On centrally nilpotent loops

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Abstract. Using a lemma on subnormal subgroups, the problem of nilpotency of multiplication groups and inner permutation groups of centrally nilpotent loops is discussed.

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R. Baer proved, among others, the following result ([1, Lemma 2.3]): a subgroup H of a group G is subnormal in G if and only if H is subnormal in the subgroup $\langle H, X\rangle$ for every denumerable subset X of G. Moreover, in the same paper, an easy counterexample shows that it is impossible to replace "denumerable" by "finite". As an extension of both this idea and another one [2], we deduce its new variant.

First, we recall some notions. For a subgroup H of a group G we put $H_0 = G$, $H_{i+1} = H^{H_i} = \langle xhx^{-1} \,|\, h \in H, x \in H_i\rangle$, $i = 0, 1, \dots$. If there exists an n such that $H_n = H^{H_{n-1}} = H$ then H is called a subnormal subgroup of depth (or defect) at most n in G. H is of depth (exactly) n if, moreover, $H_{n-1} \neq H$. In the last case, $G = H_0 \triangleright H_1 \triangleright \dots \triangleright H_{n-1} \triangleright H_n = H$ and H is nonnormal in H_{n-2} for $n > 1$.

Lemma. *Let H be a subgroup of a group G and n be a nonnegative integer. Then the following conditions are equivalent:*

(i) *H is subnormal of depth at most n in G;*
(ii) *H is subnormal of depth at most n in the subgroup $\langle H, X\rangle$ of G for every denumerable subset X of G;*
(iii) *H is subnormal of depth at most n in the subgroup $\langle H, X\rangle$ of G for every finite subset X of G.*

Proof: The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. As for (iii) $\Rightarrow$ (i), its proof can be deduced from the proof of [1, Lemma 2.1]. Nevertheless we present a direct proof here. Let us assume that the condition (iii) of Lemma is fulfilled but $H \neq H_n$. Then there is $x_0 \in H_{n-1}$ such that $x_0 H x_0^{-1} = H^{x_0} \not\subseteq H$. Since $H_{n-1} = H^{H_{n-2}}$, there exists a finite subset $X_1 \subseteq H_{n-2}$ such that $x_0 \in H^{X_1}$. Let us assume that $X_i \subseteq H_{n-i-1}$ is selected so that $X_{i-1} \subseteq H^{X_i}$. Then $H_{n-i-1} = H^{H_{n-i-2}}$ implies the existence of a subset $X_{i+1} \subseteq H_{n-i-2}$ such that $X_i \subseteq H^{X_{i+1}}$.

Now, for the finite subset $X_{n-1} \subseteq H_0$, we construct the subgroup $\langle H, X_{n-1}\rangle = K$. Since by (iii) the subgroup H is subnormal in K of depth n, we obtain $H^{K,n} = H$, where $K = H^{K,0}$, $H^{K,i+1} = H^{H_{K,i}}$, $i = 0, 1, \dots, n-1$. On the other hand, $X_{n-1} \subseteq K = H^{K,0}$. If $X_i \subseteq H^{K_{n-i-1}}$ then $X_{i-1} \subseteq H^{X_i} \subseteq H^{K_{n-i-1}} = H^{K_{n-i}}$. From this $x_0 \in H^{X_1} \in H^{H^{K,n-2}} = H^{K,n-1}$ and hence $H^{x_0} \subseteq H^{H^{X_1}} \subseteq H^{H^{K,n-1}} = H$ in contradiction to our assumption. □

Remark. For $n = 2$, there is the fourth equivalent condition:

(iv) *H is subnormal of depth at most 2 in the subgroup $\langle H, X\rangle$ of G for every subset X of G, $|X| = 1$.*

PROOF: Let, on the contrary, condition (iv) be satisfied and $H_2 \neq H$. Since H is a nonnormal subgroup in G, there is an element $x_0 \in G$ such that $x_0 H x_0^{-1} = H^{x_0} \not\subseteq N_G(H)$ (the normalizer of H in G) Then there are elements $h_0 \in H$ and $x_0 h_0 x_0^{-1} = x_1$ such that $x_1 H x_1 = H^{x_1} \subseteq H^{H_1} = H_2$ and $H \not\supseteq H^{x_1}$. Now we construct the subgroup $A = \langle H, x_0\rangle$ and then $H_{A,0} = A$, $H^{H_{A,0}} = H_{A,1} \ni x_0$ and $H^{x_1} \subseteq H^{H_{A,1}} = H_{A,2} = H$ in contradiction to our assumption. □

