The purpose of the present modest note is to check that $A = B$ is a normal abelian subgroup of G, provided that H is finite and G is infinite (notice that if H is infinite then neither A nor B need to be a subgroup of G — see [1] and [2]). The kind reader is fully referred to [1], [2] and [3] as concerns all the necessary background and many further useful details and connections.

In the rest of this note, assume that H is finite and G is infinite. If $G_1 = \langle A, B\rangle$ and $H_1 = H \cap G_1$ then $AH_1 = G_1 = BH_1$ and we show first that the centralizer K of H_1 in G_1 is of finite index in G_1.

Since $n = |H_1|$ is finite, we have $H_1 \subseteq \langle C\rangle$ for a finite subset $C \subseteq A \cup B$. Now, take $c \in C$ and put $K_c = \{x \in G_1; xc = cx\}$ and $B_u = \{b \in B; c^{-1}b^{-1}cb = u\}$ for every $u \in H_1$ (here we assume $c \in A$, the other case being similar). Then $B = \bigcup B_u$, this union is disjoint and $b_2b_1^{-1} \in K_c$ for all $b_1, b_2 \in B_u$. Further, for every $u \in H_1$ such that $B_u \neq \emptyset$, choose $b_u \in B_u$ and put $D_c = \{b_u; u \in H_1\}$. Then $G_1 = K_cD_cH_1$, $|D_cH_1| \leq n^2$ and it follows easily that the index of K_c in G_1 is at most n^2. On the other hand, $K = \bigcap K_c$ and consequently the index of K in G_1 is finite, too.

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Since H is finite, G_1 is of finite index in G and we conclude that also the index of K in G is finite. Finally, if $L = K_G$ denotes the core of K in G then G/L is a finite group. Since G is infinite, L must be so and the finiteness of H implies that we can find elements $u_1, v_1 \in L \setminus H$ such that $u_1 v_1^{-1} \notin H$. Now, $H_1 = H_1 \cap H_1^{u_1} \cap H_1^{v_1} \subseteq H \cap H^{u_1} \cap H^{v_1} = 1$ and we have proven that $H_1 = 1$. Consequently, $ab = ba$ for all $a \in A$ and $b \in B$.

Next, we show that both A and B are subgroups of G. Indeed, if $H_2 = \langle A \rangle \cap H$ then $H_2^b = H_2 \subseteq H$ for every $b \in B$ and we see that $H_2 \subseteq H_G = 1$, where H_G is the core of H in G. Thus $H_2 = 1$ and it follows easily that $\langle A \rangle = A$. Quite similarly, B is a subgroup of G.

Since the index of $A \cap B$ in G is finite, we can find $e \in A \cap B$, $e \neq 1$. If $a \in A$ and $b \in B$ are such that $a^{-1}b \in H$ then $a^{-1}b \in H \cap H^u \cap H^v$, where $u = a^{-1}$ and $v = (a^{-1}ea)^{-1}$. Assume, for a moment, that $a \neq b$. Then $u, v \in G \setminus H$, $uv^{-1} \in G \setminus H$, and therefore $H \cap H^u \cap H^v = 1$ and $a = b$, a contradiction. Thus $a = b$ and we have shown that $A = B$. Clearly, this subgroup is abelian.

It remains to show that A is normal in G. Since A is of finite index in G, the core A_G is infinite. If E denotes the centralizer of A_G, then $A \subseteq E$, E is normal in G and we have $E = AH_3$ for some $H_3 \subseteq H$. Finally, $H_3 \subseteq H^w$ for every $w \in A_G$, and therefore $H_3 = 1$ and $A = E$.

References

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[2] Drápal A., *Multiplication groups of free loops II*, Czech. Math. J. **46** (121) (1996), 201–220.

[3] Drápal A., *Multiplication groups of finite loops that fix at most two points*, submitted.

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