The equivalence of (i) and (iv) is false for $n = 3$: there is a group of order 5^{20} and exponent 5 with the properties that every 2 elements generate a subgroup of class 3 and that the group itself has class precisely 5 ([6, Theorem 4]). For $n > 3$, an expected answer is also negative.

As an immediate corollary of Lemma we obtain a new version of well known

Theorem 1 ([3, 2.19])**.** *Let Q be a loop with inner permutation group $I(Q)$ and multiplication group $M(Q)$. Then the following statements are equivalent:*

(1) *$I(Q)$ satisfies at least one (and then every) of the conditions of Lemma;*
(2) *Q is centrally nilpotent of class at most n.*

It can also be proved that the multiplication group $M(Q)$ of a centrally nilpotent loop Q is soluble ([3, Proposition 2.22]). This leads to a natural

Question. For which class of centrally nilpotent loops their multiplication groups are nilpotent?

Moreover, the question is under which hypotheses the following statements:

(3) $M(Q)$ is nilpotent of class at most m;
(4) $I(Q)$ is subnormal and nilpotent of class at most $n-1$;

are equivalent to the condition (2) of Theorem 1?

In an attempt to answer this question, we examine in a loop Q the (upper) central series

$$(\alpha) \qquad e = Z_0 \subset Z_1 \subset \ldots \subset Z_i \subset Z_{i+1} \subset \ldots \subset Z_n = Q,$$

where $Z_{i+1}/Z_i = Z(Q/Z_i)$, $i = 0, 1, \ldots, n-1$ ($Z(Q)$ denotes the center of the loop Q), which induces invariant series both in $M(Q) = G$

$$(\beta) \quad 1 = \bar{C}_0 \subset \bar{C}_1 \subset Z_1^* \subset \bar{C}_2 \subset \ldots \subset Z_i^* \subset \bar{C}_{i+1} \subset Z_{i+1}^* \subset \ldots$$
$$\ldots \subset Z_{n-1}^* \subset \bar{C}_n = G,$$

where $Z_i^* = \{ \Psi \in G \mid \Psi(x) \equiv x \,(\bmod\, Z_1), x \in Q \}$, $\bar{C}_{i+1}/Z_i^* = C(G/Z_i^*)$, $i = 0, 1, \ldots, n-1$, and in the inner permutation group $I(Q) = I$

$$(\gamma) \qquad 1 = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_i \subset I_{i+1} \subset \ldots \subset I_{n-1} = I,$$

where $I_i = I \cap Z_i^*$, $i = 0, 1, \ldots, n-2$.

When the series (α) induces also the upper central series of $M(Q)$

$$(\delta) \qquad 1 = C_0 \subset C_1 \subset C_2 \subset \ldots \subset C_i \subset C_{i+1} \subset \ldots \subset C_m = G,$$

where $C_{i+1}/C_i = C(G/C_i)$ and $C_1 = \bar{C}_1 \cong Z_1$?

Besides the trivial case $C_i = Z_i^*$, $i = 0, 1, \ldots, n-1$, when $Q \cong M(Q)$ is Abelian, a central refinement of (β) by (δ) is possible in the following situations:

(A) $Z_i^* \subsetneqq C_{i+1} = \bar{C}_{i+1}$, $i = 0, 1, \ldots, n-1$, and evidently $M(Q)$ will be nilpotent of class $m = n$;

(B) $Z_i^* = C_{2i}$, and then $\bar{C}_{i+1} = C_{2i+1}$, $i = 0, 1, \ldots, n-1$, so that $M(Q)$ will be nilpotent of class $m = 2n - 1$.

In both cases (A) and (B), we have the following conclusion:

(Γ) $Z_1^* \subseteq C_2 \Leftrightarrow Z_1^* \cap I = C_2 \cap I = I_1$, in particular $I_1 \subseteq C(I)$ and $Z_1^* = C_1 \cdot I_1$, $C_1 \cap I_1 = 1$.

In fact, every $\Psi \in Z_1^*$ has a unique representation as $\Psi = L_z \Theta$, $z \in Z_1$, $\Theta \in I_1 = Z_1^* \cap I$ and $I_1 \cap C_1 = 1$, so that the converse implication is trivial. If $Z_1^* \subseteq C_2$ then $(C_2/Z_1^*) \cap I/Z_1^* \cap I \subseteq (\bar{C}_2/Z_1^*) \cap (I/Z_1^* \cap I) = 1$, i.e. $Z_1^* \cap I = C_2 \cap I = I_1$. Now for $\Theta \in I_1$, $\eta \in I$ we have $\Theta^{-1}\eta^{-1}\Theta\eta \in (C_1 \cap I_1) = 1$, hence $\Theta \in I_1 \subseteq C(I)$.

Using (Γ) and induction on i, we can easily deduce:

(Δ) In both cases (A) and (B), the inner permutation group $I(Q) = I$ of Q is nilpotent of class (at most) $n - 1$.

Now, according to what has been said above, we can formulate

Proposition. *Under hypotheses of Theorem* 1 *and provided that either* (A) *or* (B) *is fulfilled, the following statement is valid:* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4).

Indeed, it is clear that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Since the series (α) and (β) are dual, (3) $\Rightarrow$ (2) is also correct. Moreover, the implication (4) $\Rightarrow$ (2) will be correct in a particular case of (Γ):

(Γ_0) $Z_1^* \subseteq C_2 \Rightarrow Z_1^* \cap I = Z_1^* \cap C_2 = I_1 = C(I)$.

For example, this condition is true for commutative Moufang loops ([4, Lemma 11.6, Chapter VIII]). The case (B) is realized by

Theorem 2 (cf. [4, 11.4, Chapter VIII]; [5]). *Let Q be a commutative A-loop ($I(Q) \subseteq \mathrm{Aut}(Q)$) with inner permutation group $I = I(Q)$ and multiplication group $M(Q)$. Then the following statements are equivalent:*

(I) *Q is centrally nilpotent of class at most n;*
(II) *$M(Q)$ is nilpotent of class at most $2n-1$.*

Proof: According to Proposition, it is sufficient to establish $Z_1^* = C_2$ and to use easy induction on i. For every $\Theta \in Z_1^* \cap I$, $x \in Q$ and some $z \in Z_1$, we have $\Theta(x) = xz$. Using $\Theta \in \mathrm{Aut}(Q)$ we get $\Theta^{-1} L_x \Theta = L_{\Theta(x)} = L_x L_z$ and hence $L_x^{-1}\Theta^{-1}L_x\Theta = L_z \in C_1$, i.e. $\Theta \in C_2$. According to (Γ) we have $Z_1^* \subseteq C_2$. For the proof of the inverse inclusion, writing $\Psi \in C_2$ as $\Psi = L_a\Theta$, $a = \Psi(e)$, $\Theta \in I$ and using $I \subseteq \mathrm{Aut}(Q)$, we get a chain of equalities and congruences: $L_a L_{\Theta(x)}\Theta = L_a\Theta L_x \equiv L_x L_a \Theta \pmod{C_1}$, i.e. $L_a L_{\Theta(x)} = L_x L_a L_z$ for every $x \in Q$ and suitable $z \in Z_1$. From this $\Theta(x) = L_{\Theta(x)}(e) = L_a^{-1} L_z L_x L_a(e) = L_a^{-1}(a \cdot xz) = xz$, i.e. $\Theta \in Z_1^*$. Since $L_{az} = L_a L_z$, we get $L_a L_x L_z = L_x L_a L_z$, i.e. $L_a L_x = L_x L_a$ for every $x \in Q$. Hence $a \in Z_1$ and $L_a\Theta = \Psi \in Z_1^*$. Therefore $Z_1^* = C_2$. $\square$

As an immediate consequence of Theorem 2, the case (A) is impossible for commutative A-loops.

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